Application of the Sturm-Liouville theory to classical and quantum field theory in AdS

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Joint work with Claudio Dappiaggi

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Microlocal analysis: a tool to explore the quantum world

0. Introduction

- We study systematically the classical and quantum theory of a Klein-Gordon field on anti-de Sitter (AdS) on a more mathematical precise fashion, extending the work of Avis, Isham, Storey (1978), Allen & Jacobson (1986) and others.

- We consider all suitable boundary conditions at infinity, by treating the system as a Sturm-Liouville problem, complementing the work of Wald & Ishibashi (2004).

- We propose a natural generalisation of the Hadamard condition for quantum states obeying these boundary conditions on a spacetime with a timelike boundary. This allows to properly construct a quantum theory for the Klein-Gordon field.

- We use this system as an example to study classical and quantum field theory on manifolds with boundaries.

1 Geometric preliminaries

2 Sturm-Liouville problem and boundary conditions

3 Hadamard condition for AdS

4 Conclusions

Outline

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- **Definition:** *Anti-de Sitter* AdS_{d+1} $(d \geq 2)$ is the maximally symmetric solution to the vacuum Einstein's equations with a negative cosmological constant $\Lambda < 0$. It is defined as the hypersurface in $\mathbb{R}^{d+2}$ with metric

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 - \mathrm{d}X_1^2 + \sum_{i=2}^{d+1} \mathrm{d}X_i^2$$

given by the relation

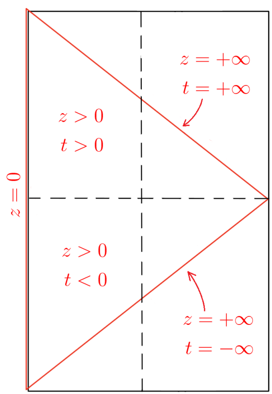
$$-X_0^2 - X_1^2 + \sum_{i=2}^{d+1} X_i^2 = -\ell^2 , \qquad \ell \doteq -\frac{d(d-1)}{\Lambda} .$$

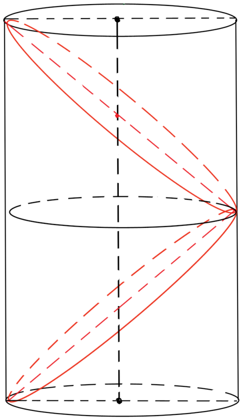
- Anti-de Sitter spacetime is **not** globally hyperbolic: it possesses a timelike boundary at spatial infinity.

- Poincaré patch (t, z, x_i), $t \in \mathbb{R}$, $z \in \mathbb{R}_{>0}$ and $x_i \in \mathbb{R}$, $i = 1, \ldots, d-1$,

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}z^2 + \delta^{ij}\mathrm{d}x_i\mathrm{d}x_j\right) .$$

The region covered by this chart is the *Poincaré fundamental domain*, PAdS_{d+1}.



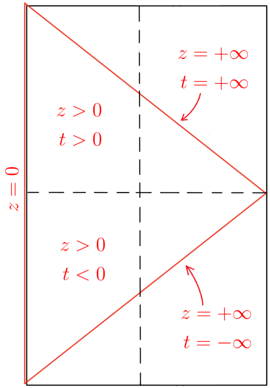


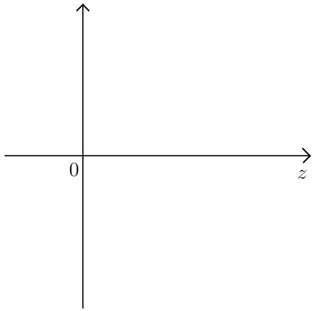
1. Geometric preliminaries

- PAdS_{d+1} can be mapped to $\mathring{\mathbb{H}}^{d+1} \doteq \mathbb{R}_{>0} \times \mathbb{R}^d$ via a conformal rescaling

$$\mathrm{d}s^2 \mapsto \frac{z^2}{\ell^2}\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \delta^{ij}\mathrm{d}x_i\mathrm{d}x_j \,.$$

We can attach a conformal boundary as the locus $z = 0$ and obtain $\mathbb{H}^{d+1} \doteq \mathbb{R}_{\geq 0} \times \mathbb{R}^d$, the *half Minkowski spacetime*.





