Construction of Hadamard states with the fermionic signature operator

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Adv. Theor. Math. Phys. **2** (1998) 963 – 985

Definition of the Dirac Sea in the Presence of External Fields

Felix Finster [1]

Dirac equation in Minkowski space

$$(i\gamma^k \partial_k - m)\,\psi = 0$$

on solutions scalar product

$$(\psi|\phi)_m := \int_{t=\text{const}} (\overline{\psi}\gamma^0\phi)(t,\vec{x})\,d\vec{x}$$

is time independent due to current conservation,
makes solution space to Hilbert space $(\mathcal{H}_m, (.|.)_m)$
(completion of solutions with spatially compact support)

Solve with Fourier transformation,

$$\psi(x) = \int \frac{d^4k}{(2\pi)^4}\, \hat{\psi}(k)\, e^{-ikx}\, d^4k$$

$$(\not{k} - m)\hat{\psi}(k) = 0$$

$$\begin{cases} k^0 > 0 & \text{positive frequency (= energy)} \\ k^0 < 0 & \text{negative frequency} \end{cases}$$

$$\mathcal{H}_m = \mathcal{H}_+ \oplus \mathcal{H}_- \qquad \text{decomposition of solution space}$$

or equivalently

$$P(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\, d^4k$$

kernel of the fermionic projector, describes Dirac sea

Now consider an external potential $\mathcal{B}(x)$

$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\psi = 0$$

How does $P(x,y)$ depend on $\mathcal{B}$?

- What is the Hadamard expansion (= light cone expansion) of $P(x,y)$?

A naive perturbation expansion

$$(i\partial\!\!\!/ - m)\,\psi = -e\,\mathcal{B}\psi$$

formal power expansion in e:

$$\psi = \sum_{p=0}^{\infty} e^p\,\psi^{(p)}$$

gives sequence of equations

$$\begin{aligned}
(i\partial\!\!\!/ - m)\,\psi^{(0)} &= 0 \\
(i\partial\!\!\!/ - m)\,\psi^{(1)} &= -\mathcal{B}\,\psi^{(0)} \\
\vdots \quad & \quad \vdots \\
(i\partial\!\!\!/ - m)\,\psi^{(p+1)} &= -\mathcal{B}\,\psi^{(p)}
\end{aligned}$$

$$(i\partial\!\!\!/ - m)\,\psi^{(p+1)} = -\mathcal{B}\,\psi^{(p)}$$

solve with Green's operators:

$$(i\partial\!\!\!/_x - m)\,s(x,y) = \delta^4(x-y)$$

$$(s\psi)(x) = \int s(x,y)\,\psi(y)\,d^4y$$

$$\psi^{(p+1)} = -s^{(p)}\,\mathcal{B}\,\psi^{(p)}$$

Problem: Green's functions not unique!

- equivalently: no initial data given

$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\psi = 0$$

The causal perturbation expansion

Consider causal Green's functions

$$s_m^{\wedge,\vee}(q) = \lim_{\nu \searrow 0} \frac{\not{q} + m}{q^2 - m^2 \mp i\nu q^0}$$

Then their difference is a homogeneous solution,

$$\begin{aligned} k_m(x,y) &:= \frac{1}{2\pi i}\big(s^\vee - s^\wedge\big)(x,y) \\ &= \int \frac{d^4k}{(2\pi)^4}\,(\not{k} - m)\,\delta(k^2 - m^2)\,\epsilon(k^0)\,e^{-ik(x-y)}\,d^4k \end{aligned}$$

satisfies distributional multiplication rule

$$k_m\,k_{m'} = \delta(m - m')\,p_m$$

where

$$p_m(x,y) = \int \frac{d^4k}{(2\pi)^4}\,(\not{k} - m)\,\delta(k^2 - m^2)\,e^{-ik(x-y)}\,d^4k$$

$$\text{define} \quad P(x,y) := \frac{1}{2}\big(p_m - k_m\big)(x,y)$$

Lemma

$$k_m \, k_{m'} = \delta(m - m') \, p_m$$

Proof.

