# Gauge theory in noncommutative geometry

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**Microlocal analysis**

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## Introduction

Noncommutative geometry provides some "microlocal" structure to spacetime, mixing the continuum with the discrete.

The Higgs field comes out as a connection 1-form, like the other bosons, but a connection between the discrete and the continuum part of the geometry.

Recent developments - e.g. twist $[D, a] \to Da - \rho(a)D$, open the way to models beyond the SM, and might help to solve old problems.

- This talk aims at explaining how gauge fields and gauge transformations are constructed in noncommutative geometry,

$$\begin{array}{rcl} \text{spacetime} & \to & \text{spectral triple,} \\ \text{gauge bundle} & \to & \text{module,} \\ \text{gauge field} & \to & \text{connection on module,} \end{array}$$

  and how these constructions adapt to the twisted case.

1. **Noncommutative geometry**
   - spectral triple
   - Connes' reconstruction theorem
2. **Gauge theory in noncommutative geometry**
   - Connection on module
   - Gauge field from Morita equivalence
   - Gauge transformation
3. **The standard model**
   - Spectral action
   - Mass of the Higgs and instability
4. **Gauge theory in twisted non-commutative geometry**
   - Twisted spectral triple
   - Twisted fluctuation and Morita equivalence
   - Twisted gauge transformation
5. **Application and open questions**
   - Twisted geometry for the standard model
   - Selfadjointness and change of signature

# 1. Noncommutative geometry

## Spectral triple

Algebra $\mathcal{A}$ acting on a Hilbert $\mathcal{H}$ together with selfadjoint $D$ such that

$$[D, a] \text{ is bounded} \quad \forall a \in \mathcal{A}.$$

Graded spectral triple: there exists $\Gamma = \Gamma^*$, $\Gamma^2 = \mathbb{I}$, such that

$$\{\Gamma, D\} = 0, \quad [\Gamma, a] = 0 \quad \forall a \in \mathcal{A}.$$

Real spectral triple: there exists antilinear operator $J$ such that

$$J^2 = \epsilon(k)\mathbb{I}, \; JD = \epsilon'(k)DJ, \; J\Gamma = \epsilon''(k)\Gamma J$$

where $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$, with $k \in \{0, 1, ..., 7\}$ the $KO$-dimension.
As well, hold the order zero and the first order conditions

$$[a, Jb^* J^{-1}] = 0, \qquad [[D, a], Jb^* J^{-1}] = 0 \qquad \forall a, b \in \mathcal{A}.$$

Example of Riemannian spin manifold $\mathcal{M}$:

$\mathcal{A} = C^\infty(\mathcal{M})$, $\quad \mathcal{H} = L^2(\mathcal{M}, S)$, $\quad D = \partial\!\!\!/ = -i\gamma^\mu \partial_\mu$, $\quad \Gamma = \gamma^5$, $\quad \mathcal{J} = C \circ cc$:

$$[\partial\!\!\!/, f]\psi = \partial\!\!\!/ f\psi - f\partial\!\!\!/\psi = (\partial\!\!\!/ f)\psi + f\partial\!\!\!/\psi - f\partial\!\!\!/\psi = (\partial\!\!\!/ f)\psi.$$

## Connes' reconstruction theorem

With other extra-conditions one has the following spectral characterization of manifolds:

Compact Riemannian manifold $\mathcal{M} \Longrightarrow$ spectral triple $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{\partial})$

$\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M}) \Longleftarrow (\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ commutative, unital

$$\begin{array}{ccc} \text{commutative spectral triple} & \rightarrow & \text{noncommutative spectral triple} \\ \updownarrow & & \downarrow \\ \text{Riemannian geometry} & & \text{non-commutative geometry} \end{array}$$

## 2. Gauge theory in noncommutative geometry

Gauge theory with gauge group $G$ $\begin{cases} \text{Fermionic fields} = \text{sections of a } G\text{-bundle } \mathcal{E}, \\ \text{Bosonic fields} = \text{connections on } \mathcal{E}. \end{cases}$

Sections of a bundle on a manifold $\mathcal{M} \iff$ a finite projective $C^\infty(\mathcal{M})$-module.

Serre-Swan

- A bundle in noncommutative geometry $\iff$ a finite projective $\mathcal{A}$-module $\mathcal{E}$.