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2. Sturm-Liouville problem and boundary conditions

2.1. Field equation as a Sturm-Liouville equation

- **Klein-Gordon equation.** Poincaré domain (PAdS_{d+1}, g), $\phi : \mathrm{PAdS}_{d+1} \to \mathbb{R}$,

$$P\phi = \left(\Box_g - m_0^2 - \xi R\right)\phi = 0\,.$$

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- **Lemma:** *In the half Minkowski spacetime* $(\mathring{\mathbb{H}}^{d+1}, \eta)$, $\Phi = (\frac{z}{\ell})^{\frac{1-d}{2}}\phi : \mathring{\mathbb{H}}^{d+1} \to \mathbb{R}$ *is a solution of*

$$P_{\mathbb{H}}\Phi = \left(\Box_\eta - \frac{m^2}{z^2}\right)\Phi = 0\,,$$

with $m^2 \doteq m_0^2 - (\xi - \frac{d-1}{4d})R$.

- **Remark:** From now on, we set $\ell = 1$.

2.1. Field equation as a Sturm-Liouville equation

- **Fourier expansion.** Fourier representation of Φ:

$$\Phi = \int_{\mathbb{R}^d} \mathrm{d}^d\underline{k}\, e^{i\underline{k}\cdot\underline{x}}\, \widehat{\Phi}_{\underline{k}}\,, \qquad \underline{x} \doteq (t, x_1, \ldots, x_{d-1})\,, \quad \underline{k} \doteq (\omega, k_1, \ldots, k_{d-1})\,,$$

where $\widehat{\Phi}_{\underline{k}}$ are solutions of the ODE

$$L\widehat{\Phi}_{\underline{k}}(z) \doteq \left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{m^2}{z^2}\right)\widehat{\Phi}_{\underline{k}}(z) = \lambda\,\widehat{\Phi}_{\underline{k}}(z)\,, \qquad \lambda \doteq \omega^2 - \sum_{i=1}^{d-1} k_i^2$$

This is a *Sturm-Liouville problem* on $z \in (0, +\infty)$ with spectral parameter λ.

- **Recall:** A Sturm-Liouville equation is of the form

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left(p(z)\frac{\mathrm{d}y}{\mathrm{d}z}\right) + q(z)y = \lambda w(z)y\,, \quad z \in (a, b)\,,$$

with $-\infty \leq a,\, b \leq \infty$, $\lambda \in \mathbb{C}$, p^{-1}, q, $w \in L^1_{\mathrm{loc}}(a, b)$ and w a weight function.

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with $-\infty \leq a,\, b \leq \infty$, $\lambda \in \mathbb{C}$, p^{-1}, q, $w \in L^1_{\mathrm{loc}}(a, b)$ and w a weight function.

- **Lemma** [Fulton (2008)]**:** *The spectrum of L contains a continuous spectrum $\sigma_{\mathrm{c}} \subset (0, \infty)$. For $\lambda \in (0, \infty)$, two eigenfunctions in $L^2(0, \infty)$ are $\sqrt{z}\, J_\nu\big(z\sqrt{\lambda}\big)$ and $\sqrt{z}\, Y_\nu\big(z\sqrt{\lambda}\big)$, with $\nu \doteq \frac{1}{2}\sqrt{1 + 4m^2} \geq 0$.*

2.1. Field equation as a Sturm-Liouville equation

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left(p(z)\frac{\mathrm{d}y}{\mathrm{d}z}\right)+q(z)y=\lambda w(z)y\,,\quad z\in(a,b)\,,$$

Definition (Endpoint classification)**:** The endpoint a is

1. *regular* if $a \in \mathbb{R}$ and $\exists\, c \in (a,b)$ s. t. p^{-1}, q, $w \in L^1_{\mathrm{loc}}(a,c]$; otherwise, it is *singular*;
2. *limit-circle* (LC) if, for some $\lambda \in \mathbb{C}$, all solutions of the equation are in $L^2(a,c]$ for some $c \in (a,b)$; it is *limit-point* (LP) otherwise.

