Spectral Theory of Vector and Tensor Fields on Schwarzschild Spacetime

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Motivation

- Goals:
  - Hawking effect for gravitons;
  - interacting gravitons on a black-hole spacetime;
  - eventually, quantum back-reaction of Hawking radiation.
- Graviton field — $p_{\mu\nu}$, its quantization — $\hat{p}_{\mu\nu}$.
- Ghost field — v_μ, its quantization — $\hat{v}_\mu$ (BRST formalism).
- Harmonic (aka *de Donder*, *Lorenz*, *wave coordinate*) gauge:
  - ${}^4\nabla^\nu \overline{p}_{\mu\nu} = 0$, where $\overline{p}_{\mu\nu} = p_{\mu\nu} - \frac{1}{2}\,{}^4g_{\mu\nu}\,\mathrm{tr}\,p$;
  - favored by BRST formalism.
- Graviton and ghost Feynman propagators:

$$G_{\mu\nu:\mu'\nu'}(x,x') = \frac{-i}{8\pi\ell_P^2}\langle T[\hat{p}_{\mu\nu}(x)\hat{p}_{\mu'\nu'}(x')]\rangle_\Psi,$$

$$G_{\mu:\mu'}(x,x') = -i\langle T[\hat{v}_\mu(x)\hat{v}_{\mu'}(x')]\rangle_\Psi.$$

Ψ — sensible quantum state, like Unruh or Hartle-Hawking.

Want an "explicit" mode expansion of $G_{\mu:\mu'}(x,x')$ and $G_{\mu\nu:\mu'\nu'}(x,x')$.

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Vector and tensor fields on Schwarzschild

- The Schwarzschild spacetime $(M, {}^4g)$ is a 4-dimensional Lorentzian manifold describing a static, spherically symmetric black hole.
- The Feynman propagators $G_{\mu:\mu'}(x,x')$ and $G_{\mu\nu:\mu'\nu'}(x,x')$ are particular Green functions, respectively, for the vector (ghost) and tensor (graviton) wave equations on Schwarzschild:

$$ {}^4\Box v_\mu = 2\,{}^4\nabla^\nu \overline{{}^4\nabla_{(\mu} v_{\nu)}} = 0, \quad {}^4\Box p_{\mu\nu} - 2\,{}^4R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} - 2\,{}^4\nabla_{(\mu}{}^4\nabla^\lambda \overline{p}_{\nu)\lambda} = 0. $$

  For tensors, it is also called the Lichnerowicz equation.
- **Goal:** write each Green function as an explicit mode sum/integral:

$$ {}^4G(x,y) \sim \int \mathrm{d}\mu_{\ell,\omega,\nu}\, \phi_{\ell,\omega}(x)\bar{\phi}_{\ell,\omega}(y) e^{-i\nu(x^0-y^0)}, $$

  where $\phi_{\ell,\omega}(x)$ are modes adapted to the static (ω, ν) and spherical (ℓ) symmetry of the black hole and $\mathrm{d}\mu_{\ell,\omega,\nu}$ is a specially chosen spectral measure that determines the Green function (and the quantum state Ψ).
- **Question:** Can $\mathrm{d}\mu_{\ell,\omega,\nu}$ be supported only on $\omega \in \mathbb{R}$?

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Separation of variables: 2+2 tensor formalism

- We follow the convenient formalism of [Martel & Poisson 2005].
- Schwarzschild ($\mathcal{M} \times S^2$) is spherically symmetric $f(r) = 1 - \frac{2M}{r}$:
  ${}^4g_{\mu\nu} = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \to \begin{pmatrix} g_{ab} & 0 \\ 0 & r^2\Omega_{AB} \end{pmatrix}$.
- Tensor indices $a, b, c, \ldots$ and ∇_a are for $(\mathcal{M}, g_{ab})$.
  Tensor indices $A, B, C, \ldots$ and D_A are for the unit sphere (S^2, Ω_{AB}).
- Vector field $v_\mu \to \begin{pmatrix} v_a \\ v_A \end{pmatrix}$, symmetric tensor $p_{\mu\nu} \to \begin{pmatrix} p_{ab} & p_{aB} \\ p_{Ab} & p_{AB} \end{pmatrix}$.
- Connection ${}^4\nabla = (\nabla, D) + \Gamma$,

$$\Gamma^\mu_{\nu\lambda} = \left[\begin{pmatrix} 0 & 0 \\ 0 & -rr^a\Omega_{BC} \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{r_b}{r}\delta^A_C \\ \frac{r_c}{r}\delta^A_B & 0 \end{pmatrix} \right].$$

- Formalism covariant with respect to changes of coordinates and metric on $(\mathcal{M}, g_{ab})$.

Spherical harmonics

- Spherical scalar, vector and tensor harmonics:

$$D_A D^A Y = -l(l+1)Y, \qquad Y_A = D_A Y, \qquad Y_{AB} = D_A Y_B + \frac{l(l+1)}{2}\Omega_{AB} Y,$$

$$\int_{S^2} \bar{Y}' Y \epsilon = \delta_{ll'}\delta_{mm'}, \qquad X_A = \epsilon_{BA} D^B Y, \qquad X_{AB} = D_A X_B + \frac{l(l+1)}{2}\epsilon_{AB} Y.$$

Simply normalized, orthogonal, tensor eigenfunctions of $D_A D^A$.