Connection: example of the tangent bundle:

$$\nabla : \Gamma^\infty(TM) \to \Gamma^\infty(TM) \otimes \Omega^1_d(\mathcal{M}),$$
$$\partial_\nu \to \Gamma^\rho_{\mu\nu} \partial_\rho \otimes dx^\mu$$

$\Omega^1_d(M) := \left\{ f^i dg_i \right\}$ the $C^\infty(\mathcal{M})$-bimodule generated by the exterior derivative $d$.

- A connection on a (right) $\mathcal{A}$-module $\mathcal{E}$: an application $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$ satisfying the Leibniz rule

$$\nabla(\eta a) = \nabla(\eta) a + \eta \otimes \delta(a) \quad \forall \eta \in \mathcal{E},\ a \in \mathcal{A},$$

where $\Omega$ is a $\mathcal{A}$-bimodule generated by a derivation $\delta$ of $\mathcal{A}$.

## Fluctuation by Morita equivalence

$\mathcal{A}$, $\mathcal{B}$ Morita equivalent $\Longleftrightarrow \mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$, $\mathcal{E}$ a Hermitian finite proj. $\mathcal{A}$-module.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, $\mathcal{E} = \mathcal{E}_R$ a right $\mathcal{A}$-module. $\mathcal{B}$ acts on

$$\mathcal{H}_R := \mathcal{E}_R \otimes_{\mathcal{A}} \mathcal{H}$$

as

$$b(\eta \otimes \psi) = b\eta \otimes \psi \qquad \forall b \in \mathcal{B}, \eta \in \mathcal{E}_R, \psi \in \mathcal{H}.$$

The "natural" action of $D$, $D^R(\eta \otimes \psi) := \eta \otimes D\psi$, is not $\mathcal{A}$-linear on $\mathcal{H}_R$:

$$D^R(\eta a \otimes \psi) - D^R(\eta \otimes a\psi) = -\eta \otimes [D, a]\psi \qquad \forall a \in \mathcal{A}.$$

- $\nabla$ a connection on $\mathcal{E}_R$ with value in the $\mathcal{A}$-bimodule of (generalized) 1-forms

  $$\Omega^1_D(\mathcal{A}) := \{a_i[D, b_i],\ a_i, b_i \in \mathcal{A}\}$$

  generated by the derivation $\delta(a) := [D, a]$.

The covariant derivative $D_R(\eta \otimes \psi) := \eta \otimes D\psi + (\nabla\eta)\psi$ is $\mathcal{A}$-linear on $\mathcal{H}_R$.

For $\mathcal{B} = \mathcal{A}$, $\mathcal{E}_R = \mathcal{A}$, then $D_R = D + A_R$ with $A_R \in \Omega^1_D(\mathcal{A})$.

Same construction with left module $\mathcal{E} = \mathcal{E}_L$, $\mathcal{H}_L = \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}_L$.

- $\nabla^\circ$ a connection on $\mathcal{E}_L$ with value in the bimodule

$$\Omega^1_D(\mathcal{A}^\circ) = \left\{ \sum_i a_i^\circ [D, b_i^\circ], \quad a_i^\circ, b_i^\circ \in \mathcal{A}^\circ \right\}$$

  generated by the derivation $\delta^\circ(a) := [D, a^\circ]$.

The covariant derivative $D_L(\psi \otimes \eta) := D\psi \otimes \eta + (\nabla^\circ \eta)\psi$ is well defined,

For $\mathcal{B} = \mathcal{A} = \mathcal{E}_L$, then $D_L = D + A^\circ = D + \epsilon' J A_L J^{-1}$ with $A^\circ \in \Omega^1_D(\mathcal{A}^\circ)$, $A_L \in \Omega^1_D(\mathcal{A})$.

Combining the two constructions yields

$$D' = D + A_R + \epsilon' J A_L J^{-1}.$$

One has that $D' J = \epsilon' J D'$ if and only if there exists $A \in \Omega^1_D(\mathcal{A})$ such that

$$D' = D_A := D + A + J A J^{-1}$$

The substitution $D \to D_A$ in the spectral triple is called a fluctuation of the metric, and $D_A$ a covariant Dirac operator.