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Proposition: *For the SL problem*

$$L\widehat{\Phi}_{\underline{k}}(z)\doteq\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{m^2}{z^2}\right)\widehat{\Phi}_{\underline{k}}(z)=\lambda\,\widehat{\Phi}_{\underline{k}}(z)\,,\qquad z\in(0,+\infty)$$

the classification for the endpoint 0 is as given in the following table

m^2	$\nu\doteq\frac{1}{2}\sqrt{1+4m^2}$	Classification of 0
$m^2=0$	$\nu=\frac{1}{2}$	Regular
$m^2\in[-\frac{1}{4},\frac{3}{4})$, $m^2\neq 0$	$\nu\in[0,1)$, $\nu\neq\frac{1}{2}$	Limit-circle (LC)
$m^2\in[\frac{3}{4},\infty)$	$\nu\in[1,\infty)$	Limit-point (LP)

The endpoint $+\infty$ is LP for all $\nu\geq 0$.

2.2. Boundary conditions

For a boundary-value problem with one or two singular endpoints, regular boundary conditions are no longer valid and need to be generalised.

Definition: Given two differentiable functions u, v, the *Wronskian* is

$$W[u,v](z) \doteq u(z)\bar{v}'(z) - \bar{v}(z)u'(z)\,.$$

The Wronskian has a finite limit at each endpoint, even if singular.

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Definition (Robin boundary condition)**:** At an endpoint a, for a solution Φ, when

1. a is regular (R), takes the form
$$\cos(\alpha)\,\Phi(a) + \sin(\alpha)\,\Phi'(a) = 0\,, \quad \alpha \in [0,\pi)\,.$$
2. a is limit-circle (LC), takes the form
$$\lim_{z\to a}\left\{\cos(\alpha)\,W[\Phi,\Psi_1](z) + \sin(\alpha)\,W[\Phi,\Psi_2](z)\right\} = 0\,, \quad \alpha \in [0,\pi)\,,$$
where $\{\Psi_1,\Psi_2\}$ is a linearly independent basis of solutions.

Remark: The LC reduces to R case if $\Psi_1(0)=0$, $\Psi_1'(0)=1$, $\Psi_2(0)=-1$ and $\Psi_2'(0)=0$.

2.2. Boundary conditions

Theorem: *The SL problem*

$$L\widehat{\Phi}_{\underline{k}}(z) \doteq \left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{m^2}{z^2}\right)\widehat{\Phi}_{\underline{k}}(z) = \lambda\,\widehat{\Phi}_{\underline{k}}(z)\,, \qquad z \in (0, +\infty)\,,$$

is well-posed when the boundary conditions at 0 are chosen as in the following table

$\nu \doteq \frac{1}{2}\sqrt{1+4m^2}$	Classification	Boundary condition at $z=0$
$\nu = \frac{1}{2}$	Regular (R)	$\cos(\alpha)\widehat{\Phi}_{\underline{k}}(0) + \sin(\alpha)\widehat{\Phi}'_{\underline{k}}(0) = 0$
$\nu \in [0,1)$, $\nu \neq \frac{1}{2}$	Limit-circle (LC)	$\cos(\alpha)W\left[\widehat{\Phi}_{\underline{k}}, \widehat{\Phi}^1_{\underline{k}}\right] + \sin(\alpha)W\left[\widehat{\Phi}_{\underline{k}}, \widehat{\Phi}^2_{\underline{k}}\right] = 0$
$\nu \in [1,\infty)$	Limit-point (LP)	Not required

where $\left\{\widehat{\Phi}^1_{\underline{k}}, \widehat{\Phi}^2_{\underline{k}}\right\}$ *is a basis of linearly independent solutions. For* $\nu > 0$*, they are given by*

$$\widehat{\Phi}^1_{\underline{k}}(z) = \sqrt{\frac{\pi}{2}}\left(\sqrt{\lambda}\right)^{-\nu}\sqrt{z}\,J_\nu\left(z\sqrt{\lambda}\right), \qquad \widehat{\Phi}^2_{\underline{k}}(z) = -\sqrt{\frac{\pi}{2}}\left(\sqrt{\lambda}\right)^{\nu}\sqrt{z}\,J_{-\nu}\left(z\sqrt{\lambda}\right).$$