Consider k_m as multiplication operator in momentum space:

$$k_m(q) = (\not{q} - m) \, \delta(q^2 - m^2) \, \epsilon(q^0)$$

Then

$$\begin{aligned}
&k_m(q) \, k_{m'}(q) \\
&= (\not{q} + m) \, \delta(q^2 - m^2) \, \epsilon(q^0) \, (\not{q} + m') \, \delta(q^2 - (m')^2) \, \epsilon(q^0) \\
&= \big(q^2 + (m + m') \, \not{q} + mm'\big) \, \delta(m^2 - (m')^2) \, \delta(q^2 - m^2) \\
&= \big(q^2 + (m + m') \, \not{q} + mm'\big) \, \frac{1}{2m} \, \delta(m - m') \, \delta(q^2 - m^2) \\
&= \delta(m - m') \, (\not{q} + m) \, \delta(q^2 - m^2) = \delta(m - m') \, p_m(q)
\end{aligned}$$

$\square$

$$\tilde{s}_m^\vee = \sum_{k=0}^\infty \left(-s_m^\vee \mathcal{B}\right)^k s_m^\vee \qquad . \tag{3.2}$$

$$\tilde{k}_m = \frac{1}{2\pi i} \left(\tilde{s}_m^\vee - \tilde{s}_m^\wedge\right) \tag{3.9}$$

for the positive and negative frequency states by taking the absolute value of $\tilde{k}_m$,

$$\tilde{p}_m \overset{\text{formally}}{:=} \sqrt{\tilde{k}_m^2} \qquad . \tag{3.10}$$

This gives a unique definition for $\tilde{p}_m$. Since $\tilde{k}_m$ is composed of eigenstates

We call the perturbation expansion of this theorem the *causal perturbation expansion*. It allows to uniquely define the Dirac sea by

$$\tilde{P}(x,y) = \frac{1}{2} \left(\tilde{p}_m - \tilde{k}_m\right)(x,y)$$

Further perturbative results

- F.F., A. Grotz, "The causal perturbation expansion revisited: Rescaling the interacting Dirac sea," arXiv:0901.0334 [math-ph], *J. Math. Phys.* **51** (2010) 072301
- F.F., J. Tolksdorf, "Perturbative description of the fermionic projector: Normalization, causality and Furry's theorem," arXiv:1401.4353 [math-ph], *J. Math. Phys.* **55** (2014) 052301

- normalization of states clarified (spatial and mass normalization)
- contour methods give a convenient way to figure out combinatorics

Non-perturbative construction

- F.F., M. Reintjes, "A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime," arXiv:1301.5420 [math-ph] *Adv. Theor. Math. Phys.* **19** (2015) 761–803

- F.F., M. Reintjes, "A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds II – Space-times of infinite lifetime," arXiv:1312.7209 [math-ph] to appear in *Adv. Theor. Math. Phys.* (2016)

Non-perturbative construction

introduce space-time inner product

$$<\psi|\phi> := \int_{\mathscr{M}} \overline{\psi(x)} \phi(x) \, d^4x$$

well-defined if $\psi \in \mathcal{H}$, $\phi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$.

basic observation: $(\psi \,|\, k_m \, \phi)_m = <\psi|\phi>$

Strategy: Represent $<.|.>$ with respect to $(.|.)_m$,

$$(\psi \,|\, \mathcal{S}_m \, \phi)_m = <\psi|\phi>$$

Thus formally, $\mathcal{S}_m = k_m$, but

- $\mathcal{S}_m$ is an operator on Hilbert space $\mathcal{H}_m$
- makes it possible to use functional analytic methods (spectral calculus for self-adjoint operators)

Space-times of finite lifetime

Let $(\mathscr{M}, g)$ be a globally hyperbolic Lorentzian manifold,

$$(\mathcal{D} - m)\psi_m = 0 \qquad \text{Dirac equation}$$

$C^\infty_{sc}(\mathscr{M}, S\mathscr{M})$ spatially compact solutions

$$(\psi_m|\phi_m)_m := 2\pi \int_{\mathcal{N}} \prec\psi_m|\nu\!\!\!/\,\phi_m\succ_x \, d\mu_{\mathcal{N}}(x) \quad \text{scalar product}$$

completion gives Hilbert space $(\mathcal{H}_m, (.|.)_m)$

$$<\psi|\phi> := \int_{\mathscr{M}} \prec\psi|\phi\succ_x \, d\mu_{\mathscr{M}} \quad \text{space-time inner product}$$