## Gauge transformation

Unitary endomorphisms of $\mathcal{E}$: $u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ such that $u^* u = \mathbb{I}$, where

$$\langle T^* \eta, \xi \rangle := \langle \eta, T\xi \rangle \quad \forall T \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}),\ \xi, \eta \in \mathcal{E}.$$

Form a group $\mathcal{U}(\mathcal{E})$, whose adjoint action on $\Omega$-value connections on $\mathcal{E}$,

$$\nabla^u := u \nabla u^* \qquad \forall u \in \mathcal{U}(\mathcal{E}),$$

yields a new $\Omega$-value connection $\nabla^u$. Hence

$$\nabla = \nabla_0 + \mathbf{A}, \quad \nabla^u = \nabla_0 + \mathbf{A}^u,$$

where $\nabla_0$ is the Grassmann connection, while the gauge potentials $\mathbf{A}$ and $\mathbf{A}^u$ are $\mathcal{A}$-linear maps $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$ (right module case) or $\mathcal{E} \to \Omega \otimes_{\mathcal{A}} \mathcal{E}$ (left module).

A gauge transformation is the map

$$\mathbf{A} \to \mathbf{A}^u.$$

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and

$$D_A = D + A + J\,A\,J^{-1} \quad \text{with} \quad A \in \Omega^1_D(\mathcal{A})$$

the covariant operator obtained by Morita equivalence.

Substituting $\nabla, \nabla^\circ$ with $\nabla^u, (\nabla^\circ)^u$, $u \in U(\mathcal{A})$ in the previous construction yields

$$D_{A^u} = D + A^u + J\,A^u\,J^{-1}$$

with

$$A^u := u[D, u^*] + uAu^*.$$

- $D_{A^u}$ is also obtained by the conjugate action of

$$\mathrm{Ad}(u) : \psi \to u\psi u^* = u(u^*)^\circ\psi = uJuJ^{-1}\psi,$$

namely

$$\mathrm{Ad}(u)\,D_A\,\mathrm{Ad}(u)^{-1} = D_{A^u}.$$

The inner automorphisms of $\mathcal{A}$ yields the gauge potentials (in the commutative case, the outer automorphisms give the diffeomorphisms of spacetime).

## 3. The standard model

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_{sm}, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_{32} + \gamma^5 \otimes D_{sm}$$

where

$$\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_{sm} = \mathbb{C}^{32=2\times2\times8} = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^C,$$

$$D_{sm} = \underbrace{\begin{pmatrix} 0_8 & M & 0_8 & 0_8 \\ M^\dagger & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & \bar{M} \\ 0_8 & 0_8 & M^T & 0_8 \end{pmatrix}}_{D_0} + \underbrace{\begin{pmatrix} 0_8 & 0_8 & M_R & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \\ M_R^\dagger & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \end{pmatrix}}_{D_R}.$$

- $M$ is a matrix whose coefficients are the Yukawa couplings of the electron, the quarks, and the neutrino (Dirac mass).
- $M_R$ contains only one non-zero entry $k_R$ (Majorana mass of the neutrino).

One also needs $\Gamma = \gamma^5 \otimes \gamma_{sm}$ and $J = \mathcal{J} \otimes J_{sm}$ with

$$\gamma_{sm} = \begin{pmatrix} \mathbb{I}_8 & & & \\ & -\mathbb{I}_8 & & \\ & & -\mathbb{I}_8 & \\ & & & \mathbb{I}_8 \end{pmatrix}, \quad J_{sm} = \begin{pmatrix} 0_{16} & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0_{16} \end{pmatrix}.$$

## Spectral action

Let $f$ be a smooth approximation of the characteristic function of $[0,1]$.
The asymptotic expansion $\Lambda \to \infty$ of

$$\mathrm{Tr}\, f\left(\frac{D_A^2}{\Lambda^2}\right)$$

yields the bosonic Lagrangian of the SM coupled with the Einstein-Hilbert action

$$\int_{\mathcal{M}} \sqrt{g} d^4x \; \Big( \frac{1}{\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^* R*$$
$$+ \frac{1}{4} G^i_{\mu\nu} \bar{G}^{\mu\nu}_i + \frac{1}{4} F^\alpha_{\mu\nu} \bar{F}^{\mu\nu}_\alpha + \frac{1}{4} B_{\mu\nu} \bar{B}^{\mu\nu}$$
$$+ \frac{1}{2} |D_\mu H|^2 - \mu_0^2 |H|^2 - \frac{1}{12} R |H|^2 + \lambda_0 |H|^4 \Big)$$

where $\lambda_0, \alpha_0, \tau_0, \kappa_0, \gamma_0$ are functions of $\Lambda$ and the momenta $f_\beta = \int_0^\infty f(v) v^{\beta-1} dv$, and we assume a unique unification scale

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4}, \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2.$$

## Mass of the Higgs and instability

The spectral action provides initial conditions at a putative unification scale.

