Why don't we formulate quantum theories on real Hilbert spaces?

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Contents

Standard Quantum Mechanics

General setting: Complex Hilbert space $(\mathcal{H}, (\cdot|\cdot))$

- *(Bounded) Observables*: Self-adjoint elements of $\mathfrak{B}(\mathcal{H})$

$$A^* = A$$

- *Possibile outcomes*: $\sigma(A) \subset \mathbb{R}$

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Preferred class of observables: Orthogonal Projections $\mathfrak{P}(\mathcal{H})$

$P \in \mathfrak{B}(\mathcal{H}),\ P^* = P,\ PP = P$

- 1-1 correspondence with closed subspaces $\mathcal{H} \supset K \mapsto P_K$
- $P_K : \psi \mapsto$ orthogonal projection of ψ on K

Standard Quantum Mechanics

Why are they so important?

- $\sigma(P) = \{0, 1\}$: two possibile outcomes: $\{False,\ True\}$
- P is a question-observable: "is it true that ....?"

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- P is a question-observable: "is it true that ....?"
- *Spectral Theorem*: let $A^* = A \in \mathfrak{B}(\mathcal{H})$

$$\begin{cases} A = \int_{\sigma(A)} \lambda \, dP^A(\lambda) \\ \sigma(A) \supset \Delta \mapsto P^A(\Delta) \in \mathfrak{P}(\mathcal{H}), \end{cases}$$

- $P_\Delta^{(A)}$: Is it true that the value of A falls within $\Delta \subset \mathbb{R}$?

Standard Quantum Mechanics

The answer to be given w.r.t. a given state of the system

- *States:* σ-addirive probability measures: $\mu : \mathfrak{P}(\mathcal{H}) \to [0,1]$

$$\begin{cases} \mu(I) = 1 \\ \mu\left(\text{s-}\sum_n P_n\right) = \sum_n \mu(P_n), \;\; P_n P_m = 0 \end{cases}$$

Interpretation of $\mu(P)$: probability that P is true if the state is μ

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Finally, we must to be able to define symmetries:

- *Symmetries:* Bijection $\alpha : \mathfrak{P}(\mathcal{H}) \to \mathfrak{P}(\mathcal{H})$
  - preserve the *lattice* structure of $\mathfrak{P}(\mathcal{H})$
  - **Wigner Theorem** $\alpha = U \cdot U^{-1}$

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This is the most general theoretical settings of a quantum particle (abstraction of Quantum Theory of wave-functions on $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C})$)

Standard Quantum Mechanics

The structure of the main character of the theory: $\mathfrak{P}(\mathcal{H})$

$$\boxed{\text{elements of } \mathfrak{P}(\mathcal{H}) \text{ are the } \underline{\textit{statements}} \text{ about the system}}$$

$\Rightarrow$ *logical connectives* (at least for commuting P, Q)

- **conjuction** $P \wedge Q := \inf\{P, Q\}$: projector on $P(\mathcal{H}) \cap Q(\mathcal{H})$
- **disjunction** $P \vee Q = \sup\{P, Q\}$: projector on $\langle P(\mathcal{H}) \cup Q(\mathcal{H}) \rangle$
- **negation** $\neg P := P^{\perp}$: projector on $P(\mathcal{H})^{\perp}$

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Logic of standard Quantum Mechanics (von Neumann)

$\mathfrak{P}(\mathcal{H})$ is a *non-distributive* bounded, σ-complete, atomic, atomistic orthocomplemented, weakly-modular lattice which satisfies the covering law.

Quantum Logic

The *lattice structure* of $\mathfrak{P}(\mathcal{H})$ is *more fundamental* than its particular realization, so why don't we **reverse our perspective?**

What if we *start* from an abstract lattice which resemble $\mathfrak{P}(\mathcal{H})$?

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Quantum Logic

The set of quantum propositions is a **bounded, σ-complete, atomic, atomistic, orthocomplemented, weakly-modular lattice** which satisfies the **covering law**

Every axioms is justified in an *operational way*

Achtung!

Some of these axioms are very strong, we will discard them later on

Quantum Logic: Piron-Solèr Theorem

Elementary Particle $\Rightarrow \mathcal{L}$ **not reducible** into sum of sublattices

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Adding other technical hypotheses it holds

$$\mathcal{L} \cong \mathfrak{P}(\mathcal{H})$$

where $\mathcal{H}$ is a Hilbert space over the field $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$

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Meaning of a quantum theory over $\mathbb{R}$ or $\mathbb{H}$?