- Vector and Tensor decompositions

$$\begin{pmatrix} p_{ab} & p_{aB} \\ p_{Ab} & p_{AB} \end{pmatrix} = \sum_{lm} \begin{pmatrix} h_{ab}^{lm} Y^{lm} & \overset{\text{even}}{r\, j_a^{lm} Y_B^{lm}} \\ r\, j_b^{lm} Y_A^{lm} & r^2 (K^{lm}\Omega_{AB} Y^{lm} + G^{lm} Y_{AB}^{lm}) \end{pmatrix} + \sum_{lm} \begin{pmatrix} 0 & \overset{\text{odd}}{r\, h_a^{lm} X_B^{lm}} \\ r\, h_b^{lm} X_A^{lm} & r^2 h_2^{lm} X_{AB}^{lm} \end{pmatrix}$$

$$\begin{pmatrix} v_a \\ v_A \end{pmatrix} = \sum_{lm} \begin{pmatrix} \overset{\text{even}}{v_a^{lm} Y^{lm}} \\ r\, u^{lm} Y_A^{lm} \end{pmatrix} + \sum_{lm} \begin{pmatrix} \overset{\text{odd}}{0} \\ r\, w^{lm} X_A^{lm} \end{pmatrix}$$

From now on, omit spherical harmonic (l, m) mode indices:

$$p = (h_{ab}, j_a, K, G \mid h_a, h_2) \quad \text{and} \quad v = (v_a, u \mid w)$$

Radial mode equation

- In static Schwarzschild (t, r) coordinates $(2M < r < \infty)$:

$$p(t,r) = p(r)e^{-i\omega t} \quad \text{and} \quad v(t,r) = v(r)e^{-i\omega t}, \quad \text{where}$$
$$p(r) = (h_{tt}, h_{tr}, h_{rr}, j_t, j_r, K, G \mid h_t, h_r, h_2),$$
$$v(r) = (v_t, v_r, u \mid w).$$

  We obtain the radial mode equations $VW_\omega v = 0$ and $L_\omega p = 0$.
- For vectors, ${}^4\Box v_\mu \rightsquigarrow VW_\omega$ consists of decoupled 3×3 (even) and 1×1 (odd) systems.
- For tensors, ${}^4\Box p_{\mu\nu} - 2\,{}^4R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} \rightsquigarrow L_\omega$ consists of decoupled 7×7 (even) and 3×3 (odd) systems.
- Indefinite quadratic-eigenvalue matrix Sturm-Liouville equation

$$E_\omega \phi := \partial_r P(r) \partial_r \phi + Q(r)\phi + i\omega A(r)\phi + \omega^2 W(r)\phi = 0,$$

  with hermitian P, Q, iA, and W, hence formally self-adjoint.

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Radial mode equation: VW_ω

Explicitly:

$$\text{(odd)} \qquad \partial_r \mathcal{B}_l r^2 f \partial_r w + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \mathcal{B}_l w + \mathcal{B}_l \frac{2M}{r} w = 0,$$

$$\text{(even)} \qquad \begin{pmatrix} -\partial_r \frac{1}{f}\, r^2 f \partial_r\, v_t \\ \partial_r f\, r^2 f \partial_r\, v_r \\ \partial_r \mathcal{B}_l\, r^2 f \partial_r\, u \end{pmatrix} + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \begin{pmatrix} -\frac{1}{f}\, v_t \\ f\, v_r \\ \mathcal{B}_l\, u \end{pmatrix}$$

$$+ i\omega \frac{2M}{f} \begin{pmatrix} v_r \\ -v_t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2f^2 & 2\mathcal{B}_l f \\ 0 & 2\mathcal{B}_l f & \mathcal{B}_l \frac{2M}{r} \end{pmatrix} \begin{pmatrix} v_t \\ v_r \\ u \end{pmatrix} = 0,$$

where $f(r) = 1 - \frac{2M}{r}$ and $\mathcal{B}_l = l(l+1)$.

Radial mode equation: L_ω (odd sector)

$$\begin{pmatrix} \partial_r(-2\frac{\mathcal{B}_l}{f}\, r^2 f\partial_r)\, h_t \\ \partial_r(2\mathcal{B}_l f\, r^2 f\partial_r)\, h_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\, r^2 f\partial_r)\, h_2 \end{pmatrix} - \mathcal{B}_l \begin{pmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{pmatrix}$$

$$+ \begin{pmatrix} -4\frac{\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 \\ 0 & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\ 0 & 2\mathcal{A}_l f & \mathcal{A}_l \end{pmatrix} \begin{pmatrix} h_t \\ h_r \\ h_2 \end{pmatrix}$$

$$-i\omega\frac{4M}{f} \begin{pmatrix} 0 & -\mathcal{B}_l & 0 \\ \mathcal{B}_l & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_t \\ h_r \\ h_2 \end{pmatrix} + \omega^2\frac{r^2}{f} \begin{pmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{pmatrix} = 0$$

where $f(r) = 1 - \frac{2M}{r}$, $\mathcal{A}_l = (l-1)l(l+1)(l+2)$ and $\mathcal{B}_l = l(l+1)$