Physical predictions by running down the parameters of the theory under the renormalization group equation. Assuming there is no new physics between the unification scale and our scale, one finds $m_H \simeq 170$ GeV $\neq 125,1$GeV

Chamseddine, Connes, Marcolli

but...

for a Higgs boson with mass $m_H \leq 130$ Gev, the quartic coupling of the Higgs field becomes negative at high energy, meaning the electroweak vacuum is meta-stable rather than stable.

$$V(H) = -\frac{\mu}{2}H^2 + \frac{\lambda}{4}H^4$$

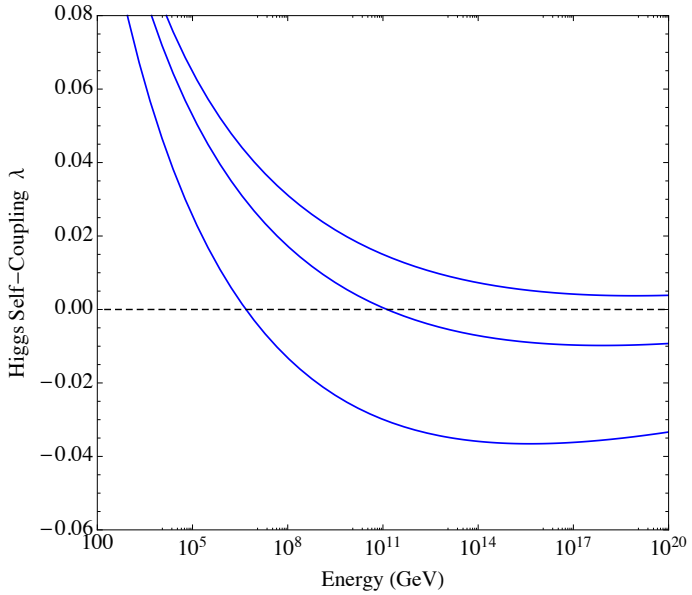



FIG. 2: Higgs self-coupling $\lambda$ as a function of energy, for different values of the Higgs mass from 2-loop RG evolution. Lower curve is for $m_H = 116\,\text{GeV}$, middle curve is for $m_H = 126\,\text{GeV}$, and upper curve is for $m_H = 130\,\text{GeV}$. All other Standard Model couplings have been fixed in this plot, including the top mass at $m_t = 173.1\,\text{GeV}$.