Corollary: *The solution of the SL problem may be written as*

$$\widehat{\Phi}_{\underline{k}}(z) = \mathcal{N}_\alpha\left[\cos(\alpha)\,\widehat{\Phi}^1_{\underline{k}}(z) + \sin(\alpha)\,\widehat{\Phi}^2_{\underline{k}}(z)\right], \qquad \begin{cases} \alpha \in [0,\pi)\,, & \nu \in [0,1)\,, \\ \alpha = 0\,, & \nu \in [1,\infty)\,. \end{cases}$$

2.3. Examples

2.3.1. Case $\nu = \frac{1}{2}$: massless, conformally coupled scalar

- Most general boundary condition: **regular Robin** boundary condition,

$$\cos(\alpha)\,\widehat{\Phi}_{\underline{k}}(0) + \sin(\alpha)\,\widehat{\Phi}'_{\underline{k}}(0) = 0\,, \quad \alpha \in [0, \pi)$$

  - $\alpha = 0$: Dirichlet boundary condition.
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- Spectrum:

$$\sigma = \begin{cases} (0, \infty)\,, & \alpha \in \{0\} \cup [\frac{\pi}{2}, \pi)\,, \\ (0, \infty) \cup \{-\cot^2(\alpha)\}\,, & \alpha \in (0, \frac{\pi}{2})\,. \end{cases}$$

  Eingenfunctions:

  - $\lambda \in (0, \infty)$: $\quad \widehat{\Phi}^1_{\underline{k}}(z) = \dfrac{\sin\left(z\sqrt{\lambda}\right)}{\sqrt{\lambda}}$, $\widehat{\Phi}^2_{\underline{k}}(z) = -\cos\left(z\sqrt{\lambda}\right)$.
  - $\lambda = -\cot^2(\alpha)$: $\widehat{\Phi}_{\underline{k}}(z) = e^{-\cot(\alpha) z}$ ("bound state" mode solution).

2.3. Examples

2.3.2. Case $\nu \in (0,1) \setminus \{\frac{1}{2}\}$

- Basis of linearly independent solutions: $\{\widehat{\Phi}^1_{\underline{k}}, \widehat{\Phi}^2_{\underline{k}}\}$.

$$\widehat{\Phi}^1_{\underline{k}}(z) = \sqrt{\frac{\pi}{2}}\,\big(\sqrt{\lambda}\big)^{-\nu}\sqrt{z}\,J_\nu\big(z\sqrt{\lambda}\big)\,, \qquad \widehat{\Phi}^2_{\underline{k}}(z) = -\sqrt{\frac{\pi}{2}}\,\big(\sqrt{\lambda}\big)^{\nu}\sqrt{z}\,J_{-\nu}\big(z\sqrt{\lambda}\big)\,.$$

- Most general boundary condition: **singular Robin** boundary condition,

$$\lim_{z\to 0}\left\{\cos(\alpha)\,W\big[\Phi, \widehat{\Phi}^1_{\underline{k}}\big](z) + \sin(\alpha)\,W\big[\Phi, \widehat{\Phi}^2_{\underline{k}}\big](z)\right\} = 0\,, \quad \alpha \in [0,\pi)\,.$$

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- Spectrum:

$$\sigma = \begin{cases} (0,\infty)\,, & \alpha\in\{0\}\cup[\frac{\pi}{2},\pi)\,,\\ (0,\infty)\cup\{-\cot^{1/\nu}(\alpha)\}\,, & \alpha\in(0,\frac{\pi}{2})\,. \end{cases}$$

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- $\lambda\in(0,\infty)$: $\widehat{\Phi}^1_{\underline{k}}(z)$, $\widehat{\Phi}^2_{\underline{k}}(z)$.
- $\lambda = -\cot^{1/\nu}(\alpha)$: $\widehat{\Phi}_{\underline{k}}(z) = \sqrt{z}\,K_\nu\left(\cot^{1/(2\nu)}(\alpha)\,z\right)$ ("bound state" mode solution).

