Topological charges of the electromagnetic quantum fields and spacelike linearity

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Outline

1. The linear e.m. quantum field
2. The universal C^*-algebra
3. Spacelike linearity, topological charges and quantum currents

- The universal $C*$-algebras of the e.m. quantum field, is a $C*$-algebra represented in any theory of the e.m. quantum field.
- Surprisingly, commutators of the field in some topologically nontrivial and spacelike seperated regions do not vanish, in general, but give rise to topological charges
- Topological charges turn out to be trivial in any regular representation in which the field is linear on test functions.
- However,regular representations of this C*-algebra in which topological charges are non-trivially represented exist, also in presence of a electric current. The corresponding fields satisfies a weak form of linearity: spcelike linearity.

The talk is based on two joint works with D.Buchholz, F.Ciolli and E.Vasselli [LMP 16] (the other will appear on LMP).

n-Forms on Minkowski spacetime

- *Minkowski spacetime*: $\mathbb{R}^4$ with signature $(+,-,-,-)$. $\perp$ *spacelike separation*.
- $\mathcal{D}_k$ set smooth k-forms with compact support in the Minkowski spacetime. f, h are spacelike separated, $f \perp h$, whenever

$$\mathrm{supp}(f) \perp \mathrm{supp}(h) \ .$$

- $d : \mathcal{D}_k \to \mathcal{D}_{k+1}$, $d^2 = 0$ *differential operator*
- $\star : \mathcal{D}_k \to \mathcal{D}_{4-k}$, $\star\star = (-)^{k+1}\, id_k$ *Hodge dual*
- $\delta : \mathcal{D}_{k+1} \to \mathcal{D}_k$, $\delta := -\star d \star$ *co-differential (gen. divergence)*

$$\delta^2 = 0 \quad , \quad \Box = \delta d + d\delta$$

- Of particular importance: $\mathcal{C}_k$ set of co-closed k-forms (divergence-free): $\delta f = 0$.
Geometrical examples: $f \in \mathcal{D}_0$ a test function; a k-simplex $\chi : [0,1]^k \to \mathbb{R}^4$, let f_χ be the k-form
$$f_\chi(y) := \int f(y-\chi) d\chi$$
then $\mathrm{supp}(f_\chi) \subseteq \mathrm{supp}(f) + \chi$ and the Stokes theorem reads
$$\delta f_\chi = f_{\partial\chi}$$
We call these forms smearing chains. Note that if χ is a cycle i.e. $\partial\chi = 0$, then $\delta f_\chi = 0$. We shall refer in this case as smearing cycles or diveregece-free forms.

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The e.m. quantum field F linear mapping $F : \mathcal{D}_2 \ni h \to F(h) \in \mathscr{A}$ to some $*$-algebra $\mathscr{A}$

(i) Causality

$$h_1 \perp h_2 \quad \Rightarrow \quad [F(h_1), F(h_2)] = 0 \ ,$$

(ii) 1^{st} Maxwell equation

$$dF(\tau) := F(\delta\tau) = 0 \ , \qquad \tau \in \mathcal{D}_3 \ .$$

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We get the 2^{nd} Maxwell equation

$$j(f) := \delta F(f) = F(df) \ , \qquad f \in \mathcal{D}_1$$

where j is the conserved current: $\delta j = 0$.

Cohomological Problem: By 1^{st} Maxwell equation $dF = 0$

F is a "quantum" closed 2-form

Exists a "quantum" 1-form A (a vector potential) which is causal and s.t.

$$F = dA \quad ?$$

Positive answer: this is possible in a "covariant and gauge independent" way, but a new causality relations arise.

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- By Local Poincaré lemma any divergence-free 1-form $f \in \mathcal{C}_1$ have co-primitives

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- So restricting to divergence-free 1-forms we may define

$$A(f) := F(\widehat{f}) \ , \qquad f \in \mathcal{C}_1 \ ,$$

  well defined i.e. independent of the choice of the co-primitive $\widehat{f}$ by 1^{st}-Maxwell eq.