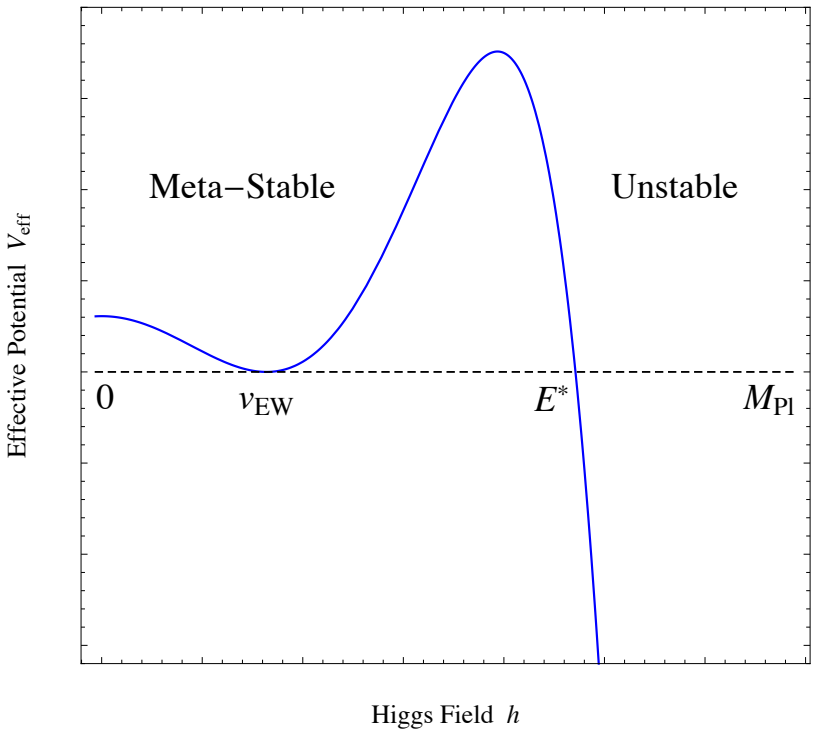
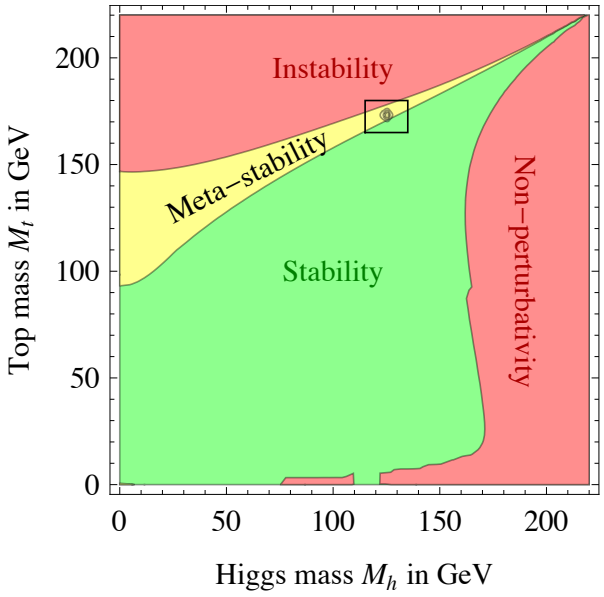
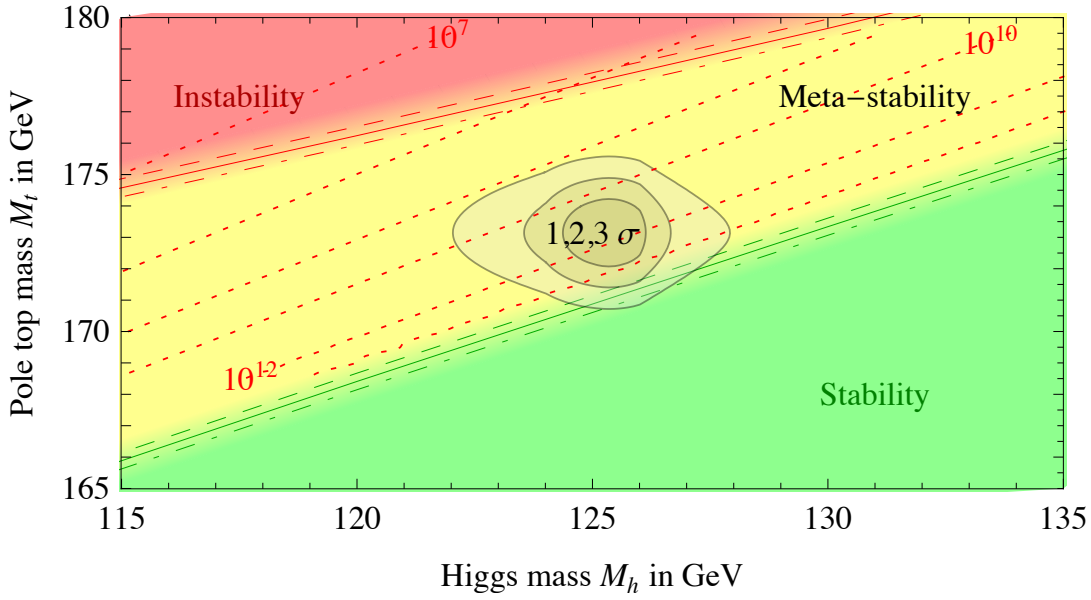



FIG. 3: Schematic of the effective potential $V_{\text{eff}}$ as a function of the Higgs field $h$. This is *not* drawn to scale; for a Higgs mass in the range indicated by LHC data, the heirarchy is $v_{EW} \ll E^* \ll M_{\text{Pl}}$, where each of these 3 energy scales is separated by several orders of magnitude.

For $h \gg v_{EW}$, $V_{eff}(h) = \frac{1}{4}\lambda(t)G(t)^4 h^4$ with $H = \frac{1}{\sqrt{2}}(0, v_{EW} + h)$, $t = \ln h/\mu$.