Space-times of finite lifetime

Definition (F-Reintjes 2013)

$(\mathscr{M}, g)$ is said to be m-finite if there is a constant $c > 0$ such that for all

$$\phi_m, \psi_m \in \mathcal{H}_m \cap C^\infty_{sc}(\mathscr{M}, S\mathscr{M}) \,,$$

the function $\prec \phi_m | \psi_m \succ_x$ is integrable on $\mathscr{M}$ and

$$|{<}\phi_m|\psi_m{>}| \leq c\, \|\phi_m\|\, \|\psi_m\|$$

(where $\|.\| = (.|.)^{\frac{1}{2}}$ is the norm on $\mathcal{H}_m$).

Then there is a unique bounded self-adjoint operator $\mathcal{S}_m$ with

$${<}\phi_m|\psi_m{>} = (\phi_m|\mathcal{S}_m\, \psi_m)_m$$

(Fréchet-Riesz theorem)

Space-times of inifinite lifetime

recall formula in Minkowski space

$$k_m \, k_{m'} \;=\; \delta(m-m')\, p_m \qquad . \tag{3.21}$$

This formalism has some similarity with the bra/ket notation in quantum mechanics, if the position variable $\vec{x}$ is replaced by the mass parameter m

distribution in mass, integrate over mass parameters

Family of Dirac solutions $\Psi = (\psi_m)_{m \in I}$ with $I = (m_L, m_R) \not\ni 0$
nice class of solutions: $\Psi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$

$$(\Psi|\Phi) := \int_I (\psi_m|\phi_m)_m \, dm \quad \text{scalar product}$$

completion gives Hilbert space $(\mathcal{H}, (.|.))$
choose dense subspace $\mathcal{H}^\infty$ (e.g. $\mathcal{H}^\infty = C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$)

$$\mathfrak{p} \;:\; \mathcal{H}^\infty \to C^\infty_{sc}(\mathscr{M}, S\mathscr{M}) \,, \qquad \mathfrak{p}\Psi = \int_I \psi_m \, dm \,.$$

Space-times of inifinite lifetime

Definition (F-Reintjes 2013)

The Dirac operator $\mathcal{D}$ has the strong mass oscillation property in the interval $I = (m_L, m_R)$ with domain $\mathcal{H}^\infty$, if there is a constant $c > 0$ such that

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| \leq c \int_I \|\phi_m\|_m \, \|\psi_m\|_m \, dm \qquad \forall\, \psi, \phi \in \mathcal{H}^\infty$$

Theorem (F-Reintjes 2013)

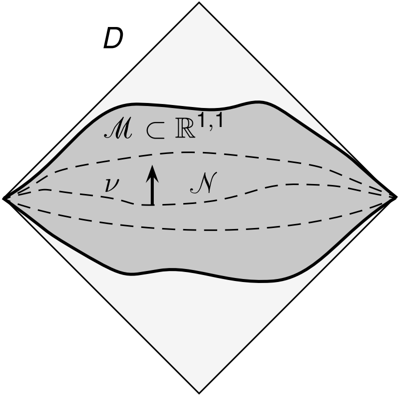
The following statements are equivalent:

(i) *The strong mass oscillation property holds.*

(ii) *There is a unique family of bounded self-adjoint operators $\mathcal{S}_m \in \mathrm{L}(\mathcal{H}_m)$ such that*

$${<}\mathfrak{p}\psi|\mathfrak{p}\phi{>} = \int_I (\psi_m \,|\, \mathcal{S}_m \, \phi_m)_m \, dm \qquad \forall\, \psi, \phi \in \mathcal{H}^\infty$$

- F.F., O. Müller, “Lorentzian spectral geometry for globally hyperbolic surfaces,” arXiv:1411.3578 [math-ph] *Adv. Theor. Math. Phys.* **20** (2016) 751–820

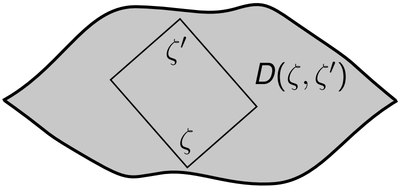


Theorem (F-Müller 2014)