The intrinsic vector potential is a linear mapping $\mathcal{C}_1 \ni f \mapsto A(f) \in \mathcal{A}$ s.t.

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$$f_1 \bowtie f_2 \quad \Rightarrow \quad [A(f_1), A(f_2)] = 0$$

where $f_1 \bowtie f_2$ means that the supports of f_1 and f_2 are contained, respectively, in two contractible and spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ (for instance double cones).

- The e.m. field $F = dA$
- The 1^{st} Maxwell equation $dF = d^2 A = 0$
- The conserved current: $j = \delta F = \delta dA$.
- Covariance: $\gamma_P : \mathcal{C}_1 \to \mathcal{C}_1$ with $(\gamma_P f)^\mu := (Pf)^\mu \circ P^{-1}$ then

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Basic question: understand strong causality. Clearly $f_1 \bowtie f_2 \Rightarrow f_1 \perp f_2$. But

$$f_1 \perp f_2 \quad \Rightarrow \quad [A(f_1), A(f_2)] = \ ? \ .$$

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- The converse does not hold in general:

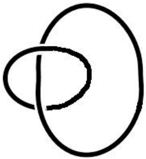

Figure: Spacelike separated linked curves at the subspace $t = 0$

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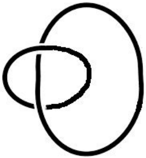

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$$h \in \mathcal{D}_2,\ \delta h = f_1 - f\ , \mathrm{supp}(h) \perp \mathrm{supp}(f_2) \quad \Rightarrow \quad [A(f_1), A(f_2)] = [A(f), A(f_2)]$$

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- Translation invariance

$$f_1 \perp f_2 \quad \Rightarrow \quad [A(f_{1,x}), A(f_{2,x})] = [A(f_1), A(f_2)]\ , \qquad \forall x \in \mathbb{R}^4$$

- Dilation invariance

$$f_1 \perp f_2 \quad \Rightarrow \quad [A(\tau_\lambda(f_1)), A(\tau_\lambda(f_2))] = \lambda^{-6}\,[A(f_1), A(f_2)]\ , \qquad \forall \lambda > 0$$

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- Centrality (topological charges ?) by translation invariance

$$\boxed{f_1 \perp f_2 \quad \Rightarrow \quad [[A(f_1), A(f_2)]\ ,\ A(f)] = 0\ , \qquad \forall f \in \mathcal{C}_1}$$

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The universal C*-algebra of the e.m. quantum field

Let $\mathcal{U}$ be the group generated by $U : \mathbb{R} \times \mathcal{C}_1 \ni (a, f) \to U(a, f)$ s.t.

(i) $U(a, f)^* = U(-a, f) \quad , \quad U(0, f) = 1 \quad , \quad U(a, f)\, U(b, f) = U(a + b, f)$;

(ii) $f_1 \bowtie f_2 \quad \Rightarrow \quad U(a_1, f_1)\, U(a_2, f_2) = U(1, a_1 f_1 + a_2 f_2)$;

(iii) $f_1 \perp f_2 \quad \Rightarrow \quad \lfloor U(a, f), \lfloor U(a_1, f_1), U(a_2, f_2) \rfloor \rfloor = 1$

where $\lfloor , \rfloor$ is the group commutator. The Poincaré group acts on $\mathcal{U}$: $P(a, f) := (a, Pf)$ for any $P \in \mathcal{P}_+^\uparrow$. The universal C*-algebra of the e.m. field $\mathfrak{U}$ is the full group C^*-algebra of $\mathcal{U}$.