Radial mode equation: L_ω (even sector)

$$\begin{pmatrix} \partial_r(-2\,r^2 f\partial_r)\,h_{tr} \\ \partial_r(-2\frac{\mathcal{B}_l}{f}\,r^2 f\partial_r)\,j_t \\ \partial_r(\frac{1}{f^2}\,r^2 f\partial_r)\,h_{tt} \\ \partial_r(f^2\,r^2 f\partial_r)\,h_{rr} \\ \partial_r(2\,r^2 f\partial_r)\,K \\ \partial_r(2\mathcal{B}_l f\,r^2 f\partial_r)\,j_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\,r^2 f\partial_r)\,G \end{pmatrix} - \mathcal{B}_l \begin{pmatrix} -2\,h_{tr} \\ -2\frac{\mathcal{B}_l}{f}\,j_t \\ \frac{1}{f^2}\,h_{tt} \\ f^2\,h_{rr} \\ 2\,K \\ 2\mathcal{B}_l f\,j_r \\ \frac{\mathcal{A}_l}{2}\,G \end{pmatrix}$$

$$\begin{pmatrix} \frac{2(f^2+1)}{f} & -4\mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\ -4\mathcal{B}_l & -\frac{4\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4M^2}{2f^3r^2} & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{r} & \frac{2}{f}\frac{2M}{r} & 0 & 0 \\ 0 & 0 & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{e} & \frac{f(\frac{4M^2}{r^2}-8f^2)}{2} & 4f(1-\frac{3M}{r}) & 4\mathcal{B}_l f^2 & 0 \\ 0 & 0 & \frac{2}{f}\frac{2M}{r} & 4f(1-\frac{3M}{r}) & -4(1-\frac{4M}{r}) & -4\mathcal{B}_l f & 0 \\ 0 & 0 & 0 & 4\mathcal{B}_l f^2 & -4\mathcal{B}_l f & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\ 0 & 0 & 0 & 0 & 0 & 2\mathcal{A}_l f & \mathcal{A}_l \end{pmatrix} \begin{pmatrix} h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G \end{pmatrix}$$

$$-i\omega\frac{4M}{f}\begin{pmatrix} 0 & 0 & -\frac{1}{f} & -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathcal{B}_l & 0 \\ \frac{1}{f} & 0 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G \end{pmatrix} + \omega^2\frac{r^2}{f}\begin{pmatrix} -2\,h_{tr} \\ -2\frac{\mathcal{B}_l}{f}\,j_t \\ \frac{1}{f^2}\,h_{tt} \\ f^2\,h_{rr} \\ 2\,K \\ 2\mathcal{B}_l f\,j_r \\ \frac{\mathcal{A}_l}{2}\,G \end{pmatrix} = 0$$

where $f(r) = 1 - \frac{2M}{r}$, $\mathcal{A}_l = (l-1)l(l+1)(l+2)$ and $\mathcal{B}_l = l(l+1)$

Spectral theory of the radial mode equation

- How to wrie the Green function $E_\omega^{-1}(r,r')$ for the operator pencil
$$E_\omega\phi := \partial_r P(r)\partial_r\phi + Q(r)\phi + i\omega A(r)\phi + \omega^2 W(r)\phi$$
in spectral representation (mode sum/integral)?
Use ideas of Weyl-Titchmarsh-Kodaira (1910–1950), Keldysh (1951).
[Weidmann (Springer, 1987)] [Gohberg-Kaashoek-Lay (1976)] [Markus (AMS, 1988)]
- The spectrum is $\sigma(E_\omega) = \mathbb{C} \setminus \rho(E_\omega)$. For $\omega \in \rho(E_\omega)$ in the resolvent set, E_ω^{-1} is bounded. Need to choose a function space/domain!
- Linearize $E_\omega \rightsquigarrow \boldsymbol{E}_\omega$, prove analyticity of $\boldsymbol{E}_\omega^{-1}$ over $\rho(E_\omega)$:
$$\boldsymbol{E}_\omega = \begin{bmatrix} \partial_r P\partial_r + Q & 0 \\ 0 & -W \end{bmatrix} + \omega \begin{bmatrix} iA & W \\ W & 0 \end{bmatrix}, \quad \boldsymbol{E}_\omega^{-1} = \begin{bmatrix} E_\omega^{-1} & \omega E_\omega^{-1} \\ \omega E_\omega^{-1} & \omega^2 E_\omega^{-1} - W^{-1} \end{bmatrix}$$
- Integrate over a positive simple contour γ about $\sigma(E_\omega)$:
$$\boldsymbol{E}_\nu^{-1} = \oint_\gamma \frac{\mathrm{d}\omega}{2\pi i}\frac{1}{\omega-\nu}\boldsymbol{E}_\omega^{-1}, \quad \begin{bmatrix} 0 & W^{-1} \\ W^{-1} & -W^{-1}iAW^{-1} \end{bmatrix} = \oint_\gamma \frac{\mathrm{d}\omega}{2\pi i}\boldsymbol{E}_\omega^{-1}$$
- Decompose $E_\omega^{-1}(r,r') = \mathring{E}_\omega^{-1}(r,r') + m^{jj'}(\omega)\phi_{\omega,j}(r)\phi_{-\omega,j'}(r')$, where $\mathring{E}_\omega^{-1}$ is analytic! Then $m(\omega) \rightsquigarrow \mathrm{d}\mu_\omega$ — the spectral measure.