The main difficulty regards Noether Theorem:

- Time evolution: $t \mapsto V_t$
- Dynamical symmetry: $s \mapsto U_s$ such that $V_t U_s V_t^{-1} = U_s$
- Stone Theorem: $U_s = e^{sA}$ with $A^* = -A$

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- Stone Theorem: $U_s = e^{sA}$ with $A^* = -A$

$$\Longrightarrow V_t A V_t^{-1} = A$$

Quantum Logic: Pirón-Solèr Theorem

In the $\mathbb{C}$-case we multiply everything by i and get $((iA)^* = iA)$

$$V_t(iA)V_t^{-1} = iA$$

$\boxed{\text{Noether: dynamical symmetry } s \mapsto U_s \Rightarrow \text{conserved observable } (iA)}$

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What about the other cases introduced by Piron-Soler?

- $\mathbb{R}$: there are no imaginary units
- $\mathbb{H}$: the operator jA is not well-defined

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Solution: Imaginary Operator

$J \in \mathfrak{B}(\mathcal{H})$ such that $J^* = -J \quad JJ = -I$

Quantum Logic: Piron-Solèr Theorem

Find J such that $JU_s = U_sJ$ and $JV_t = V_tJ$ then

$$\boxed{\text{Noether is safe: } (JA)^* = JA \text{ and } V_t(JA)V_t^{-1} = JA}$$

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Question marks

- Does such a J exists for any *fixed* dynamical symmetry?
- If it exists, is it the same for *all* the dynamical symmetries?

Now we stick to the real and complex cases and face them in a very general fashion. Let us see what happens!

Real Quantum Mechanics

Strategy

1. Take Piron-Solèr thesis simply as a clue on the nature of $\mathcal{L}$
2. Forget the unnatural axioms
3. Weaken the Piron-Solèr thesis
4. find a good candidate which satisfies the natural axioms

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Our framework

Let $\mathcal{H}$ be over $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ a von Neumann algebra.

- Quantum propositions are the elements of $\mathfrak{P}(\mathcal{M})$
- Observables are the self-adjoint elements of $\mathcal{M}$
- States are probability measures $\mu : \mathfrak{P}(\mathcal{M}) \to [0, 1]$
- Symmetries are automorphisms $\alpha : \mathfrak{P}(\mathcal{M}) \to \mathfrak{P}(\mathcal{M})$

Real Quantum Mechanics

We are interested in relativistic elementary particles

Relativistic system not decomposable into subsystems

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1. $\mathcal{M}$ is **irreducible** as a consequence of
   - *non existence of Super Selection Rules*
   - *existence of a Maximal Set of Commuting Observables:*

Real Quantum Mechanics

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Relativistic system not decomposable into subsystems

1. $\mathcal{M}$ is **irreducible** as a consequence of
   - *non existence of Super Selection Rules*
   - *existence of a Maximal Set of Commuting Observables:*
2. **Poincaré symmetry** $\mathcal{P} \ni g \mapsto \alpha_g \in Aut(\mathfrak{P}(\mathcal{M}))$
   - *no proper fixed points:* $\begin{cases} \alpha_g(P) = P \quad \forall P \Rightarrow P = 0, I \\ \text{otherwise } \mathfrak{P}(\mathcal{H})_P \text{ closed under } \alpha \end{cases}$
   - *faithfulness*
   - *weak continuity:* $g \mapsto \mu(\alpha_g(P))$ is continuous $\forall \mu, P$

Real Quantum Mechanics

Let us stick to the complex case:

- Irreducibility: $\mathcal{M} = \mathcal{M}'' = \{\mathbb{C}I\}' = \mathfrak{B}(\mathcal{H})$
- Wigner: $\alpha_g = U_g \cdot U_g$
- Bargmann: $g \mapsto U_g$ strongly-continuous faithful unitary repres.
- $g \mapsto U_g$ is irreducible, hence $\{U_g \mid g \in \mathcal{P}\}'' = \mathfrak{B}(\mathcal{H})$

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$$\boxed{\text{The particle is completely characterized by } \mathcal{P} \ni g \mapsto U_g}$$

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This agrees with the definition of Wigner for elementary particles

A (complex) **elementary particle** is an irreducible strongly-continuous faithful unitary representation of $\mathcal{P}$ on a complex Hilbert space.

Real Quantum Mechanics: naive approach

What about the real case?

Naive approach: mimic the complex case

A **real elementary particle** is an irreducible strongly-continuous faithful unitary representation of $\mathcal{P}$ on a real Hilbert space.