M. P. Hertzberg, *A correlation between the Higgs mass and dark matter*, arXiv:1210.3624




Degrassi, Di Vita, Elias-Miro, Espinosa, Guidice, Isidori and A. Sturmia, *Higgs mass and Vacuum Stability in the SM at NNLO*, arXiv:1205.6497

The instability of the electroweak vacuum can be cured by introducing a new scalar field $\sigma$:[†]

$$V(H,\sigma) = \frac{1}{4}(\lambda H^4 + \lambda_\sigma \sigma^4 + 2\lambda_{H\sigma} H^2 \sigma^2).$$

In the spectral triple of the standard model, turning into a field the neutrino Majorana mass, $k_R \to k_R \sigma$, yields the required field, and alters the running of the parameters so that to make the computation of $m_H$ compatible with 125 Gev.

Chamseddine, Connes 2012

The field $\sigma$ cannot be obtained by a fluctuation of the Dirac operator, since

$$[\gamma^5 \otimes D_R, a] = 0 \quad \forall a, b \in \mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}.$$

But it can be obtained by a twisted fluctuation

$$[\gamma^5 \otimes D_R, a]_\rho \neq 0.$$

Devastato, Lizzi, P.M.

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[†] Elias-Miro, Espinosa, Guidice, Lee and Sturmia, *Stabilization of the Electroweak Vacuum by a Scalar Threshold effect*, JHEP **1206** (2012) 031; Degrassi, Di Vita, Elias-Miro, Espinosa, Guidice, Isidori and A. Sturmia, *Higgs mass and Vacuum Stability in the SM at NNLO, arXiv:1205.6497*; Chian-Shu Chen and Yong Tang, *Vacuum Stability, Neutrinos and Dark matter*, JHEP **1204** (2012) 019; Oleg Lebedev, *On Stability of the Higgs Potential and the Higgs Portal*, JHEP, arXiv:1203.0156.

# 4.Gauge theory for twisted spectral triple

## Twisted spectral triples

Given a triple $(\mathcal{A}, \mathcal{H}, D)$, instead of asking the commutators $[D, a]$ to be bounded, one asks the boundedness of the twisted commutator Connes, Moscovici 2008

$$[D, a]_\rho := Da - \rho(a)D \qquad \text{for some} \quad \rho \in \mathrm{Aut}(\mathcal{A}).$$

- Makes sense mathematically. Relevant to deal with type III algebras.

Twist compatible with the real structure. We define a twisted fluctuation of $D$ as

$$D_{A_\rho} := D + A_\rho + J\, A_\rho\, J^{-1}$$

where $A_\rho$ is an element of the set of twisted 1-forms

$$\Omega^1_D(\mathcal{A}, \rho) := \{a_i[D, b_i]_\rho, a_i, b_i \in \mathcal{A}\}$$

such that $D_{A_\rho}$ is selfadjoint.

## Twisted fluctuations and Morita equivalence

$(\mathcal{A}, \mathcal{H}, D; \rho)$ a twisted spectral triple, right $\mathcal{A}$-module $\mathcal{E}_R = p\mathcal{A}^N$, $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}_R)$.

Need an operator on $\mathcal{H}_R = \mathcal{E}_R \otimes_{\mathcal{A}} \mathcal{H}$ whose non-linearity can be cured by an $\Omega^1_D(\mathcal{A}, \rho)$-value connection. Consider

$$((\rho \otimes \mathbb{I}) \circ D^R)(\eta \otimes \psi) = \rho(\eta) \otimes D\psi \quad \forall \eta \in \mathcal{E}_R, \psi \in \mathcal{H},$$

where

$$\rho(\eta) := p \begin{pmatrix} \rho(\eta_1) \\ \vdots \\ \rho(\eta_N) \end{pmatrix} \quad \forall \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} \in \mathcal{E}_R,\ \eta_i \in \mathcal{A}.$$

Then

$$((\rho \otimes \mathbb{I}) \circ D^R)\,(\eta a \otimes \psi - \eta \otimes a\psi) = -\rho(\eta) \otimes [D, a]_\rho\, \psi.$$

**Proposition** Landi, P.M. 2017

Let $\nabla_\rho$ be an $\Omega^1_D(\mathcal{A}, \rho)$-value connection on $\mathcal{E}_R$. Then the operator

$$\tilde{D}_R := (\rho \otimes \mathbb{I}) \circ (D^R + \nabla_\rho)$$

is linear on $\mathcal{H}_R$. In case $\mathcal{B} = \mathcal{A} = \mathcal{E}_R$, one obtains

$$\tilde{D}_R = D + A^R_\rho \ \text{ with } \ A^R_\rho \in \Omega^1_D(\mathcal{A}, \rho).$$

Similar construction for left module.