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- **Quantum state.** We investigate if one can define a quantum state which obeys the aforementioned boundary conditions and which satisfies a natural analogue of the *Hadamard condition* of globally hyperbolic spacetimes.

- A state ω is fully characterised by its *two-point distribution*

$$\omega_2(x, x') \doteq \omega(\phi(x)\phi(x')) \,.$$

In $\mathring{\mathbb{H}}^{d+1}$, the conformally related quantity is

$$\omega_2^{\mathbb{H}}(x, x') = (zz')^{\frac{1-d}{2}} \omega_2(x, x') \,,$$

a solution of $(P_\eta \otimes \mathbb{I})\, \omega_2^{\mathbb{H}} = (\mathbb{I} \otimes P_\eta)\, \omega_2^{\mathbb{H}} = 0$.

- Fourier-Bessel transform along $\underline{x} \ni \mathbb{R}^d$

$$\omega_2^{\mathbb{H}}(x, x') = \lim_{\epsilon \to 0^+} \int_0^\infty \mathrm{d}q\, q \int_0^\infty \mathrm{d}k\, k \, \frac{e^{-i\sqrt{k^2+q^2}(t-t'-i\epsilon)}}{\sqrt{2\pi(k^2+q^2)}} \left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr)\, \widehat{G}_{\underline{k}}(z, z') \,,$$

where

$$(L \otimes \mathbb{I})\, \widehat{G}_{\underline{k}} = (\mathbb{I} \otimes L)\, \widehat{G}_{\underline{k}} = \lambda\, \widehat{G}_{\underline{k}} \,, \qquad L = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{m^2}{z^2} \,.$$

3.1. Two-point distribution in AdS

Finding $\widehat{G}_{\underline{k}}$, and thus $\omega_2^{\mathbb{H}}$, is a problem of eigenfunction expansion of the δ-distribution (c.f. Titchmarsh 1962).

Proposition: *The two-point distribution $\omega_2^{\mathbb{H}} \in \mathcal{D}'(\mathring{\mathbb{H}}^{d+1} \times \mathring{\mathbb{H}}^{d+1})$ for different values of $\nu \in \mathbb{R}_{>0}$ has integral kernel as follows:*

- *If $\nu \in [1, \infty)$, with no boundary conditions at 0,*

$$\omega_2^{\mathbb{H}}(x, x') = \lim_{\epsilon \to 0+} \mathcal{N}\sqrt{zz'} \int_0^\infty \mathrm{d}k\, k \left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr) \int_0^\infty \mathrm{d}q\, q\, \frac{e^{-i\sqrt{k^2+q^2}(t-t'-i\epsilon)}}{\sqrt{2\pi(k^2+q^2)}} J_\nu(qz) J_\nu(qz')\,.$$

- *If $\nu \in (0, 1)$ with Robin boundary conditions such that $c \doteq \cot(\alpha) < 0$,*

$$\omega_2^{\mathbb{H}}(x, x') = \lim_{\epsilon \to 0+} \mathcal{N}\sqrt{zz'} \int_0^\infty \mathrm{d}k\, k \left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr) \int_0^\infty \mathrm{d}q\, q\, \frac{e^{-i\sqrt{k^2+q^2}(t-t'-i\epsilon)}}{\sqrt{2\pi(k^2+q^2)}}$$
$$\times \frac{\left[cJ_\nu(qz) - q^{2\nu} J_{-\nu}(qz)\right]\left[cJ_\nu(qz') - q^{2\nu} J_{-\nu}(qz')\right]}{c^2 - 2cq^{2\nu}\cos(\nu\pi) + q^{4\nu}}\,.$$

Remark: There is **no** ground state for Robin boundary conditions with $c > 0$ and for $\nu = 0$ due to the bound states, as AdS symmetry is not preserved.