The VN algebra of the system is $\mathcal{M}_U := \{U_g \mid g \in \mathcal{P}\}''$

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- Stone Theorem: $U_{g_t} = e^{tP_0}$ with $P_0^* = -P_0$
- Polar Decomposition: $P_0 = J|P_0|$ where
  1. $|P_0| \geq 0$ e $|P_0|^* = |P_0|$: **energy operator**
  2. $J^* = -J$ and J partial isometry

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Naive approach: main result

J is an imaginary operator and $J \in \mathcal{M}_U \cap \mathcal{M}'_U$

Real Quantum Mechanics: naive approach

Definition (Complexification)

The real Hilbert space $\mathcal{H}$ equipped with

1. complex multiplication $(a + ib)v := (aI + bJ)v$
2. hermitean scalar product $(u|v)_J := (u|v) - i(u|Jv)$

is a complex Hilbert space, denoted by $\mathcal{H}_J$

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Proposition

A $\mathbb{R}$-linear operator A is $\mathbb{C}$-linear iff $AJ = JA$. If this is true:

1. A is unitary on $\mathcal{H}$ iff it is unitary on $\mathcal{H}_J$
2. A is (anti) selfadjoint on $\mathcal{H}$ iff it is (anti) selfadjoint on $\mathcal{H}_J$

Real Quantum Mechanics: naive approach

This has several consequences:

1. $J \in \mathcal{M}'_U$ hence $\mathcal{M}_U \subset \mathfrak{B}(\mathcal{H}_J)$. Actually $\boxed{\mathcal{M}_U = \mathfrak{B}(\mathcal{H}_J)}$
2. every U_g is $\mathbb{C}$-linear and unitary
3. $U : g \mapsto U_g \begin{cases} \text{real irreduciblity } \Rightarrow \text{complex irreduciblity} \\ \text{strong-continuity is preserved } (\|\cdot\|_J = \|\cdot\|) \\ \text{faithfulness is preserved} \end{cases}$

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$\Rightarrow$ recover the standard definition of complex elementary particle

Conclusion

naive *real* theory is a fake: equivalent to standard *complex* theory

Real Quantum Mechanics: correct approach

Deficiency of this approach

Existence of such a $g \mapsto U_g$ is too strong a requirement!

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As said before we should start with

1. irreducible von Neumann algebra $\mathcal{M}$
2. Poincaré symmetry $\mathcal{P} \ni g \mapsto \alpha_g \in Aut(\mathcal{L}(\mathcal{M}))$

<u>This is the only physical assumption we can made!</u>

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This is the only physical assumption we can made!

- in the $\mathbb{C}$-case this immediately leads to $g \mapsto U_g$
- in the $\mathbb{R}$-case this is not obvious: in general $\mathcal{M} \neq \mathfrak{B}(\mathcal{H})$ and Wigner result does not apply

Real Quantum Mechanics: correct approach

Proposition

One and only one of the following holds for $\mathcal{M}$ irreducible

1. $\mathcal{M}' = \{aI, a \in \mathbb{R}\}$
2. $\mathcal{M}' = \{aI + bJ,\ a, b \in \mathbb{R}\} \quad (J \in \mathcal{M})$
3. $\mathcal{M}' = \{aI + bJ + cK + dJK,\ a, b, c, d \in \mathbb{R}\} \quad (J, K \notin \mathcal{M})$

where J, K are imaginary operators, with $JK = -KJ$

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Proposition

It holds respectively

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2. $\mathcal{M} = \mathfrak{B}(\mathcal{H}_J) \quad \mathfrak{P}(\mathcal{M}) = \mathfrak{P}(\mathcal{H}_J)$

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3. $\mathcal{M} = \mathfrak{B}(\mathcal{H}_{JK}) \quad \mathfrak{P}(\mathcal{M}) = \mathfrak{P}(\mathcal{H}_{JK})$

Real Quantum Mechanics: correct approach

$\boxed{\alpha_g \text{ automorphism over } \mathfrak{P}(\mathcal{H}_{\mathbb{K}}) \text{ with } \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \text{ respectively.}}$

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$\Rightarrow$ standard Wigner et al. results apply:

Theorem

There exists an irreducible strongly-continuous faithful unitary representation $g \mapsto U_g$ on $\mathcal{H}, \mathcal{H}_J, \mathcal{H}_{JK}$, respectively, such that

$$\alpha_g = U_g \cdot U_g^*$$

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Some remarks:

1. on $\mathcal{H}_J$ association $\alpha_g \leftrightarrow U_g$ up to "phases" $e^{\alpha J}$ for $\alpha \in \mathbb{R}$
2. on $\mathcal{H}$ and $\mathcal{H}_{JK}$ association $\alpha_g \leftrightarrow U_g$ up to "phases" ± 1
3. **phases:** unitary elements of the **center** $\mathcal{Z} = \mathcal{M} \cap \mathcal{M}'$

Real Quantum Mechanics: correct approach

We must make another important physical assumption

Physical assumption

Elementary particle characterized by its maximal symmetry group

Its observables must come from its representation somehow.