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Universality. For instance if $\mathcal{W}$ is the Weyl algebra of the free electromagnetic intrinsic e.m. potential and π_F is its Fock representation of the Fock space $\mathcal{H}_F$ then

$$\widetilde{\pi}_F(U(a, f)) := \pi_F(W(a, f)) \;, \qquad (a, f) \in \mathbb{R} \times \mathcal{C}_1$$

gives a representation of $\mathfrak{U}$ on $\mathcal{H}_F$

States and representations: recovering the intrisic vector potential

A regular vacuum state of the algebra $\mathfrak{U}$ is a pure and Poincaré invariant state ω s.t.

- strong regularity
$$a_1, \ldots, a_n \mapsto \omega(U(a_1, f_1) \cdots U(a_n, f_n))$$
are smooth with tempered derivatives at 0
- $\mathcal{P}_+^\uparrow \ni P \to \omega(A\alpha_P(B))$ continuous ;
- spectral condition
$$\mathbb{R}^4 \ni p \to \int e^{ipx} \omega(A\alpha_x(B))\, d^4x \in \overline{V}_+$$

Consequences:

ω is a regular vacuum state; $(\Omega, \pi, \mathcal{H})$ be the GNS of ω.

- Strong regularity $\Rightarrow$ exist selfadjoint operators $A_\pi(f)$ with common stable core $\mathcal{D} \subseteq \mathcal{H}$ such that
$$\pi(U(a, f)) = e^{iaA_\pi(f)} \ .$$
- Spectral condition $\Rightarrow$ ω is determined by the generating functional
$$f \mapsto \omega(U(1, f)) \ , \quad f \in \mathcal{C}_1 \ ,$$
(analyticity and EOW theorem)

In general: the generators A_π associated with a regular vacuum state are not linear on test functions !
If a regular vacuum state ω satisfies condition L i.e.

$$\frac{d}{dt}\omega(VU(t,f_1)U(t,f_2)U(-t,f_1+f_2)W)|_{t=0} = 0$$

then

$$a_1A_\pi(f_1) + a_2A_\pi(f_2) = A_\pi(a_1f_1 + a_2f_2) \ on\ \mathcal{D}$$

i.e. $\mathcal{C}_1 \ni f \mapsto A_\pi(f)$ satisfies all the Wightaman axioms.

Meaningful states.

- Zero current $j = 0$. ω_0 reg. vacuum state with proprty L and s.t.

$$j_\pi(f) = A_\pi(\delta df) = 0 \ , \qquad \forall f \in \mathcal{C}_1 \ .$$

then

$$\omega_0(U(1,f)) = e^{-W(f,f)/2} \ , \qquad f \in \mathcal{C}_1$$

where $W(f,f)$ is the 2-point function of the free electromagnetic field i.e. A_π free electromagnetic field in Fock representation

- Classical current (central current). ω reg. vacuum state with proprty L and s.t.

$$[j_\pi(g), A_\pi(f)] = 0 \ , \qquad g \in \mathcal{D}_1, \ f \in \mathcal{C}_1$$

then

$$\omega(U(1,f)) = e^{ij_\pi(G_0(f))}\omega_0(U(1,f))$$

where G_0 Green's function of $\Box$ (we recover the results by Streater [RJMP 14])

Questions

- Does exists regular vacuum states of the universal C^*-algebra $\mathfrak{U}$ describing the intrinsic vector potential with a quantum current i.e. a non central j ?
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Both these question have a positive answer if the fields satisfies a weak form of linearity, i.e. spacelike linearity:

$$f_1 \bowtie f_2 \;\Rightarrow\; A(f_1) + A(f_2) = A(f_1 + f_2)$$

In particular the corresponding regular vacuum ω violate property L

Outline

Non existence of topological charges in case of linearity

- Let ω_0 be the regular vacuum state with **property L** of the algebra $\mathfrak{A}$.
- The corresponding intrinsic vector potential A is a Wightman fields (linear on test functions in particular)