3.1. Two-point distribution in AdS

Proposition: *Let:*

$$G_1(x,x') = \lim_{\epsilon\to 0^+} \frac{F\left(\frac{d}{2}+\nu, \frac{1}{2}+\nu; 1+2\nu; \left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{-2}\right)}{\left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{\frac{d}{2}+\nu}},$$

$$G_2(x,x') = \lim_{\epsilon\to 0^+} \frac{F\left(\frac{d}{2}-\nu, \frac{1}{2}-\nu; 1-2\nu; \left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{-2}\right)}{\left[\cosh\left(\frac{\sqrt{2\sigma_\epsilon}}{2}\right)\right]^{\frac{d}{2}-\nu}},$$

where $\sigma_\epsilon \doteq \sigma + 2i\epsilon(t-t') + \epsilon^2$ *and* F *is the Gaussian hypergeometric function.*

The integral kernel of the two-point distribution for the ground state on $PAdS_{d+1}$ *is*

$$\omega_2(x,x') = \begin{cases} \mathcal{N}\, G_1(u)\,, & \nu \in [1,\infty)\,, \\ \mathcal{N}_\alpha \left[\cos(\alpha)\, G_1(u) + \sin(\alpha)\, G_2(u)\right]\,, & \nu \in (0,1)\,, \end{cases}$$

where $\mathcal{N}$ *and* $\mathcal{N}_\alpha$ *are normalization constants and* $\alpha \in (\frac{\pi}{2}, \pi)$.

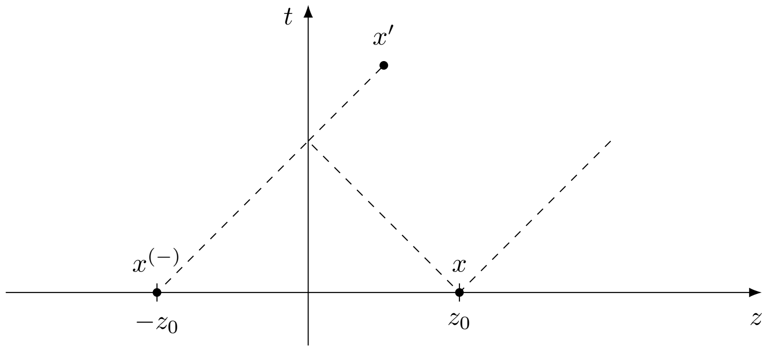
3.2. Hadamard condition in AdS

Proposition: *Let $H(x,x')$ be the Hadamard parametrix in $PAdS_{d+1}$ and let $H^{(-)}(x,x') \doteq \iota_z H(x,x')$, where $\iota_z(x,x') \doteq (\underline{x}, -z; \underline{x}', z')$. Then, if $\alpha \neq \frac{3\pi}{4}$, the two-point distribution $\omega_2(x,x')$ is such that*

$$\omega_2(x,x') - H(x,x') - i(-1)^{-\nu}\frac{\cos(\alpha) + (-1)^{-2\nu}\sin(\alpha)}{\cos(\alpha)+\sin(\alpha)} H^{(-)}(x,x')$$

lies in $C^\infty(PAdS_{d+1} \times PAdS_{d+1})$.

