Feynman Propagators

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Joint work with Jan Dereziński

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Klein–Gordon operator (with external potential)

$$
\begin{aligned}
P &:= \Box_A + m^2 \\
&= |g|^{-\frac{1}{2}} (\mathrm{i}\partial_\mu - A_\mu) g^{\mu\nu} |g|^{\frac{1}{2}} (\mathrm{i}\partial_\nu - A_\mu) + m^2
\end{aligned}
$$

Conventions:

- g is a Lorentzian metric with signature $(-+++)$
- $|g| = |\det g_{\mu\nu}|$
- $m \geq 0$
- $A = \overline{A}$

Propagators

An operator G is a **bisolution** of P if it satisfies

$$P \circ G = 0 \quad \text{and} \quad G \circ P = 0.$$

An operator G is an **inverse** of P if it satisfies

$$P \circ G = \mathbb{1} \quad \text{and} \quad G \circ P = \mathbb{1}.$$

A (distinguished) bisolution or inverse will be called **propagator**.

Types of propagators

	classical	non-classical / quantum
inverse	G^{+}: forward propagator G^{-}: backward propagator	G^{F}: Feynman propagator $G^{\mathrm{F}} = G^{(+)} + G^{-} = G^{(-)} + G^{+}$
bisolution	G^{PJ}: Pauli-Jordon propagator $G^{\mathrm{PJ}} = G^{+} - G^{-}$	$G^{(+)}$: pos. frequency bisolution $G^{(-)}$: neg. frequency bisolution

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Some recent work on Feynman propagators:

- Gell-Redman, Haber, Vasy: Commun. Math. Phys. 342, 333–384 (2016)
- Gérard, Wrochna: arXiv:1609.00192 [math-ph]

I

Evolution and Propagators

Setting

- $g = -\beta\,\mathrm{d}t \otimes \mathrm{d}t + g_\Sigma$ on $M = \mathbb{R} \times \Sigma$
- $\{g_\Sigma(t)\}_t$ are quasi-isometric Riemannian metrics on Σ
- $A(t) = \big(V(t), \vec{A}(t)\big)$ with V small compared to m^2

For this presentation:

- $\beta = 1$
- $-\Delta_{\vec{A}} + m^2$ is essentially self-adjoint on $C_c^\infty(\Sigma)$ wrt. $L^2(\Sigma, \mathrm{d}g_\Sigma(t))$

Klein–Gordon operators:

$$
\begin{aligned}
P &= |g|^{-\frac{1}{2}}(\mathrm{i}\partial_\mu - A_\mu)g^{\mu\nu}|g|^{\frac{1}{2}}(\mathrm{i}\partial_\nu - A_\mu) + m^2 \\
&= -|g_\Sigma|^{-\frac{1}{2}}(\mathrm{i}\partial_t - V)|g_\Sigma|^{\frac{1}{2}}(\mathrm{i}\partial_t - V) - \Delta_{\vec{A}} + m^2
\end{aligned}
$$

First order equation

Rewrite the Klein–Gordon equation as a first order equation:

$$P_1(t) := -\mathrm{i}\partial_t + B(t), \quad B(t) := \begin{pmatrix} W(t) & \mathbb{1} \\ L(t) & \overline{W}(t) \end{pmatrix}$$

Coefficients:

- $L := -|g|^{\frac{1}{4}} \Delta_{\vec{A}} |g|^{-\frac{1}{4}} + m^2$
- $W := V + \frac{\mathrm{i}}{4} |g|^{-1} (\partial_t |g|)$

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Proposition

If E is a bisolution or inverse for P_1, then

$$G = |g|^{-\frac{1}{4}} E_{12} |g|^{\frac{1}{4}}$$

is a bisolution or inverse for P.