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Physical assumption

Elementary particle characterized by its maximal symmetry group

Its observables must come from its representation somehow.

How? There is a natural way this can be achieved in

$$\boxed{\mathfrak{P}(\mathcal{M}) \subset \{\{U_g \mid g \in \mathcal{P}\} \cup \mathcal{Z}\}''}$$

Phases must be included: only α_g has physical meaning and

$$\alpha_g \cong \text{" } U_g \text{ up to phases"}$$

Real Quantum Mechanics: correct approach

1. Real commutant: $\mathcal{M} = \mathfrak{B}(\mathcal{H})$
   - $\mathfrak{P}(\mathcal{H}) = \mathfrak{P}(\mathcal{M}) \subset \{U_g \mid g \in \mathcal{P}\}'' \subset \mathfrak{B}(\mathcal{H})$ from which
   $$\mathfrak{B}(\mathcal{H}) = \{U_g \mid g \in \mathcal{P}\}'' =: \mathcal{M}_U$$
   - $g \mapsto U_g$ irreducible faithful strongly-cont. unitary repr. on $\mathcal{H}$

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Naive result: exists imaginary operator $J_0 \in \mathcal{M}_U \cap \mathcal{M}'_U$

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CONTRADICTION! $\mathcal{M}_U' = \mathfrak{B}(\mathcal{H})' = \mathbb{R}I$ and $J_0^* = -J_0 \neq 0$

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2. Quaternionic commutant: $\mathcal{M} = \mathfrak{B}(\mathcal{H}_{JK})$

   Different treatment same conclusion: CONTRADICTION!

Conclusions

- A Wigner elementary particle on a real Hilbert space is equivalent to a standard Wigner elementary particle on a complex Hilbert space
- Trying a more natural and abstract approach we end up with three mutually exclusive possibilities:
  1. Wigner elementary particle on real Hilbert space
  2. Wigner elementary particle on complex Hilbert space
  3. Wigner elementary particle on quaternionic Hilbert space

  The extremal possibilities lead to a contradiction: only the complex option survives.

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  2. Wigner elementary particle on complex Hilbert space
  3. Wigner elementary particle on quaternionic Hilbert space

  The extremal possibilities lead to a contradiction: only the complex option survives.

  THE THEORY IS NECESSARILY COMPLEX
- Even though we discard some unnatural axioms and weaken Piron-Solér thesis we eventually recover it.

Work in progress

$\boxed{\text{What about the quaternionic case suggested by Piron-Soler?}}$

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STRATEGY: mimic the discussion done for the real case

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2. consider an irreducible von Neumann algebra $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$
3. see what happens...

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What is a quaternionic von Neumann algebra?

Current situation

1. we chose a real subalgebra $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ such that $\mathcal{M}'' = \mathcal{M}$
2. mimicked the real theory
3. three mutually exclusive possibilities came out.
4. it remains to study them in detail...

And to conclude, some bibliography...

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Thank you for the attention!

Appendix: quaternionic commutant case

$\boxed{\text{Quaternionic commutant } \mathcal{M} = \mathfrak{B}(\mathcal{H}_{JK}),\ \mathfrak{P}(\mathcal{M}) = \mathfrak{P}(\mathcal{H}_{\mathcal{J}\mathcal{K}})}$

1. $\mathfrak{P}(\mathcal{H}_{JK}) \subset \{U_g \mid g \in \mathcal{P}\}''$
2. $g \mapsto U_g$ is quaternionic irreducible on $\mathcal{H}_{JK}$

Take $J \in \mathcal{M}'$ (or K) and define $\mathcal{H}_J$ the usual way. 1) and 2) imply

Theorem

The map $g \mapsto U_g$ is a **irreducible** strongly-continuous faithful unitary representation on $\mathcal{H}_J$

Appendix: quaternionic commutant case

Let us work on $\mathcal{H}_J$: Take the time-translation $t \mapsto g_t$

- Stone Theorem $U_{g_t} = e^{tP_0}$ with $P_0^* = -P_0$
- Polar decomposition: $P_0 = J_0|P_0|$
  1. $|P_0| \geq 0$ and $|P_0|^* = |P_0|$
  2. $J_0^* = -J_0$ and J_0 is a partial isometry

Naive result (complex version)

It holds $J_0 = \pm iI = \pm J$

Let us go back to $\mathcal{H}$: it still holds $U_{g_t} = e^{tP_0}$ and $P_0 = J_0|P_0|$

Properties of Polar Decomposition

$K \in \{U_g | g \in \mathcal{P}\}' \Rightarrow Ke^{tP_0} = e^{tP_0}K \Rightarrow KP_0 = P_0K \Rightarrow KJ_0 = J_0K$

IMPOSSIBLE! because $JK = -KJ$