A reasonable hypothesis

- At the very least, the explicit form of the equations gives us the asymptotics [Wasow (Intersci., 1965)] for a solution basis ϕ_j of $E_\omega \phi = 0$:

$$\phi_j(r) \sim \begin{cases} e^{\pm i\omega r_*} Z_\pm(f) y_j^{(2M)} & r \to 2M \\ e^{\pm i\omega r_*} \frac{1}{r} y_j^{(\infty)} & r \to \infty \end{cases},$$

  where y_j are constant coefficients, $Z_\pm(f)$ are Laurent polynomial matrices in $f = 1 - \frac{2M}{r}$, and $r_* = r + 2M \log(\frac{r}{2M} - 1)$ is the tortoise coordinate, so that $e^{\sigma r_*} \sim f^\sigma$.
- Theorem: There exists a function space $\mathcal{H}$, complete w.r.t an inner product $(\phi, \phi) = \int_{2M}^{\infty} \phi^\dagger \tilde{W} \phi \, \mathrm{d}r$, and a domain $D_\omega \subset \mathcal{H}$ that is a core for a closed operator realization of E_ω such that, for $\Im\omega \neq 0$,
  - exactly half of the ϕ_j are admissible at $r \to \infty$,
  - exactly half of the ϕ_j are admissible at $r \to 2M$.
- Hypothesis: for $\Im\omega \neq 0$, no $\sum_j a_j \phi_j$ is admissible at both ends. Then the spectrum would be purely real and we would be done!

Example: asymptitics, VW_e (even)

$$\begin{pmatrix} v_t \\ v_r \\ u \end{pmatrix} \sim \sum_{\pm} Z_{\pm}(f) f^{\pm 2i\omega M} \overset{(r\to 2M)}{y_{\pm}^{(2M)}} \quad \text{or} \quad \sum_{\pm} \frac{1}{r} e^{\pm i\omega r_*} \overset{(r\to\infty)}{y_{\pm}^{(\infty)}},$$

where $f = 1 - \frac{2M}{r}$, $r_* = r + 2M \log(\frac{r}{2M} - 1)$ and

$$Z_{\pm}(f) = \begin{pmatrix} 1 & \pm f & 0 \\ \mp\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{l(l+1)+1}{4\omega M(4\omega M \pm i)} \begin{pmatrix} f & 0 & 0 \\ \mp 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The function space $\mathcal{H}$ consists of measurable $\phi = (v_t, v_r, u)$ such that

$$(\phi, \phi) = \int_{2M}^{\infty} \mathrm{d}r \left[\frac{1}{f} |v_t|^2 + f|v_r|^2 + \frac{1}{f}|u|^2 \right] < \infty,$$

$$\text{and} \quad D_\omega = \left\{ \phi \in \mathcal{H} : \int_{2M} \mathrm{d}r \frac{1}{f} \left| \frac{i\omega r}{f} v_t + r\partial_r f v_r \right|^2 < \infty \right\}.$$

The key idea

- How can we obtain information about the spectrum of E_ω?
- When in doubt, turn to the physics literature and discover. . . a sea of formulas!
- But also the claim [Berndtson (PhD, 2007)] [Rosa-Dolan (2012)] that each of $VW_\omega v = 0$ and $L_\omega p = 0$ are "equivalent" to a decoupled system of, respectively, 4 or 10 scalars $(\phi_i)_i$ satisfying generalized Regge-Wheeler equations $\mathcal{D}_{s_i,\omega}\phi_i = 0$, for spins $s_i \in \{0, 1, 2\}$:

$$\mathcal{D}_{s,\omega}\psi := \partial_r f \partial_r \psi - \frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}\psi + \frac{\omega^2}{f}\psi$$

- Good news: each $\mathcal{D}_{s,\omega}$ is a standard, scalar, self-adjoint Sturm-Liouville operator with purely real spectrum!
- **Q:** What is the precise meaning of "equivalent"?
  Q: How can this information help with the spectral problem of E_ω?

Core equivalence result

Recall, by E_ω we denote either of VW_ω or L_ω, or their odd or even parts.

Theorem (Berndtson, Rosa-Dolan, IK)

Each E_ω is equivalent to a system of Regge-Wheeler ($\mathcal{D}_{s_i,\omega}$) equations.

(a) There exist differential operators making this diagram commute, with $\xrightarrow{k'_\omega} \bullet \xrightarrow{\bar{k}'_\omega}$ exact on solutions:

$$\begin{tikzcd}
\bullet \arrow[r, "\mathcal{D}_{s_i,\omega}"'] & \bullet \\
\bullet \arrow[u, "k'_\omega"] \arrow[r, "E_\omega"'] & \bullet \arrow[u, "g'_\omega"'] \\
\bullet \arrow[u, "\bar{k}'_\omega"] \arrow[r, "\mathcal{D}_{s_i,\omega}"] & \bullet \arrow[u, "\bar{g}'_\omega"']
\end{tikzcd}$$

(b) Allowing formal inverses ($\mathcal{D}^{-1}_{s,\omega}$), the diagram from **(a)** converts to a commutative square, with vertical maps mutual inverses, up to corrections (h_ω, $\bar{h}_\omega$):

$$\begin{tikzcd}
\bullet \arrow[r, "E_\omega"'] \arrow[d, shift right, "k_\omega"'] & \bullet \arrow[l, dashed, bend right, "h_\omega"'] \arrow[d, shift right, "g_\omega"'] \\
\bullet \arrow[r, "\mathcal{D}_{s_i,\omega}"'] \arrow[u, shift right, "\bar{k}_\omega"'] & \bullet \arrow[l, dashed, bend left, "\bar{h}_\omega"] \arrow[u, shift right, "\bar{g}_\omega"']
\end{tikzcd}$$

Homological algebra: chain maps and homotopy equivalences!