Energy space

Let $u, v \in C_c^\infty(\Sigma; \Omega^{\frac{1}{2}}) \oplus C_c^\infty(\Sigma; \Omega^{\frac{1}{2}})$. Notation: $\Omega^{\frac{1}{2}}$ is the half-density bundle

Energy products:

$$(u \,|\, v)_{\mathrm{en},t} := (u \,|\, H(t)v), \qquad H(t) := \begin{pmatrix} L(t) & \overline{W}(t) \\ W(t) & \mathbb{1} \end{pmatrix}$$

Charge form:

$$(u \,|\, Qv) := (u_1 \,|\, v_2) + (u_2 \,|\, v_1), \qquad Q := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

Energy spaces:

$$\mathscr{H}_{\mathrm{en},t} := \text{Hilbert space with energy product } (\,\cdot\,|\,\cdot\,)_{\mathrm{en},t}$$

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Proposition

$B(t)$ is self-adjoint with respect to $\mathscr{H}_{\mathrm{en},t}$.

[Hermiticity is quickly checked: $H(t)B(t) = H(t)QH(t) = B(t)^* H(t)$]

Evolution

Using known results on evolution equations, we find

Theorem

Suppose that g, A, m satisfy Lipschitz continuity conditions (in time) and become asymptotically static.

Then there exists a unique family of operators $\{U(t,s)\}_{t,s\in\mathbb{R}}$ such that

1. $U(t,t) = \mathbb{1}$
2. $U(t,r)U(r,s) = U(t,s)$
3. $\||B(t)|^{\alpha}U(t,s)|B(t)|^{-\alpha}\|_{\mathrm{en},t} \leq C_t$ for $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$
4. $\mathrm{i}\partial_t U(t,s)u = \quad B(t)U(t,s)u$ for $u \in \mathscr{H}_{\mathrm{en},t}$
5. $\mathrm{i}\partial_s U(t,s)u = -U(t,s)B(s)u$ for $u \in \mathscr{H}_{\mathrm{en},t}$

Classical propagators

Using the evolution $U(t,s)$, the kernels of the classical propagators for E are given by their kernels:

$$
\begin{aligned}
E^{\mathrm{PJ}}(t,s) &:= U(t,s) \\
E^{+}(t,s) &:= \theta(t-s)U(t,s) \\
E^{-}(t,s) &:= -\theta(s-t)U(t,s)
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Proposition

For $s > \frac{1}{2}$, the propagators G^{PJ}, $G^{\pm}$ defined by

$$(G^{\bullet}f)(t) = \int_{\mathbb{R}} |g(t)|^{-\frac{1}{4}} E^{\bullet}(t,s)_{12} |g(s)|^{\frac{1}{4}} f(s)\,\mathrm{d}s$$

are bounded from $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$. Note: $\langle t\rangle = (1+t^2)^{\frac{1}{2}}$

Spectral projections

Provided $|W|^2$ is sufficiently small compared to m^2, $B(t)$ will have a spectral gap around 0.

Proposition

For each time t there exist **spectral projections** $\Pi_t^{(\pm)}$ such that

1. $\Pi_t^{(+)} + \Pi_t^{(-)} = \mathbb{1}$
2. $\Pi_t^{(\pm)} B(t) = B(t) \Pi_t^{(\pm)}$
3. $\sigma\left(\Pi_t^{(\pm)} B(t)\right) \subset \mathbb{R}_\pm$

They are positive / negative with respect to the charge form:

$$\pm(u \,|\, Q \Pi_t^{(\pm)} u) \geq 0$$

Non-classical propagators

Now, using the spectral projections $\Pi_t^{(\pm)}$, the non-classical propagators can be defined:

$$
\begin{aligned}
E_r^{(\pm)}(t,s) &:= \pm U(t,r)\Pi_r^{(\pm)}U(r,s) \\
E_r^{\mathrm{F}}(t,s) &:= \theta(t-s)E_r^{(+)}(t,s) + \theta(s-t)E_r^{(-)}(t,s) \\
&= E_r^{(+)}(t,s) + E^{-}(t,s)
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For $s > \frac{1}{2}$, the propagators $G_t^{(\pm)}$, G_t^{F} defined by

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are bounded from $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$.

In and out bisolutions

The non-classical propagators $G_t^{(\pm)}$, G_t^{F} depend on the choice of the time variable and are thus highly non-unique. Moreover, typically they do not yield Hadamard states.