- $\nabla^\circ_\rho$ a connection on $\mathcal{E}_L$ with value in the $\mathcal{A}$-bimodule

$$\Omega^1_D(\mathcal{A}^\circ, \rho^\circ) := \left\{ \sum_i a_i^\circ [D, b_i^\circ]_{\rho^\circ}, \quad a_i^\circ, b_i^\circ \in \mathcal{A}^\circ \right\}.$$

generated by the derivation of $\mathcal{A}$: $\delta^\circ(a) := [D, a^\circ]_{\rho^\circ}$, with $\rho^\circ(a^\circ) := \rho(a)^\circ$.

**Proposition** Landi, P.M. 2017

The operator

$$\tilde{D}_L := (\mathbb{I} \otimes \rho) \circ (D^L + \nabla^\circ_\rho)$$

is well defined on $\mathcal{H}_L$. In case $\mathcal{B} = \mathcal{A} = \mathcal{E}_L$, one obtains

$$\tilde{D}_L = D + A^\circ_\rho = D + \epsilon' J A^L_\rho J^{-1}$$

with $A^\circ_\rho \in \Omega^1_D(\mathcal{A}^\circ, \rho)$, $A^L_\rho \in \Omega^1_D(\mathcal{A}, \rho)$.

Combining the two constructions yields

$$\tilde{D}' = D + A^R_\rho + \epsilon' J A^L_\rho J^{-1}$$

with $A^R_\rho$, $A^L_\rho$ two elements of $\Omega^1_D(\mathcal{A}, \rho)$, a priori distinct.

**Proposition** Landi, P.M. 2017

One has that $D'J = \epsilon' D' J$ if and only if there exists $A'_\rho \in \Omega^1_D(\mathcal{A}, \rho)$ such that

$$\tilde{D}' = D_{A_\rho} : D + A'_\rho + \epsilon' J A'_\rho J^{-1}.$$

- Twisted fluctuations arise by Morita equivalence, in the same way as non twisted ones. The only difference is that the "natural action" of $D$ on $\mathcal{H}_{R,L}$,

$$\eta \otimes \psi \to \eta \otimes D\psi, \quad \psi \otimes \eta \to D\psi \otimes \eta$$

needs to be twisted by $\rho$,

$$\eta \otimes \psi \to \rho(\eta) \otimes D\psi, \quad \psi \otimes \eta \to D\psi \otimes \rho(\eta).$$

## Twisted gauge transformation

Let $(\mathcal{A}, \mathcal{H}, D; \rho)$ be a real twisted spectral triple and

$$D_{A_\rho} = D + A_\rho + J\, A_\rho\, J^{-1} \quad \text{with } A_\rho \in \Omega^1_D(\mathcal{A}, \rho)$$

the twisted-covariant operator obtained by Morita equivalence.

**Proposition** Landi, P.M. 2017

Substituting $\nabla_\rho$, $\nabla^\circ_\rho$ with $\nabla^u_\rho$, $(\nabla^\circ_\rho)^u$ yields

$$D_{A^u_\rho} = D + A^u_\rho + J\, A^u_\rho J^{-1}$$

where

$$A^u_\rho := \rho(u)[D, u^*]\rho + \rho(u)Au^*.$$

Furthermore,

$$D_{A^u_\rho} = \rho(U) D_{A_\rho} U^{-1} \text{ for } U = \mathrm{Ad(u)}.$$

- The law of transformation of the twisted-gauge potential $A_\rho \to A^u_\rho$ is simply the twisted version of the usual transformation $A \to A^u$.
- The same is true for the conjugate action of $U = \mathrm{Ad}(u) = uJuJ^{-1}$,

