

# A taste of microlocal analysis on supermanifold

**Simone Murro**

*Department of Mathematics  
University of Regensburg*

**Microlocal analysis: a tool to explore a quantum world**

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# Motivations and goals of my talk

- Supersymmetric QFTs are interesting because of their unexpected renormalization properties
  - ↪ Non-renormalization theorems [Grisaru, Rocek, Siegel; Seiberg; ...]
- Usual framework is very restrictive: super-QFTs on super-Minkowski space

**Q:** Do the non-renormalization theorems survive on curved supermanifolds?

**A:** We don't know yet! Analyzing this question requires heavy machinery such as perturbative locally covariant QFT [Brunetti, Fredenhagen, ...]

- Up to now, the structure of supersymmetry transformations in locally covariant QFT is under control [Hack, Hanisch, Schenkel]
- **The goal of my talk:**
  - 1 Introduce a supergeometric generalization of the wavefront set
  - 2 Generalize to supermanifolds the ordinary pullback theorem for distributions

# Outline of the Talk

- Basic aspects of supermanifolds
- The polarization bundle
- Super pseudodifferential operators
- The super wavefront set
- The pullback theorem for superdistributions

Based on:

- C. Dappiaggi, H. Gimperlein, S. M., A. Schenkel, ([arXiv:1512.07823](https://arxiv.org/abs/1512.07823))

# Basic aspects of supermanifolds

Equivalent definition of smooth manifolds:

- 1 A topological space  $M$  which locally looks like  $\mathbb{R}^m$  with smooth transition maps
- 2 A sheaf  $\mathcal{M} = (M, \mathcal{O}_M)$  of commutative algebras which locally looks like  $(\mathbb{R}^m, C_{\mathbb{R}^m}^\infty)$

**Def:** A **supermanifold** is a sheaf  $\mathcal{M} = (M, \mathcal{O}_M)$  of **supercommutative superalgebras** which locally looks like  $\mathbb{R}^{m|n} := (\mathbb{R}^m, C_{\mathbb{R}^m}^\infty \otimes \wedge^\bullet \mathbb{R}^n)$

- A morphism  $\Phi : \mathcal{M} \rightarrow \mathcal{N}$  is now a pair  $(\phi : M \rightarrow N, \Phi^* : \mathcal{O}_N(V) \rightarrow \mathcal{O}_M(\phi^{-1}(V)))$

**Def:** We call **superdomain**  $U^{m|n} := (U \subset \mathbb{R}^m, C_U^\infty \otimes \wedge^\bullet \mathbb{R}^n) \subseteq \mathbb{R}^{m|n}$

- The section over every  $U \subseteq \mathbb{R}^m$  are given by  $C_U^\infty \otimes \wedge^\bullet \mathbb{R}^n =: \mathcal{O}_{\mathbb{R}^{m|n}}(U)$

$$f = \sum_{I \in \mathbb{Z}_2^n} f_I \theta^I := \sum_{(i_1, \dots, i_n) \in \mathbb{Z}_2^n} f_{(i_1, \dots, i_n)} \theta^{1^{i_1}} \dots \theta^{n^{i_n}},$$

**Ex.** Let us consider  $\mathbb{R}^{1|1}$  and the superdistribution  $u = u_1 + \delta \theta$  with  $u_1 \in C^\infty(\mathbb{R})$ :

$u^2$  is well defined!

# The polarization bundle

**Def:** The **polarization bundle** is defined as  $\mathcal{P}^* U^{m|n} := T^* U \times \wedge^\bullet \mathbb{C}^n \longrightarrow T^* U$

$$\begin{array}{ccc}
 \mathcal{P}^* V^{m'|n'}|_{T^*_{\phi(x)} V} & \xrightarrow{\mathcal{P}^* \Phi} & \mathcal{P}^* U^{m|n}|_{T^*_x U} \\
 \downarrow \pi & & \downarrow \pi \\
 T^*_{\phi(x)} V & \xrightarrow{T^* \phi} & T^*_x U
 \end{array}$$

