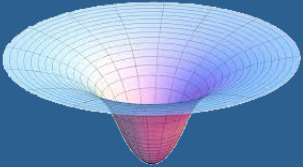
Renormalization of vector fields in locally covariant AQFT

Joint work with I. Khavkine and V. Moretti



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Outline

1. Motivations
2. Technical tools: Peetre's theorem
3. Renormalization of Vector fields in AQFT

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Motivations

Most interesting **observables** in field theory are local and nonlinear (in the field):

- Field squared A^2
- Stress-energy tensor $T_{\mu\nu}$
- Currents $\bar{\psi}\gamma^\mu\psi$

In **Minkowskian QFT** these observables are defined using **normal ordering**. For a scalar field, in position space:

$$:\varphi^2(x) := \lim_{x_1 \to x_2} \varphi(x_1)\varphi(x_2) - \langle 0|\varphi(x_1)\varphi(x_2)|0\rangle \mathbb{I}$$

In **curved spacetime** it is not so simple: There is no preferred vacuum state.

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Motivation

In CST the best we can do is to replace $|0\rangle$ with a Hadamard state:

$$:\varphi^2(x) := \lim_{x_1 \to x_2} \varphi(x_1)\varphi(x_2) - \underbrace{\omega(\varphi(x_1)\varphi(x_2))}_{\omega \text{ instead of } |0\rangle} \mathbb{I}$$

Since this expression is ambiguous, it is necessary to classify all ambiguities: **Hollands&Wald** studied the **scalar field** case

$$\widetilde{:\varphi^k:}(x) =: \varphi^k : (x) + \sum_{l=0}^{k-2} \binom{k}{l} C_{k-l}(x) : \varphi^l : (x)$$

$$C_k(x) \equiv C_k\left[g_{ab}(x), R_{abcd}(x), \dots, \nabla_{(e_1} \cdots \nabla_{e_{k-2})} R_{abcd}(x), m^2, \xi\right]$$

each C_k is a **scalar** that depends polynomially on m^2, on the Riemann tensor R and its derivatives and depends analytically on ξ

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Motivation

Among other axioms, Hollands&Wald require **analytic dependence** of Wick powers on the set of analytic metrics.

This requirement has been always considered as very unnatural:

- **Physically unnatural**: special behavior on analytic metrics
- **Technical difficulties**: analytic families of Hadamard states
- Analyticity is used to establish that C_k are **differential operators** of finite order

Recently Khavkine&Moretti elaborated a new proof dropping the analyticity requirement

We adopt the same approach to study the renormalization of vector fields.

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Peetre's theorem and other results

Peetre's theorem: If E, F are smooth bundles, a map $\Gamma(E) \to \Gamma(F)$ that is local must be a smooth differential operator (generally non linear) of locally bounded order.

Thomas theorem: A tensor function of $\mathbf{g}$ and its derivatives at x is covariant under diffeomorphisms fixing x iff it is a tensor tensorially constructed out of $\mathbf{g}$, $\mathbf{R}$, $\partial\mathbf{R}$,...

$\longrightarrow$ Old result but **new** proof for tensor valued functions

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Peetre's theorem and other results

Theorems on scaling: A function which is homogeneous under physical scaling (dilatation of parameters) and smooth at 0 under coordinate scaling is a polynomial.

$\longrightarrow$ Polynomial with respect to the parameters $(m^2, \ldots)$

Definition (Equivariant tensors)

A map $T_l^k \ni \mathbf{s} \mapsto \mathbf{t}(\mathbf{s}) \in T_{l'}^{k'}$ is equivariant with respect to the action of $GL(n)$ if $\tilde{u}\mathbf{t}(\mathbf{s}) = \mathbf{t}(u\mathbf{s})$

Theorem on equivariant tensors: The only tensors equivariant under the action of $GL(n)$ are tensors polynomially built with $\mathbf{g}$.

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Locally covariant vector and tensor fields

Let $T^k_l M$ be the usual tensor bundle.
Let $\mathscr{E}(T^k_l M)$, $\mathscr{D}(T^k_l M)$ be the space of smooth and smooth compactly supported sections of T^k_l respectively.

Definition

A **locally covariant quantum tensor field** A **of order** k is an algebra-valued distribution

$$A_{(M,\mathbf{h})} : \mathscr{D}(T^k M) \to \mathcal{W}(M, \mathbf{h})$$

which respects the inclusions and isomorphisms induced by isometries.