$$UD_A U^{-1} \to \rho(U) D_{A_\rho} U^{-1}.$$

## 5. Applications and open questions

### Twisted geometry for the standard model

$$(\mathcal{A} \otimes \mathbb{C}^2,\ \mathcal{H}, D; \rho)$$

with $\mathcal{A}, \mathcal{H}, D$ as in the standard model, and

$$\rho(z_1, z_2) = z_2, z_1, \quad z_1, z_2 \in \mathbb{C}.$$

Spectral action computed with the twisted covariant operator

$$D_{A_\rho} = D + A_\rho + JA_\rho J^{-1}$$

where $A_\rho$ is a twisted 1-form yields

Devastato, Lizzi, P.M. 2014, 2016

- A model beyond the SM that spontaneously breaks to the SM.
- The fluctuations around the standard model are encoded by
  - the scalar field $\sigma$, coming from the twisted fluctuation of $\gamma^5 \otimes D_R$;
  - a vector field $X_\mu$, coming from the twisted fluctuation of $\partial\!\!\!/ \otimes \mathbb{I}_{32}$.

## Selfadjointness and the change of signature

In the non-twisted case $(\mathcal{A}, \mathcal{H}, UD_A U^{-1})$ is a spectral triple.

In the twisted case, $D_{A^u_\rho} = \rho(U) D_{A_\rho} U^{-1}$ has no reason to be selfadjoint.

- May not be a problem for the spectral action, defining it as

$$\lim_{\Lambda \to \infty} f\left(\frac{D^*_{A^u_\rho} D_{A^u_\rho}}{\Lambda^2}\right).$$

- Quid fermionic action ?

Possible solution: restrict to transformation such that $\rho(U) = U$.

In case of minimal twist $(\mathcal{A}, \mathcal{H}, D) \to (\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D; \rho)$, one has

$$\rho(z_1, z_2) = (z_2, z_1) \quad z_1, z_2 \in \mathbb{C}.$$

Unitaries of $\mathbb{C}^2$ are $u = (e^{i\theta_1}, e^{i\theta_2})$, and $\rho(u) = u$ iff $\theta_1 = \theta_2$. Hence

$$\mathcal{U}_\rho(\mathbb{C}^2) := \left\{ u \in \mathbb{C}^2, u^* u = \mathbb{I}, \rho(u) = u \right\} \simeq U(1).$$

- Twisting the standard model would mean adding an extra $U(1)$-field.

Another possibility is to require $\rho(U)^* = U^{-1}$ without imposing $U$ unitary.
In case of minimal twist, one has

$$\rho \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} = \begin{pmatrix} z_2 & 0 \\ 0 & z_1 \end{pmatrix} = X \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} X \quad \text{with} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Requiring $\rho(u)^* = u^{-1}$, i.e $\rho(u^*)u = \mathbb{I}$, means requiring $Xu^*Xu = \mathbb{I}$.
Working in the base where $X$ is diagonal ($X \to \eta = \mathrm{diag}(1, -1)$, $u \to v$) yields

$$\eta v^* \eta v = \mathbb{I} \text{ that is } v \in U(1,1).$$

- In the case of the standard model, could mean passing from $\mathrm{Spin}(4) = SU(2) \times SU(2)$ to $U(1,1) \times SU(2)$: twisting = changing the signature ?

Gauge transformation for twisted spectral triples,
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with A. Devastato and F. Lizzi, Fortschritte der Physik **62** 9-10 (2014) 863-868.

## Covariant Dirac operator

A fluctuation of $D$ by $\mathcal{A}$ is

$$D_A = D + A + J\,A\,J^{-1}$$

with

$$A = \sum_i a_i\,[D, b_i] = A^* \quad a^i, b^i \in \mathcal{A}.$$

- $D_A$ is called the covariant Dirac operator.

$$\left.\begin{array}{rcl} \mathcal{A} & = & C^\infty(\mathcal{M}) \otimes \mathcal{A}_F \\ \mathcal{H} & = & L_2(\mathcal{M}, S) \otimes \mathcal{H}_F \\ D & = & \partial\!\!\!/ \otimes \mathbb{I}_{32} + \gamma^5 \otimes D_F \end{array}\right\} \Longrightarrow A = \gamma^5 \otimes H - i \sum_\mu \gamma^\mu \otimes A_\mu.$$

- $H$: scalar field on $\mathcal{M}$ with value in $\mathcal{A}_F$ $\rightarrow$ Higgs.
- $A_\mu$: 1-form field with value in $Lie(U(\mathcal{A}_F))$ $\rightarrow$ gauge field.