- Any superalgebra morphism  $\Phi^* : \mathcal{O}_{V^{m'|n'}} \rightarrow \mathcal{O}_{U^{m|n}}$  can be factorized uniquely

$$(\Phi_V^*)_j^i = \begin{cases} \phi^* \circ (D^\Phi)_j^i & \text{if } j - i \geq 0 \text{ even} \\ 0 & \text{else} \end{cases}$$

- We define the mapping  $\mathcal{P}^* \Phi$  component-wise by

$$(\phi(x), k', \lambda') \mapsto \begin{cases} (x, T^* \phi(k'), \sigma_{\frac{j-i}{2}}(D^\Phi)_j^i(\phi(x), k') \cdot \lambda') & \text{if } j - i \geq 0 \text{ even} \\ (x, T^* \phi(k'), 0) & \text{else} \end{cases}$$

- The polarization mapping is compatible with compositions

$$\mathcal{P}^*(\Phi' \circ \Phi) = (\mathcal{P}^* \Phi) \circ (\mathcal{P}^* \Phi')$$

# Super pseudodifferential operators

**Def:** A **super pseudodifferential operator** is linear map  $A : \mathcal{O}_c(U) \rightarrow \mathcal{O}(U)$

$$A_j^i : C_c^\infty(U) \otimes \wedge^i \mathbb{R}^n \rightarrow C^\infty(U) \otimes \wedge^j \mathbb{R}^n$$

are (matrices of) pseudodifferential operators

**Def:** A super pseudodifferential operator  $A$  on  $U^{m|n}$  is of order  $l$  if

$$s\Psi\mathrm{DO}^l(U^{m|n}) := \left\{ A : C_c^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \rightarrow C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n : A_j^i \in \Psi\mathrm{DO}^{\frac{j-i}{2}+l}(U) \right\} .$$

**Prop:** Let  $A \in s\Psi\mathrm{DO}^l(U^{m|n})$  and  $B \in s\Psi\mathrm{DO}^{l'}(U^{m|n})$ . Then:

- a)  $B \circ A \in s\Psi\mathrm{DO}^{l+l'}(U^{m|n})$  and  $\sigma_{l+l'}(B \circ A) = \sigma_{l'}(B) \circ \sigma_l(A)$
- b) If  $\Phi : U^{m|n} \rightarrow V^{m|n}$  is a supermanifold isomorphism, then

$$\begin{aligned} \Phi_V^{*-1} \circ A \circ \Phi_V^* &\in s\Psi\mathrm{DO}^l(V^{m|n}) \\ \sigma_l(\Phi_V^{*-1} \circ A \circ \Phi_V^*) &= (\mathcal{P}^* \Phi^{-1}) \circ \sigma_l(A) \circ (\mathcal{P}^* \Phi) . \end{aligned}$$

## Example: Wess-Zumino model

- $\mathcal{M} = (M, C_M^\infty \otimes \wedge^\bullet \mathbb{R}^2)$ , where  $M$  is a smooth 3-dimensional Lorentzian manifold.
- The equation of motion  $P : \mathcal{O}_M \rightarrow \mathcal{O}_M$  of the 3|2-dimensional Wess-Zumino model

$$P = \begin{pmatrix} m & 0 & -1 \\ 0 & i\not{\nabla} + m & 0 \\ \square & 0 & m \end{pmatrix},$$

- The operator  $P \in s\Psi\mathrm{DO}^1(X)$  is of order 1, and in local coordinates

$$\sigma_1(P)(x, k) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -\gamma^\mu(x) k_\mu & 0 \\ -k_\mu k_\nu g^{\mu\nu}(x) & 0 & 0 \end{pmatrix}$$

- $\sigma_1(P)(x, k)$  is invertible for all  $(x, k) \in T^*M \setminus \mathbf{0}$  which are not null covector

$$\sigma_1(P)(x, k)^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{k_\mu k_\nu g^{\mu\nu}(x)} \\ 0 & -\frac{\gamma^\mu(x) k_\mu}{k_\mu k_\nu g^{\mu\nu}(x)} & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