Remark: If $\nu = \frac{1}{2}$, we recover the *method of images.*

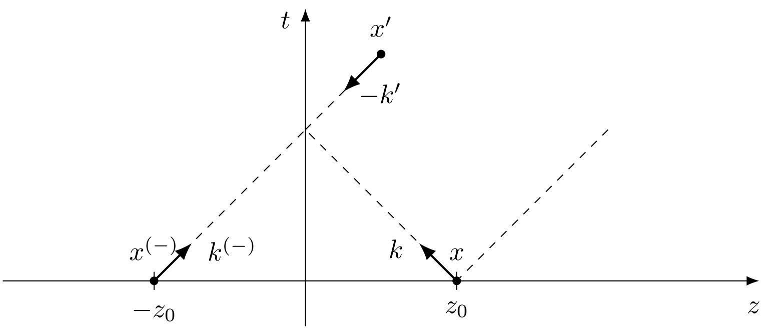


3.2. Hadamard condition in AdS

Theorem: *The wavefront set of the two-point distribution $\omega_2^{\mathbb{H}}$ in $\mathring{\mathbb{H}}^{d+1}$ is given by*

$$WF(\omega_2^{\mathbb{H}}) = \left\{(x,k;x',k') \in T^*(\mathring{\mathbb{H}}^{d+1})^{\times 2} \setminus \{0\} : (x,k) \sim_\pm (x',k'),\ k \triangleright 0\right\}$$

- $\sim_\pm$: $\exists$ *null geodesics* $\gamma, \gamma^{(-)} : [0,1] \to \mathring{\mathbb{H}}^{d+1}$ *with*
  - $\gamma(0) = x = (\underline{x}, z)$, $\gamma^{(-)}(0) = x^{(-)} = (\underline{x}, -z)$ *and* $\gamma(1) = x'$;
  - $k = (k_{\underline{x}}, k_z)$ $(k^{(-)} = (k_{\underline{x}}, -k_z))$ *is coparallel to* γ $(\gamma^{(-)})$ *at 0;*
  - $-k'$ *is the parallel transport of* k $(k^{(-)})$ *along* γ $(\gamma^{(-)})$ *at 1;*
- $k \triangleright 0$: k *is future-directed.*



3.2. Hadamard condition in AdS

Definition: We call a state $\omega^{\mathbb{H}}$ a **Hadamard state** for a scalar field in $\mathring{\mathbb{H}}^{d+1}$ if its two-point distribution has a wavefront set as above.

This definition can be read as a generalization at the level of states of F-locality.

Proposition: Any Hadamard state ω for a scalar field on $\mathring{\mathbb{H}}^{d+1}$ is such that $\omega_{2,D}$, the restriction to any globally hyperbolic subregion $D \subset \mathring{\mathbb{H}}^{d+1}$ of the two-point distribution $\omega_2^{\mathbb{H}}$, has a wavefront set of Hadamard form

$$WF(\omega_{2,D}) = \left\{(x,k;x',k') \in T^*(D \times D) \setminus \{0\} : (x,k) \sim (x',k'),\, k \triangleright 0\right\}.$$

Remarks:

- These results are in full agreement with Wrochna (2016).
- We have not proved a *local to global* analogue of the Radzikowski result for globally hyperbolic spacetimes. We conjecture that it holds for fixed boundary conditions and field parameters.
- With the definition of Hadamard states above, it is possible to construct a global algebra of Wick polynomials in AdS.

Outline

1 Geometric preliminaries

2 Sturm-Liouville problem and boundary conditions

3 Hadamard condition for AdS

4 Conclusions

4. Conclusions

- We studied the classical and quantum field theory of a massive scalar field on AdS on a more mathematical rigorous way. We treated the classical dynamics as a singular Sturm-Liouville problem, determining all the suitable boundary conditions at infinity, which only depend on the mass of the field.

- We obtained the two-point distributions for states obeying these boundary conditions and showed that, besides the usual singularity at the coincidence limit, there exists only one extra singularity given by the method of images, independently of the mass of the field. This suggests a natural generalisation of the Hadamard condition to spacetimes with timelike boundaries.

- Next steps:
  - extend this formalism to a larger class of spacetimes with boundaries;
  - relate the states constructed in AdS to states on an QFT at the boundary in order to construct Hadamard states in asymptotically AdS spacetimes.

THANK YOU FOR YOUR ATTENTION!