Implications for the spectrum (a)

- **Geometric Corollary** ($l \geq 2$): The differential operators from **(a)** preserve asymptotics:

$$(r \to 2M) \quad e^{\pm i\omega r_*} \qquad Z_\pm(f) f^{\pm 2i\omega M} \qquad e^{\pm i\omega r_*}$$

$$\ker \mathcal{D}_{s_i,\omega} \xrightarrow{\bar{k}'_\omega} \ker E_\omega \xrightarrow{k'_\omega} \ker \mathcal{D}_{s_i,\omega}$$

$$(r \to \infty) \quad e^{\pm i\omega r_*} \qquad \frac{1}{r} e^{\pm i\omega r_*} \qquad e^{\pm i\omega r_*}$$

Hence $E_\omega \colon \mathcal{H} \to \mathcal{H}^*$ has real spectrum, where $\mathcal{H} = L^2(\tilde{W}\,\mathrm{d}r)$ and $\mathcal{H}^* = L^2(\tilde{W}^{-1}\,\mathrm{d}r)$:

the self-adjointness of $\mathcal{D}_{s,\omega}$ on $L^2(\mathrm{d}r_*)$ shows that no solution ϕ of $E_\omega \phi = 0$ belongs to $D_\omega \subset \mathcal{H}$ (is admissible both at $r \to \infty$ and $r \to 2M$).

- ($l < 2$): WIP

Implications for the spectrum (b)

$$\begin{tikzcd}
H^{-1}(\tilde{W}\,\mathrm{d}r) \arrow[r, "E_\omega"'] \arrow[d, shift right, "k_\omega"'] & H^{1}(\tilde{W}^{-1}\,\mathrm{d}r) \arrow[l, dashed, bend right, "h_\omega"'] \arrow[d, shift right, "g_\omega"'] \\
L^2(\mathrm{d}r_*) \arrow[r, "\mathcal{D}_{s_i,\omega}"] \arrow[u, shift right, "\bar{k}_\omega"'] & L^2(\mathrm{d}r_*) \arrow[l, dashed, bend left, "\bar{h}_\omega"] \arrow[u, shift right, "\bar{g}_\omega"']
\end{tikzcd}$$

- **Analytical Corollary** ($l \geq 2$): Regge-Wheeler operators $\mathcal{D}_{s_i,\omega}$ spectrally dominate $E_\omega \colon H^{-1}(\tilde{W}\,\mathrm{d}r) \to H^{1}(\tilde{W}\,\mathrm{d}r)$ (on weighted Sobolev spaces):

$$E_\omega^{-1} = \bar{k}_\omega \circ D_{s_i,\omega}^{-1} \circ g_\omega + h_\omega,$$

  l.h.s is bounded whenever each operator on the r.h.s is bounded. Therefore*, $\sigma(E_\omega) \subset \sigma(\mathcal{D}_{s_i,\omega})$ and is purely real.
- ***Caveat**: In the even L_ω case, the equivalence maps $(\bar{k}_\omega, g_\omega, h_\omega)$ do have poles at $\omega = \pm i \frac{(l-1)l(l+1)(l+2)}{12M} =: \pm i\omega_*$. So, the Analytical Corollary only implies $\sigma(L_\omega) \subset \mathbb{R} \cup \{\pm i\omega_*\}$.
- ($l < 2$): WIP

Equivalence of spectral problems up to homotopy

- Homological formulation of the spectral problem for e_ω. [J.L.Taylor (1970)] [Gromov-Shubin (1991)]
  Resolvent: $\rho(e_\omega) = \{\omega \in \mathbb{C} \mid e_\omega$ sits in a split exact sequence$\}$.
  Spectrum: $\sigma(e_\omega) = \mathbb{C} \setminus \rho(e_\omega)$.

$$0 \longrightarrow V \xrightarrow{e_\omega} W \longrightarrow 0 \ .$$

  Replace V by the domain $D(e_\omega) \subset V$ if e_ω is unbounded.
- Equivalence up to (chain) homotopy of e_ω and $\bar{e}_\omega$:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "e_\omega"'] \arrow[d, shift right, "k_\omega"'] & W \arrow[l, dashed, bend right, "h_\omega"'] \arrow[r] \arrow[d, shift right, "g_\omega"'] & 0 \\
0 \arrow[r] & \bar{V} \arrow[r, "\bar{e}_\omega"] \arrow[u, shift right, "\bar{k}_\omega"'] & \bar{W} \arrow[l, dashed, bend left, "\bar{h}_\omega"] \arrow[r] \arrow[u, shift right, "\bar{g}_\omega"'] & 0
\end{tikzcd} ,$$

$$\bar{e}_\omega \circ k_\omega = g_\omega \circ e_\omega, \quad \bar{k}_\omega \circ k_\omega = \mathrm{id} - h_\omega \circ e_\omega, \quad \bar{g}_\omega \circ g_\omega = \mathrm{id} - e_\omega \circ h_\omega,$$
$$e_\omega \circ \bar{k}_\omega = \bar{g}_\omega \circ \bar{e}_\omega, \quad k_\omega \circ \bar{k}_\omega = \mathrm{id} - \bar{h}_\omega \circ \bar{e}_\omega, \quad g_\omega \circ \bar{g}_\omega = \mathrm{id} - \bar{e}_\omega \circ \bar{h}_\omega.$$