- Because  $\sigma_1(P)(x, k)$  is invertible for non-null cotangent vector we call  $P$  **hyperbolic**

# The super wavefront set

- The **super wavefront set** (of order  $l$ ) of a superdistribution  $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$

$$sWF^l(u) := \bigcap_{\substack{A \in s\Psi DO^l(U^{m|n}) \\ \text{s.t. } Au \text{ smooth}}} \left\{ (x, k, \lambda) \in \widehat{\mathcal{P}}^* U^{m|n} : \sigma_l(A)(x, k)(\lambda) = 0 \right\} \subseteq \widehat{\mathcal{P}}^* U^{m|n} .$$

$$\text{where } \pi : \widehat{\mathcal{P}}^* U^{m|n} := \pi^{-1}(T^* U \setminus \mathbf{0}) \longrightarrow T^* U \setminus \mathbf{0}$$

**Prop:** Let  $u = \sum_{l \in \mathbb{Z}_2^n} u_l \theta^l \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  and  $A \in s\Psi DO^l(U^{m|n})$ :

- $sWF^l(u) = sWF^{l'}(u)$  for all  $l, l'$
- $\pi(sWF^l(u) \setminus ((T^* U \setminus \mathbf{0}) \times \{0\})) = \bigcup_{l \in \mathbb{Z}_2^n} WF(u_l)$
- $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  is smooth if and only if  $sWF(u) = (T^* U \setminus \mathbf{0}) \times \{0\}$
- $sWF(Au) \supseteq \sigma_l(A)(sWF(u)) := \{(x, k, \sigma_l(A)(x, k)(\lambda)) : (x, k, \lambda) \in sWF(u)\}$
- If  $\Phi : U^{m|n} \rightarrow V^{m|n}$  is a supermanifold isomorphism

$$sWF(\Phi_V^*(u)) = \mathcal{P}^* \Phi(sWF(u))$$



## Example II: the polarization

- $U^{m|2}$  and the superdistribution  $u = v + v \theta^1 \theta^2$
- The super pseudodifferential operator of order 0

$$Au = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 0 \\ v \end{pmatrix} = 0$$

- The super principal symbol of order 0 of  $A$  reads as

$$\sigma_0(A)(x, k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

- The super wavefront set  $sWF(u) \subseteq (T^*U \setminus \mathbf{0}) \times \{\lambda \in \wedge^\bullet \mathbb{C}^2 : \lambda_{(1,1)} = 0\}$
- Our super wavefront sets picks out the leading singularities and assigns higher weight to the components with a lower number of  $\theta$ -powers.
- This feature generalizes to superdomains in higher odd-dimensions  $U^{m|n}$ .

# The pullback theorem for superdistributions

- The normal set of a smooth map  $\phi : U \subseteq \mathbb{R}^m \rightarrow V \subseteq \mathbb{R}^{m'}$  is

$$N_\phi := \left\{ (\tilde{\chi}(x), k') \in T^*V : x \in U, T^*\tilde{\chi}(k') = 0 \right\} \subset T^*V$$

- Hormander proved that the pullback map  $\phi^* : C^\infty(V) \rightarrow C^\infty(U)$  admits a *unique continuous extension* to those distributions  $u \in \mathcal{D}'(V)$  which satisfy

$$\text{WF}(u) \cap N_\phi = \emptyset.$$

- Let us now consider a supermanifold morphism  $\Phi : U^{m|n} \rightarrow V^{m'|n'}$

1 the superalgebra morphism factorized as  $\Phi_V^* = \phi^* \circ D^\Phi$

2  $D^\Phi u \in \mathcal{D}'(V) \otimes \wedge^\bullet \mathbb{R}^n$  is *always well-defined*

3  $\phi^* D^\Phi u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  exists if we assume

$$\pi \left( s\text{WF}(D^\Phi u) \setminus ((T^*V \setminus \mathbf{0}) \times \{0\}) \right) \cap N_\phi = \emptyset,$$