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Axioms of Wick powers

Let A be a vector field. Our result is compatible with all prescriptions that satisfy the following conditions:

- **Locality and covariance**: Each Wick power $:A^k:$ is a locally covariant quantum symmetric tensor field of order k.
- **Low power**: $:A: = A$
- **Scaling**: $(\mathbf{g}, m^2, \xi, A) \mapsto (\lambda^{-2}\mathbf{g}, \lambda^2 m^2, \xi, \lambda^{d_A} A)$

$$\Longrightarrow :A^k: \mapsto \lambda^k :A^k: + \lambda^k O(\log \lambda)$$

- **Commutator**: $\left[A^k_{\mu_1 \cdots \mu_k}(x), A_\nu(y)\right] = ikA^{k-1}_{(\mu_1 \cdots \mu_{k-1}}(x)\Delta_{\mu_k)\nu}(x, y)$
- **Smoothness**: with respect to x, m^2 and ξ

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Theorem (Main result)

Let $\{\widetilde{A}^k\}$ and $\{A^k\}$ be two families ($k \in \mathbb{N}$) of Wick powers

$$\widetilde{A}^k_{\mu_1\cdots\mu_k}(x) = A^k_{\mu_1\cdots\mu_k}(x) + \sum_{l=0}^{k-2} \binom{k}{l} C_{(\mu_1\cdots\mu_{k-l}}(x) A^l_{\mu_1\cdots\mu_l)}(x)$$

with $C_{\mu_1\cdots\mu_{k-l}}$ fully symmetric (tensors) for all k, l.

$$C_k(x) \equiv C_k\left[g_{ab}(x), R_{abcd}(x), \ldots, \nabla_{(e_1}\cdots\nabla_{e_{k-2})} R_{abcd}(x), m^2, \xi\right]$$

each C_k is a **tensor** *that depends polynomially on m^2, on the Riemann tensor R and its derivatives and depends analytically on ξ*

Sketch of proof: Part I

- We considered only Wick powers of A, without products with derivatives of A or time ordered products
- Consider finite renormalization $\alpha_k = \widetilde{:A^k:} - :A^k:$
- Use **Low power** axiom and **commutators** to show that

$$\alpha_k = \sum_l \binom{k}{l} C_{k-l} :A^l:$$

with $C_{k-l} = C_{k-l}[\mathbf{g}, m^2, \xi]$

- We can now use **smoothness** to show that

$$C_{k-l} : (\mathbf{g}, m^2, \xi) \mapsto \mathscr{E}(T_{k-l})$$

Remark. All these results do not depend on the dynamics.

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Sketch of proof: Part II

- Use **locality** and **Peetre theorem** to show that C_j are differential operators of locally bounded order.
- Use **Thomas theorem** to prove that C_{k-l} are $(k-l)$-tensors build with metric and curvatures.
- Use results on **equivariance** and **scaling** to show that C_{k-l} are polynomial with respet to the metric and all parameters (except ξ).

Example: the squared field

For example, in 4-dimension, if $k = 2$ we have

$$\widetilde{A}^2_{\mu\nu} = A^2_{\mu\nu} + \alpha g_{\mu\nu} m^2 + \beta g_{\mu\nu} R + \gamma R_{\mu\nu}$$

α, β, γ are physical parameters that must be measured.

Remark. An important result obtained studying the renormalization in a locally covariant approach is that

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Conclusion

- In a seminal work, Hollands&Wald studied the renormalization of scalar fields in the framework of locally covariant QFT.
- They use an unnatural analyticity hypothesis.
- Khavkine&Moretti streamlined the proof for scalar fields without using the analyticity hypothesis.
- We successfully use the same scheme to study the renormalization of vector fields.

Future development

- Currently we are working also on the renormalization of spinor fields
- It remains to generalize the result to tensor fields, products with derivatives and time ordered products

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Bibliography

[HoWa01] **S. Hollands, R.M. Wald**, "*Local Wick polynomials and time ordered products of Quantum Fields in curved spacetime*", Commun. Math. Phys. 223 (2001)

[KhMo16] **I. Khavkine, V. Moretti** "*Analytic dependence is an unnecessary requirement in renormalization of locally covariant QFT*", Commun. Math. Phys. 334 (2016)

I. Khavkine, A.M., V. Moretti, *in preparation*

Thanks for your attention