- When $k_\omega, g_\omega, h_\omega, \bar{k}_\omega, \bar{g}_\omega, \bar{h}_\omega$ are bounded, the resolvent sets agree, $\rho(e_\omega) = \rho(\bar{e}_\omega)$ because
$$e_\omega^{-1} = \bar{k}_\omega \circ \bar{e}_\omega^{-1} \circ g_\omega + h_\omega \quad \text{and} \quad \bar{e}_\omega^{-1} = k_\omega \circ e_\omega^{-1} \circ \bar{g}_\omega + \bar{h}_\omega.$$
- Mutatis mutandis for spectral domination.

A toy example: equivalence up to homotopy

Consider the following diagram of scalar differential operators:

$$
\begin{tikzcd}
0 \arrow[r] & \bullet \arrow[r, "\partial_r^2+\omega^2"] \arrow[d, shift right, "\partial_r"'] & \bullet \arrow[l, bend right, dashed, "\frac{1}{\omega^2}"'] \arrow[r] \arrow[d, shift right, "\partial_r"'] & 0 \\
0 \arrow[r] & \bullet \arrow[r, "\partial_r^2+\omega^2"] \arrow[u, shift right, "\frac{-\partial_r}{\omega^2}"'] & \bullet \arrow[l, bend left, dashed, "\frac{1}{\omega^2}"] \arrow[r] \arrow[u, shift right, "\frac{-\partial_r}{\omega^2}"'] & 0
\end{tikzcd}
,
$$

which satisfy the identities

$$
(\partial_r^2+\omega^2)\partial_r = \partial_r(\partial_r^2+\omega^2),
$$

$$
(\partial_r^2+\omega^2)\frac{-\partial_r}{\omega^2} = \frac{-\partial_r}{\omega^2}(\partial_r^2+\omega^2),
$$

$$
\frac{-\partial_r}{\omega^2}\partial_r = 1-\frac{1}{\omega^2}(\partial_r^2+\omega^2),
$$

$$
\frac{-\partial_r}{\omega^2}\partial_r = 1-(\partial_r^2+\omega^2)\frac{1}{\omega^2},
$$

$$
\partial_r\frac{-\partial_r}{\omega^2} = 1-\frac{1}{\omega^2}(\partial_r^2+\omega^2),
$$

$$
\partial_r\frac{-\partial_r}{\omega^2} = 1-(\partial_r^2+\omega^2)\frac{1}{\omega^2}.
$$

We will say that the top and bottom lines are equivalent up to (chain) homotopy.

A toy example: spectral domination

Consider the previous example with specific function spaces:

$$\begin{tikzcd}
0 \arrow[r] & H^{-1} \arrow[r, "\partial_r^2+\omega^2"'] \arrow[d, shift right, "\partial_r"'] & H^1 \arrow[l, dashed, bend right, "\frac{1}{\omega^2}"'] \arrow[r] \arrow[d, shift right, "\partial_r"'] & 0 \\
0 \arrow[r] & H^0 \arrow[r, "\partial_r^2+\omega^2"] \arrow[u, shift right, "\frac{-\partial_r}{\omega^2}"'] & H^0 \arrow[l, dashed, bend left, "\frac{1}{\omega^2}"] \arrow[r] \arrow[u, shift right, "\frac{-\partial_r}{\omega^2}"'] & 0
\end{tikzcd},$$

where H^k is the Sobolev space of degree k. As usual, unbounded operators are defined on dense domains.

We can conclude that $\partial_r^2 + \omega^2$ spectrally dominates $\partial_r^2 + \omega^2$, that is $\sigma(\partial_r^2 + \omega^2) \subset \sigma(\partial_r^2 + \omega^2)$, because

$$(\partial_r^2 + \omega^2)^{-1} = \frac{-\partial_r}{\omega^2}(\partial_r^2 + \omega^2)^{-1}\partial_r + \frac{1}{\omega^2}$$

and all the operators on the r.h.s are bounded whenever $\omega \notin \sigma(\partial_r^2 + \omega^2)$.

Boundedness is achieved by letting $\partial_r^2 + \omega^2$ map from low regularity (H^{-1}) to high regularity (H^1).