Thm. Let γ_1, γ_2 be simple closed curves and $\mathcal{O}_1$,$\mathcal{O}_2$ double cones such that

$$\mathcal{O}_1 + \gamma_1 \quad \perp \quad \mathcal{O}_2 + \gamma_2$$

For any pair $f_1, f_2 \in \mathcal{C}_1$ with $\mathrm{supp}(f_1) \subset \mathcal{O}_1 + \gamma_1$ and $\mathrm{supp}(f_2) \subset \mathcal{O}_2 + \gamma_2$ we have

$$[A(f_1), A(f_2)] = [A(f_2), A(f_1)] \quad \Rightarrow \quad [A(f_1), A(f_2)] = 0$$

The example from non-linearity

- Given a 2-form G define

$$\bar{G}^{\mu\nu} := \int G^{\mu\nu}(x)\, d^4x \quad , \quad \bar{G}^2 := \bar{G}^{\mu\nu}\, \bar{G}_{\mu\nu} \ .$$

$\bar{G}^2$ in an invariant and we say that G is of Electric type whenever $\bar{G}^2 > 0$ and of Magnetic type if $\bar{G}^2 < 0$.

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- Let ω_0 be the regular vacuum state with property L of the algebra $\mathfrak{U}$ with zero conserved current j. We know

$$\omega_0(U(1,g)) = e^{iA_0(g)} \ , \qquad g \in \mathcal{C}_1$$

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- If $g \in \mathcal{C}_1$ has connected support, let G be any co-primitive of g and define

$$A_T(g) := \theta_+(\bar{G}^2)A_0(\delta G) + \theta_-(\bar{G}^2)A_0(\delta \star G)$$

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- As $\delta G = g$ and since the conserved current is 0 $A_0(\delta * G)$ **does not depend on the choice of the co-primitive**, the definition is well posed.

- For any $g_1, g_2 \in \mathcal{C}_1$ with connected support

$$[A_T(g_1), A_T(g_2)] = \left(\theta_+(\bar{G}_1^2)\theta_+(\bar{G}_2^2) + \theta_-(\bar{G}_1^2)\theta_-(\bar{G}_2^2)\right) \cdot [A_0(g_1), A_0(g_2)] + \\ + \left(\theta_+(\bar{G}_1^2)\theta_-(\bar{G}_2^2) - \theta_-(\bar{G}_1^2)\theta_+(\bar{G}_2^2)\right) [A_0(g_1), A_0(\delta \star G_2)]$$

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- Key observation: Roberts has shown that

$$[A_0(g_1), A_0(\delta \star G_2)] = c \cdot \mathbb{1} \quad , \quad c \neq 0$$

for a particular class of divergence-free 1-forms $g_1, g_2 \in \mathcal{C}$ whose supports are spacelike separated and (nontrivially) linked together.

- For any $g_1, g_2 \in \mathcal{C}_1$ with connected support

$$[A_T(g_1), A_T(g_2)] = \Big(\theta_+(\bar{G}_1^2)\theta_+(\bar{G}_2^2) + \theta_-(\bar{G}_1^2)\theta_-(\bar{G}_2^2)\Big) \cdot [A_0(g_1), A_0(g_2)] + \\ + \Big(\theta_+(\bar{G}_1^2)\theta_-(\bar{G}_2^2) - \theta_-(\bar{G}_1^2)\theta_+(\bar{G}_2^2)\Big) \, [A_0(g_1), A_0(\delta \star G_2)]$$

- $g_1 \bowtie g_2 \quad \Rightarrow \quad [A_T(g_1), A_T(g_2)] = 0$
- $g_1 \perp g_2$ then

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- Key observation: Roberts has shown that

$$[A_0(g_1), A_0(\delta \star G_2)] = c \cdot \mathbb{1} \quad , \quad c \neq 0$$

for a particular class of divergence-free 1-forms $g_1, g_2 \in \mathcal{C}$ whose supports are spacelike separated and (nontrivially) linked together.

- A suitable modification $g_1', and g_2'$ leads

$$[A_T(g_1'), A_T(g_2')] = [A_0(g_1), A_0(\delta * G_2)] = c\mathbb{1}.$$

and we have central elements.