For massless Dirac equation on globally hyperbolic surfaces of finite lifetime,

$$\mathrm{tr}\big(\mathcal{S}^2\big) = \frac{\mu(\mathscr{M})}{4\pi^2}$$

$$\mathrm{tr}\big(\mathcal{S}^4\big) = \frac{1}{8\pi^4} \int_{\mathscr{M}} d\mu(\zeta) \int_{J(\zeta)} \exp\left(\frac{1}{4} \int_{D(\zeta,\zeta')} R \, d\mu \right) d\mu(\zeta')$$



Construction of quasi-free quantum states

$$\chi_{(-\infty,0)}(\mathcal{S}_m) \qquad \text{or} \qquad \frac{1}{1+e^{\beta m \pi \mathcal{S}}}$$

are positive operators on one-particle Hilbert space

Theorem (Araki 1970)

There is an algebra of smeared fields generated by $\Psi(g)$, $\Psi^*(f)$, and a pure quasi-free state ω such that:

$$\big\{ \Psi(g), \Psi^*(f) \big\} = \,<\!g^* \,|\, k_m\, f\!>\,, \quad \ldots \qquad (CAR)$$

$$\omega\big(\Psi(g)\, \Psi^*(f) \big) = - \iint_{\mathcal{M} \times \mathcal{M}} g(x) P(x,y) f(y)\, d^4x\, d^4y$$

$$P = \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \qquad \textit{resp.} \qquad P = \frac{1}{1+e^{\beta m \pi \mathcal{S}}}\, k_m$$

The first state is referred to as fermionic projector (FP) state.

Symmetries

The FP state respects all symmetries of space-time:

Theorem (F-Reintjes, in preparation)

If $\mathcal{G}$ is a group of symmetries on $S\mathscr{M}$, then ω is invariant under the action of $\mathcal{G}$, i.e.

$$\omega\big(\Psi(g)\,\Psi^*(f)\big) = \omega\big(\Psi(\mathfrak{g}^* g)\,\Psi^*(\mathfrak{g}^* f)\big) \qquad \text{for all } \mathfrak{g} \in \mathcal{G}$$

Sketch of proof in finite life-time.

Let U be group representation on Dirac solutions with

$$\begin{aligned} & <U\psi|U\phi> = <\psi|\phi> \\ \Longrightarrow \quad & (U\psi|\mathcal{S}_m U\phi)_m = (\psi|\mathcal{S}_m \phi)_m \\ \Longrightarrow \quad & U^* \mathcal{S}_m U = \mathcal{S}_m \end{aligned}$$

$\square$

Ultrastatic space-times

$$\mathscr{M} = \mathbb{R} \times \mathscr{N}$$

with $\mathscr{N}$ complete Riemannian manifold,

$$ds^2 = dt^2 - g_{\mathscr{N}}$$

Theorem (F-Reintjes 2013)

$$\sigma(\mathscr{S}_m) = \{1, -1\} \,,$$

and eigenspaces reproduce frequency splitting.

Corollary

The FP state is a Hadamard state.

A space-time slab

- C.J. Fewster, B. Lang, “Pure quasifree states of the Dirac field from the fermionic projector,” arXiv:1408.1645 [math-ph], *Class. Quantum Grav.* **32** (2015) 095001

$$\mathscr{M} = (0, T) \times \mathscr{N}$$
$$ds^2 = dt^2 - g_{\mathscr{N}}$$

Theorem (Fewster-Lang 2014)

The FP state is in general not a Hadamard state.

But “softened” construction (inspired by Brum-Fredenhagen) gives a Hadamard state.

External potential in Minkowski space

- F.F., S. Murro, C. Röken, C., "The fermionic projector in a time-dependent external potential: Mass oscillation property and Hadamard states," arXiv:1501.05522 [math-ph], *J. Math. Phys.* **57** (2016) 072303

back to problem at the beginning:

$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\psi = 0$$

Theorem (F-Murro-Röken 2015)

Assume that $\mathcal{B}$ is smooth and

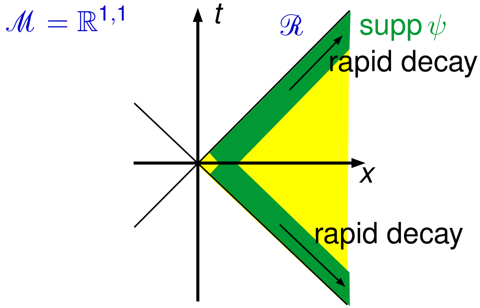
$$\int_{-\infty}^{\infty} |\partial_t^p \mathcal{B}(t)|_{C^0}\, dt < \infty \qquad \textit{for all } p \in \mathbb{N}$$

$$\int_{-\infty}^{\infty} |\mathcal{B}(t)|_{C^0}\, dt < \sqrt{2} - 1$$

Then the FP state is a Hadamard state.