Example: VW_ω

- The complete decoupling is a relative of the Helmholtz decomposition of vector fields (grad + curl). There are three steps:
  - Split vector field into longitudinal and transverse parts, $\bullet_L$ and $\bullet_T$.
  - Put transverse equation into involutive form, $\bigcirc_\omega$.
  - Decouple to Regge-Wheeler equations, $(\mathcal{D}_{s_i,\omega})$, $(s_i) = (0, 0, 1 \mid 1)$.

$$
\begin{tikzcd}
0 \arrow[r] & \bullet \arrow[r, "VW_\omega"'] \arrow[d, shift right] & \bullet \arrow[l, dashed, bend right] \arrow[r] \arrow[d, shift right] & 0 \\
0 \arrow[r] & \bullet_L \oplus \bullet_T \arrow[r, "\mathcal{D}_0 \oplus VW_\omega"] \arrow[u, shift right] \arrow[d, shift right] & \bullet_L \oplus \bullet_T \arrow[r] \arrow[u, shift right] \arrow[d, shift right] & 0 \\
0 \arrow[r] & \bullet_L \oplus \bullet \arrow[r, "\mathcal{D}_0 \oplus \bigcirc_\omega"] \arrow[u, shift right] \arrow[d, shift right] & \bullet_L \oplus \bullet \arrow[r] \arrow[u, shift right] \arrow[d, shift right] & 0 \\
0 \arrow[r] & \bullet \arrow[r, "(\mathcal{D}_{s_i})"'] \arrow[u, shift right] & \bullet \arrow[l, dashed, bend left] \arrow[r] \arrow[u, shift right] & 0
\end{tikzcd}
$$

- The L_ω case is structurally similar, with significantly more complicated formulas.

VW_ω (even): gradient and divergence

We need the (v_t, v_r, u) component versions of gradient (K) and divergence (T) operators, ${}^4\nabla_\mu \psi$ and ${}^4\nabla^\mu v_\mu$.

The operators $\quad K_\omega := \frac{1}{r^2}\begin{pmatrix} -i\omega r \\ r^2\partial_r \frac{1}{r} \\ 1 \end{pmatrix}, \quad K'_\omega := \begin{pmatrix} \frac{i\omega r}{f} \\ fr^2\partial_r \frac{1}{r} \\ \mathcal{B}_l \end{pmatrix}$

and $\quad T_\omega := \begin{pmatrix} \frac{i\omega r}{f} & \frac{1}{r}\partial_r fr^2 & -\mathcal{B}_l \end{pmatrix}, \quad T'_\omega := \frac{1}{r^2}\begin{pmatrix} -i\omega r & r\partial_r & -1 \end{pmatrix}$

satisfy $\quad VW^e_\omega \circ K_\omega = K'_\omega \circ \mathcal{D}_{0,\omega}, \quad T'_\omega \circ VW^e_\omega = \mathcal{D}_{0,\omega} \circ T_\omega.$

Hence, we can define the idempotent projectors

$$P_L := K_\omega \mathcal{D}^{-1}_{0,\omega} T_\omega \qquad P'_L := K'_\omega \mathcal{D}^{-1}_{0,\omega} T'_\omega,$$

$$P_T := \mathrm{id} - P_L, \qquad P'_T := \mathrm{id} - P'_L,$$

Onto the purely transverse ($\bullet_T$) and purely longitudinal ($\bullet_L$) vector fields.

VW_ω (even): decouple the divergence

The first equivalence square decouples the purely longitudinal from the purely transverse vector fields:

$$
\begin{tikzcd}
0 \arrow[r] & \bullet \arrow[r, "VW^e_\omega"'] \arrow[d, shift right, "T_\omega \oplus P_T"'] & \bullet \arrow[l, dashed, bend right, "0"'] \arrow[r] \arrow[d, shift right, "T'_\omega \oplus P'_T"'] & 0 \\
0 \arrow[r] & \bullet_L \oplus \bullet_T \arrow[r, "\mathcal{D}_0 \oplus VW^e_\omega"] \arrow[u, shift right, "K_\omega \mathcal{D}^{-1}_{0,\omega} \oplus \mathrm{id}"'] & \bullet_L \oplus \bullet_T \arrow[l, dashed, bend left, "0"] \arrow[r] \arrow[u, shift right, "K'_\omega \mathcal{D}^{-1}_{0,\omega} \oplus \mathrm{id}"'] & 0
\end{tikzcd}
$$

Evidently, on longitudinal modes ($v_\mu = {}^4\nabla_\mu \frac{1}{r}\psi_0$),

$$VW^e_\omega v_\mu = 0 \quad \text{translates to} \quad \mathcal{D}_{0,\omega}\psi_0 = 0.$$

VW_ω (even): adjoin the transversality condition

Now we adjoin the transversality (Lorenz) condition (${}^4\nabla^\mu v_\mu = 0$) in its component form ($T_\omega v = 0$).

It turns out to be convenient to put the (overdetermined) system (T_ω, VW^e_ω) into involutive form, which becomes determined but mixed order ($\partial_r v_t, \partial_r v_r, \partial_r^2 u$):

$$\bigcirc_\omega \begin{pmatrix} v_t \\ v_r \\ u \end{pmatrix} := \begin{pmatrix} -\frac{i\omega r}{f} & 0 & 0 & 0 \\ -fr^2\partial_r\frac{1}{r} & 0 & 1 & 0 \\ -\mathcal{B}_l & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_\omega \\ VW^e_\omega \end{pmatrix} \begin{pmatrix} v_t \\ v_r \\ u \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\omega^2 r^2}{f} & -\frac{i\omega}{f}\partial_r r^2 f & \mathcal{B}_l\frac{i\omega r}{f} \\ -i\omega r^2\partial_r & \omega^2 r^2 - \mathcal{B}_l f & \mathcal{B}_l f\partial_r r \\ -\mathcal{B}_l\frac{i\omega r}{f} & -\mathcal{B}_l r\partial_r f & \partial_r\mathcal{B}_l r^2 f\partial_r + \mathcal{B}_l\frac{1}{f}(\omega^2 r^2 + f\frac{2M}{r}) \end{pmatrix} \begin{pmatrix} v_t \\ v_r \\ u \end{pmatrix}$$

Of course, the transformation from (T_ω, VW^e_ω) to $\bigcirc_\omega$ is invertible, also by a differential operator.