**Thm:** The pullback map  $\Phi_V^* : \mathcal{O}_V \rightarrow \mathcal{O}_U$  admits a **unique continuous extension** to those superdistributions  $u \in \mathcal{D}'(V) \otimes \wedge^\bullet \mathbb{R}^{n'}$  which satisfy

$$\left( \bigcup_{I \in \mathbb{Z}_2^n} \text{WF}((D^\chi u)_I) \right) \cap N_\phi$$

- Another condition which would guarantee the existence of  $\Phi_V^* u$  is

$$\pi\left(s\text{WF}(u) \setminus ((T^*V \setminus \mathbf{0}) \times \{0\})\right) \cap N_\phi = \bigcup_{J \in \mathbb{Z}_2^{n'}} \text{WF}(u_J) \cap N_\phi = \emptyset$$

- ... but this is a strong requirement

$$\begin{aligned} \text{WF}((D^x u)_I) \cap N_\phi &= \text{WF}\left(\sum_{J \in \mathbb{Z}_2^{n'}} (D^x)_I{}^J u_J\right) \cap N_\phi \\ &\subseteq \bigcup_{J \in \mathbb{Z}_2^{n'}} \text{WF}((D^x)_I{}^J u_J) \cap N_\phi \subseteq \bigcup_{J \in \mathbb{Z}_2^{n'}} \text{WF}(u_J) \cap N_\phi = \emptyset \quad (1) \end{aligned}$$

- Consider the supermanifold morphism  $\Phi : \{*\} \rightarrow U^{m|n}$

$$\chi_U^* : C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \longrightarrow \mathbb{R}, \quad f = \sum_{I \in \mathbb{Z}_2^n} f_I \theta^I \longmapsto f_{(0, \dots, 0)}(\phi(*))$$

- We can extend  $\Phi_U^*$  to *all* superdistributions with smooth lowest component  $u_{(0, \dots, 0)}$
- Because  $N_\phi = T_{\phi(*)}^* U$ , *the condition (1) is violated* if any  $u_I$  is singular at this point
- In contrast, **our condition is verified** because just involves the lowest component

$$D^\Phi = (1 \ 0 \ \dots \ 0) \text{ and hence } D^\Phi u = u_{(0, \dots, 0)}$$

# The dessert: product of superdistributions

- The super diagonal mapping  $\Delta : U^{m|n} \longrightarrow U^{m|n} \times U^{m|n} \simeq (U \times U)^{2m|2n}$

1  $\tilde{\Delta} : U \rightarrow U \times U$  defined as  $x \mapsto (x, x)$

2  $\Delta_{U \times U}^* : C^\infty(U \times U) \otimes \wedge^\bullet \mathbb{R}^n \otimes \wedge^\bullet \mathbb{R}^n \rightarrow C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$

$$\Delta_{U \times U}^* = \tilde{\Delta}^* \circ D^\Delta = (\tilde{\Delta}^* \otimes \text{id}_{\wedge^\bullet \mathbb{R}^n}) \circ (\text{id}_{C^\infty(U \times U)} \otimes \mu)$$

3  $N_{\tilde{\Delta}} = \left\{ ((x, x), (k, -k)) \in T^*(U \times U) : (x, k) \in T^*U \right\}$

- Given  $u, v \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ , their product (if it exists) is given by

$$\begin{aligned} u \cdot v &:= \Delta_{U \times U}^*(u \otimes v) = (\tilde{\Delta}^* \circ D^\Delta) \left( \sum_{I, J \in \mathbb{Z}_2^n} u_I \otimes v_J (\theta^I \otimes \theta^J) \right) = \\ &= \tilde{\Delta}^* \left( \sum_{I, J \in \mathbb{Z}_2^n} u_I \otimes v_J (\theta^I \theta^J) \right) = \sum_{I, J \in \mathbb{Z}_2^n} \tilde{\Delta}^*(u_I \otimes v_J) (\theta^I \theta^J) \end{aligned}$$

**Cor:** The product  $u v \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  exists whenever

$$\pi \left( s\text{WF}(D^\Delta(u \otimes v)) \setminus ((T^*(U \times U) \setminus \mathbf{0}) \times \{0\}) \right) \cap N_{\tilde{\Delta}} = \emptyset$$