- If g has an (infinite) countable connected components $\{g_k\}$, we have

$$G^\sharp = \left(\sum_{k=1}^{\infty} \sharp G_k \right) \in \mathcal{D}_2 \quad , \quad \delta G_k = g_k \quad , \quad \sharp := \begin{cases} id \, , & \bar{G}^2 > 0 \\ \star \, , & G^2 < 0 \, . \end{cases}$$

and note that

$$g_1 \bowtie g_2 \quad \Rightarrow \quad (G_1 + G_2)^\sharp = G_1^\sharp + G_2^\sharp$$

Setting

$$A_T(g) := A_0(\delta G^\sharp) \; , \qquad \forall g \in \mathcal{D}_2$$

A_T is spacelike linear but not linear.

Thm. Let

$$\omega_T(U(a,g)) := \omega_0(e^{iaA_T(g)}) = \omega_0(e^{iaA_0(\delta G^\sharp)}) \; , \qquad a \in \mathbb{R}, g \in \mathcal{C}_1$$

is a regular vacuum state for the algebra $\mathfrak{U}$ and there are spacelike separated 1-forms g_1, g_2 whose central theoretic commutator does not vanish in the representation induced by ω_T. Thus topological charges appear in this representation.

Quantum Currents

Let J be a causal, covariant conserved current

$$\delta J(g) = J(dg) = 0 \ , \qquad g \in \mathcal{D}_0$$

which is a Wightman field of some Hilbert space H_J and vacuum vector Ω_J (for instance. the conserved current associated with the free Dirac field.)

Def. The intrinsic vector potential A_J is defined as follows: for any $f \in \mathcal{C}_1$ with **connected support** we let

$$A_J(f) := \begin{cases} J(f^\circ) \ , & \delta df^\circ = f \\ 0 \ , & \mathit{otherwise} \end{cases}$$

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Well posedness: by local Poincaré Lemma and conservation law of j if $\tilde{f} \in \mathcal{D}_1$ with $\delta d\tilde{f} = f$ then

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Well posedness implies A_J is covariant and causal. Moreover

$$\boxed{\delta dA_J(f) = A_J(\delta df) = J(f)}$$

J is the conserved current of A_j

- For general $f \in \mathcal{C}_1$, decompose into a sum $f = \sum_k f_k$ of functions having disjoint connected supports.
- Denotes by $\{f'_m\} \subset \{f_k\}$ such that $f'_m = \delta dh_m$ for some $h_m \in \mathcal{D}_1$.
- Define $f' = \sum_m f'_m \in \mathcal{C}_1$. It turns out that

$$f' = \delta dh' \ , \qquad h' \in \mathcal{D}_1$$

- Then the general definition the intrinsic vector potential is

$$A_J(f) := \begin{cases} J(h') \ , & h' \in \mathcal{D}_1, \ \delta dh' = f' \\ 0 \ , & f' = 0 \end{cases}$$

A_J is not linear but spacelike linear. It is covariant and causal:

$$f_1 \perp f_2 \ \Rightarrow \ [A_j(f_1), A_j(f_2)] = 0 \ \text{ No topological charges}$$

and

$$\delta dA_J(f) = J(f), \qquad f \in \mathcal{D}_1 \ .$$

Defining

$$\omega_J(U(a,f)) := (\Omega_j, e^{iaA_J(f)}\Omega_j) \ , \qquad f \in \mathcal{C}_1$$

we get a regular vacuum state s.t. the corresponding intrinsic vector potential A_J in t
GNS representation is spacelike linear and has conserved current J.

Topologial charges with quantum currents

Take ω_T and ω_J the states with topological charges and quantum current defined before. Defining

$$\omega_{TJ}(U(a,f)) := \omega_T(U(a,f)) \cdot \omega_J(U(a,f))$$

we get a regular vacuum state of $\mathfrak{U}$ with nontrivial topological charges and quantum conserved current J.