VW_ω (even): adjoin the transversality condition

The next equivalence square relates VW^e_ω acting on purely transverse vector fields to the joint system (T_ω, VW^e_ω), acting on unconstrained vector fields:

$$\begin{tikzcd}
0 \arrow[r] & \bullet_T \arrow[r, "VW^e_\omega"'] \arrow[d, shift right, "\mathrm{id}"'] & \bullet_T \arrow[l, bend right, dashed, "0"'] \arrow[r] \arrow[d, shift right, "g_\omega"'] & 0 \\
0 \arrow[r] & \bullet \arrow[r, "\bigcirc_\omega"] \arrow[u, shift right, "P_T"'] & \bullet \arrow[l, bend left, dashed, "\bar{h}_\omega"] \arrow[u, shift right, "\bar{g}_\omega"'] \arrow[r] & 0
\end{tikzcd},$$

where

$$\bar{h}_\omega = K_\omega \mathcal{D}^{-1}_{0,\omega} \begin{pmatrix} -\frac{f}{i\omega r} & 0 & 0 \end{pmatrix},$$

$$g_\omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \bar{g}_\omega = P'_T \frac{1}{i\omega r} \begin{pmatrix} -i\omega r & r\partial_r & -1 \\ -fr^3 \partial_r \frac{f}{r^2} & i\omega r & 0 \\ -\mathcal{B}_l f & 0 & i\omega r \end{pmatrix}.$$

VW_ω (even): decouple to Regge-Wheeler

The final equivalence square decouples $\bigcirc_\omega v = 0$ into two Regge-Wheeler equations $\mathcal{D}_{0,\omega}\phi_0 = 0$ and $\mathcal{D}_{1,\omega}\phi_1 = 0$:

$$\begin{tikzcd}
0 \arrow[r] & \bullet \arrow[r, "\bigcirc_\omega"'] \arrow[d, shift right, "k_\omega"'] & \bullet \arrow[l, bend right, dashed, "h_\omega"'] \arrow[r] \arrow[d, shift right, "g_\omega"'] & 0 \\
0 \arrow[r] & \bullet \arrow[r, "\mathcal{D}_{0,\omega}\oplus\mathcal{D}_{1,\omega}"] \arrow[u, shift right, "\bar{k}_\omega"'] & \bullet \arrow[l, bend left, dashed, "\bar{h}_\omega"] \arrow[r] \arrow[u, shift right, "\bar{g}_\omega"'] & 0
\end{tikzcd},$$

where $(\phi_0, \phi_1)^T = k_\omega(v_t, v_u, u)^T$,

$$k_\omega = \begin{pmatrix} i\omega r & -\mathcal{B}_l f & \mathcal{B}_l f \partial_r r \\ 0 & -f & f\partial_r r \end{pmatrix}, \qquad g_\omega = \frac{1}{r^2}\begin{pmatrix} i\omega r f & -\mathcal{B}_l - r\partial_r f & r\partial_r f \\ 0 & -1 & \frac{r^2}{\mathcal{B}_l}\partial_r \frac{f}{r} \end{pmatrix},$$

$$\bar{k}_\omega = \frac{1}{\omega^2 r^2}\begin{pmatrix} -i\omega r & \mathcal{B}_l i\omega r \\ r^2\partial_r \frac{1}{r} & -\mathcal{B}_l r\partial_r \\ 1 & -\mathcal{B}_l - fr\partial_r \end{pmatrix}, \qquad \bar{g}_\omega = \frac{1}{\omega^2}\begin{pmatrix} -\frac{i\omega r}{f} & \mathcal{B}_l \frac{i\omega r}{f} \\ 0 & -\mathcal{B}_l f \\ 0 & -\mathcal{B}_l r\partial_r f \end{pmatrix},$$

$$h_\omega = \frac{1}{\omega^2 r^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{f}{\mathcal{B}_l} \end{pmatrix}, \qquad \bar{h}_\omega = \frac{f}{\omega^2}\begin{pmatrix} 0 & \mathcal{B}_l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Discussion

- Complete separation of variables for Lichnerowicz and vector wave equations on Schwarzschild.
- Equivalence with decoupled Regge-Wheeler equations.
- **Example:** the vector wave equation is equivalent to 4 generalized Regge-Wheeler equations ($\mathcal{D}_{s_i}$), with spins $(s_i) = (0, 0, 1 \mid 1)$.
- The Lichnerowicz equation is equivalent to 10 generalized Regge-Wheeler equations ($\mathcal{D}_{s_i}$), with spins $(s_i) = (0, 0, 1, 0, 0, 1, 2 \mid 1, 1, 2)$.
  Similar to vector wave equation, but more complicated.
- Need to deal with low angular modes ($l \leq 1$) separately. (WIP)
- At the mode level, the equivalence maps are given (mostly) by differential operators. What is their relation with Debye potentials at the spacetime level?

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Thank you for your attention!