Rinder space-time

- F.F., S. Murro, S., C. Röken, “The fermionic signature operator and quantum states in Rindler space-time,” arXiv:1606.03882 [math-ph] (2016)



$$\psi \in C^\infty_{sc}(\mathscr{R}, S\mathscr{R}) \quad \Longrightarrow \quad \big| {<}\psi|\phi{>} \big| \leq c(\psi)\, \|\phi\| \quad \forall \phi \in \mathcal{H}$$

$$\big| {<}\psi|\phi{>} \big| = (\mathcal{S}\psi|\phi) \quad \forall \phi \in \mathcal{H}$$

Defines $\mathcal{S} : C^\infty_{sc}(\mathscr{R}, S\mathscr{R}) \to \mathcal{H}$ densely defined operator

Rinder space-time

$$\mathcal{S} = -\frac{H}{\pi m}$$

where H is Hamiltonian in Rindler time (unbounded!)
(i.e. generator of Lorentz boosts)

Theorem (F-Murro-Röken 2016)

In two-dimensional Rindler space-time, the FP state is a Hadamard state.

- construction of thermal states is possible
- in four-dimensional Rindler space-time, FP state is a new state (spin couples to transversal momenta).

  Open questions:
  - Is it Hadamard?
  - What is its physical significance?

A plane electromagnetic wave

- F.F. and M. Reintjes, “The fermionic signature operator and Hadamard states in the presence of a plane electromagnetic wave,” arXiv:1609.04516 [math-ph], to appear in *Ann. Henri Poincaré* (2017)

$$(i\partial\!\!\!/ + A\!\!\!/ - m)\,\psi = 0$$
$$A = A(t+x)$$

A smooth, but no decay assumptions at infinity!

separation ansatz:

$$\psi_m(t,x,y,z) = e^{-ik_2 y - ik_3 z}\, e^{-iu(t-x)}\, \chi^m_{k_2,k_3,u}(t+x)$$

$$\mathcal{S}_m = \epsilon(u) \quad \text{multiplication operator}$$

Theorem (F-Reintjes 2016)

The FP state is a Hadamard state.

Proof of the Hadamard property in an external field

In Minkowski vacuum, $\mathcal{S}_m$ has eigenvalues ± 1.
The corresponing splitting of the solution space

$$\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^-$$

coincides with frequency splitting.
In the presence of the external potential (denoted by tilde),

$$\tilde{\mathcal{S}} = \begin{pmatrix} \tilde{\mathcal{S}}^+_+ & \tilde{\mathcal{S}}^+_- \\ \tilde{\mathcal{S}}^-_+ & \tilde{\mathcal{S}}^-_- \end{pmatrix}$$

$$\tilde{\mathcal{S}}^{\mathrm{D}} := \tilde{\mathcal{S}}^+_+ + \tilde{\mathcal{S}}^-_- \,, \qquad \Delta\tilde{\mathcal{S}} := \tilde{\mathcal{S}}^+_- + \tilde{\mathcal{S}}^-_+$$

Then

$$\tilde{\mathcal{S}}_m = \tilde{\mathcal{S}}^{\mathrm{D}} + \Delta\tilde{\mathcal{S}}$$

Proof of the Hadamard property in an external field

General strategy:

1. Use that $\mathcal{B}$ is not too large to prove that

$$\big\|\tilde{\mathcal{S}}^{\mathrm{D}} - \mathcal{S}_m\big\|,\ \big\|\tilde{\mathcal{S}}_m - \mathcal{S}_m\big\| \leq \frac{1}{2}$$

   Then

$$\sigma\big(\tilde{\mathcal{S}}^{\mathrm{D}}\big), \sigma\big(\tilde{\mathcal{S}}_m\big) \subset \Big[-\frac{3}{2}, -\frac{1}{2}\Big] \cup \Big[\frac{1}{2}, \frac{3}{2}\Big]$$

2. Use contour representation (Cauchy integral)

$$\chi_{(-\infty,0)}\big(\tilde{\mathcal{S}}_m\big) = -\frac{1}{2\pi i} \oint_{\partial B_1(-1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, d\lambda$$

3. Apply resolvent identity

$$(\tilde{\mathcal{S}}_m - \lambda)^{-1} = (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1} - \big(\tilde{\mathcal{S}}_m - \lambda\big)^{-1}\, \Delta\tilde{\mathcal{S}}\, (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1}$$

   to obtain

$$\chi^{\pm}(\tilde{\mathcal{S}}_m) = \chi^{\pm}(H) + \frac{1}{2\pi i} \oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, \Delta\tilde{\mathcal{S}}\, (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1}\, d\lambda$$

Proof of resolvent identity

Lemma

$$(\tilde{\mathfrak{S}}_m - \lambda)^{-1} = (\tilde{\mathfrak{S}}^{\mathrm{D}} - \lambda)^{-1} - \big(\tilde{\mathfrak{S}}_m - \lambda\big)^{-1} \,\Delta\tilde{\mathfrak{S}}\, (\tilde{\mathfrak{S}}^{\mathrm{D}} - \lambda)^{-1}$$

Proof.

$$\tilde{\mathfrak{S}}_m = \tilde{\mathfrak{S}}^{\mathrm{D}} + \Delta\tilde{\mathfrak{S}}$$

rewrite this as

$$\big(\tilde{\mathfrak{S}}^{\mathrm{D}} - \lambda\big) = \big(\tilde{\mathfrak{S}}_m - \lambda\big) - \Delta\tilde{\mathfrak{S}}$$

Mulitply from right by $\big(\tilde{\mathfrak{S}}^{\mathrm{D}} - \lambda\big)^{-1}$
and from leftt by $\big(\tilde{\mathfrak{S}}_m - \lambda\big)^{-1}$. $\square$

4. Show that the remaining contour integral

$$\oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1} \, \Delta\tilde{\mathcal{S}} \, (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1} \, d\lambda$$

has a smooth kernel. To this end, apply:

Lemma

Let $A \in \mathrm{L}(\mathcal{H}_m)$ such that for all $p, q \in \mathbb{N}$,

$$H^q \, A \, H^p \;:\; C_0^\infty(\mathcal{N}, S\mathcal{M}) \to C^\infty(\mathcal{N}, S\mathcal{M}) \quad \textit{is bounded}$$

Then A has a smooth integral kernel.

Proof of the Hadamard property in an external field

Key technical tool: Explicit representation of $\tilde{\mathcal{S}}_m$:

Lemma

$$
\begin{aligned}
\tilde{\mathcal{S}}_m &= \mathcal{S}_m \\
&- \frac{i}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0) \left[\mathcal{S}_m \, U_m^{t_0,t} \, \mathcal{V}(t) \, \tilde{U}_m^{t,t_0} - \tilde{U}_m^{t_0,t} \, \mathcal{V}(t) \, \mathcal{S}_m \, U_m^{t,t_0} \right] dt \\
&+ \frac{1}{2} \left(\int_{t_0}^{\infty} \int_{t_0}^{\infty} + \int_{-\infty}^{t_0} \int_{-\infty}^{t_0} \right) \tilde{U}_m^{t_0,t} \, \mathcal{V}(t) \, \mathcal{S}_m \, U_m^{t,t'} \, \mathcal{V}(t') \, \tilde{U}_m^{t',t_0} \, dt \, dt'
\end{aligned}
$$

Here write the Dirac equation in Hamiltonian form

$$
i\partial_t \psi_m = \tilde{H} \psi_m \qquad \text{with} \qquad \tilde{H} := -\gamma^0 \left(i\vec{\gamma}\vec{\nabla} + \mathcal{B} - m \right) = H + \mathcal{V}
$$

U_m^{t,t_0}: unitary time evolution without interaction

$\tilde{U}_m^{t,t_0}$: unitary time evolution with interaction

- rewrite products $H^q \, \Delta\tilde{\mathcal{S}}_m \, H^p$ as iterated commutators
  estimate these commutators

Proof of the Hadamard property in an external field

To derive this representation combine

- Lippmann-Schwinger equation

$$\psi_m|_t = U_m^{t,t_0}\psi_m^0 + i \int_{t_0}^t U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m\right)\big|_\tau \, d\tau$$

- Multiplication rules for distributions in Minkowski space like

$$k_m\, k_{m'} = \delta(m - m')\, p_m$$

Outlook: Causal fermion systems

- light-cone expansion gives explicit expansion about the light cone

$$
\begin{aligned}
\chi_L\, P(x,y) = \;& \frac{i}{2}\, \chi_L\, e^{-i\Lambda_L^{xy}}\, \not{\xi}\, T^{(-1)} \\
& - \frac{1}{2}\, \chi_L\, \not{\xi}\, \xi_i \int_x^y [0,0\,|\,1]\, j_L^i\; T^{(0)} \\
& + \frac{1}{4}\, \chi_L\, \not{\xi} \int_x^y F_L^{ij}\, \gamma_i \gamma_j\; T^{(0)} \\
& - \chi_L\, \xi_i \int_x^y [0,1\,|\,0]\, F_L^{ij}\, \gamma_j\; T^{(0)} \\
& - \chi_L\, \xi_i \int_x^y [0,1\,|\,1]\, \not{\partial} j_L^i\; T^{(1)} \\
& - \chi_L \int_x^y [0,2\,|\,0]\, j_L^i\, \gamma_i\; T^{(1)} + \cdots
\end{aligned}
$$

Here

$$T_a(q) = \Theta(-q^0)\, \delta(q^2 - a)$$

$$T^{(l)} = \left(\frac{d}{da} \right)^l T_a \Big|_{a=0} - \text{(counter terms)}$$

and

$$\int_x^y [l, r \mid n]\, dz\, f(z) := \int_0^1 d\alpha\, \alpha^l (1-\alpha)^r (\alpha - \alpha^2)^n\, f(\alpha y + (1-\alpha) x)$$

$P(x, y)$ is of Hadamard form
$\Longleftrightarrow$ only bounded line integrals appear.

Outlook: Causal fermion systems

General observation: $P(x,y)$ determines the bosonic potential.

Build in particles and anti-particles states:

$$P(x,y) = P_m^{\text{sea}}(x,y) - \frac{1}{2\pi} \sum_{k=1}^{n_{\mathrm{p}}} \psi_k(x)\overline{\psi_k(y)} + \frac{1}{2\pi} \sum_{l=1}^{n_{\mathrm{a}}} \phi_l(x)\overline{\phi_l(y)}$$

- $P(x,y)$ describes the physical system completely

General idea (goes back to 1990):

- formulate physical equations directly with $P(x,y)$

What is a causal fermion system?

- approach to fundamental physics
- a new consistent physical theory
- promising candidate for a unified physical theory

- novel approach to describe space and space-time, as well as structures therein:
  "quantum space-time," "quantum geometry"
- dynamics described by causal action principle
  - background-free, no space-time presupposed
  - space-time emerges by minimizing the causal action

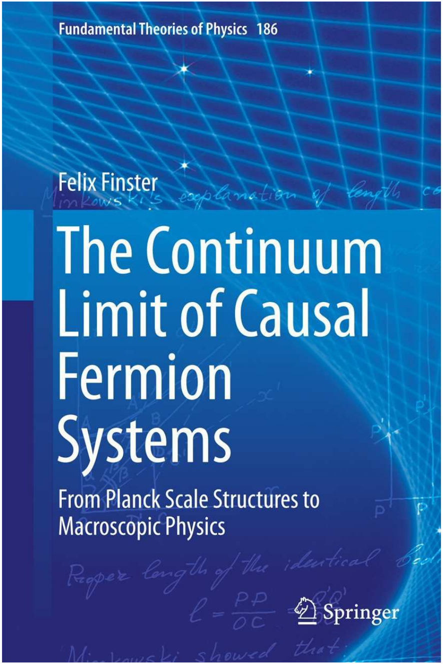
Results of the theory

Continuum limit
(classical fields coupled to second-quantized Dirac field):

- interactions of the standard model (electroweak + strong)
- general relativity
- quantum mechanics

Other limiting case ("microscopic mixing")

- quantum field theory (second-quantized bosonic fields)
  no ultraviolet problems



Fundamental Theories
of Physics **186**

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548+xi pages

arXiv:1605.04742 [math-ph]