In and out bisolutions

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Better choice: The 'in' and 'out' bisolutions

$$E_{-\infty}^{(\pm)}(t,s) := \lim_{s\to-\infty} \pm U(t,r)\Pi_r^{(\pm)}U(r,s)$$
$$E_{+\infty}^{(\pm)}(t,s) := \lim_{s\to+\infty} \pm U(t,r)\Pi_r^{(\pm)}U(r,s)$$

and the corresponding Feynman propagators

$$E_{\pm\infty}^{\mathrm{F}}(t,s) := E_{\pm\infty}^{(+)}(t,s) + E^{-}(t,s)$$

Under appropriate conditions the resulting state is Hadamard.

See also: Gérard, Wrochna: arXiv: 1609.00190 [math-ph]

II

Distinguished Propagators

Self-adjointness (ultrastatic/static)

Clearly P is Hermitian with respect to the Hilbert space $L^2(M)$.

An application of Nelson's commutator theorem shows:

Theorem

If P is static, it is essentially self-adjoint on $C_c^\infty(M)$.

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Easy case: Suppose that $V = 0$. Then

$$P = \partial_t^2 \otimes \mathbb{1} + \mathbb{1} \otimes (-\Delta_{\vec{A}} + m^2)$$

in the sense of $L^2(M) = L^2(\mathbb{R}) \otimes L^2(\Sigma)$ and self-adjointness is almost automatic.

Resolvent limit (ultrastatic/static)

Theorem

If P is static,

$$G^{\mathrm{F}} = \operatorname*{s-lim}_{\varepsilon \searrow 0} (P - \mathrm{i}\varepsilon)^{-1}$$

in the sense of operators from $\langle t \rangle^{-s} L^2(M)$ to $\langle t \rangle^{s} L^2(M)$ for $s > \frac{1}{2}$.

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Easy case: Suppose that $V = 0$. Then

$$\operatorname*{s-lim}_{\varepsilon \searrow 0} \langle t \rangle^{-s} (\partial_t^2 + \lambda \pm \mathrm{i}\varepsilon)^{-1} \langle t \rangle^{-s}, \quad s > \frac{1}{2},$$

is bounded in $L^2(\mathbb{R})$ for $\lambda \in \mathbb{R} \setminus \{0\}$. Then use that, for $m^2 \geq c$,

$$(P - \mathrm{i}\varepsilon)^{-1} = \int_{-\infty}^{-c} (\partial_t^2 - \lambda - \mathrm{i}\varepsilon)^{-1} \otimes \mathrm{d}E(\Delta_{\vec{A}} - m^2; \lambda)$$

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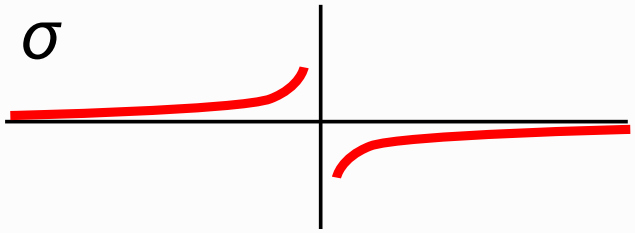
General case: Define

$$B_{\mathrm{i}\varepsilon} := \begin{pmatrix} V & \mathbb{1} \\ L - \mathrm{i}\varepsilon & V \end{pmatrix}.$$



$B_{\mathrm{i}\varepsilon}$ is a so-called bisectorial operator. We can construct projections $\Pi^{\pm}_{\mathrm{i}\varepsilon}$ and an associated propagator

$$E^{\mathrm{F}}_{\mathrm{i}\varepsilon}(t,s) = \theta(t-s)\mathrm{e}^{-\mathrm{i}(t-s)B_{\mathrm{i}\varepsilon}}\Pi^{(+)}_{\mathrm{i}\varepsilon} - \theta(s-t)\mathrm{e}^{-\mathrm{i}(t-s)B_{\mathrm{i}\varepsilon}}\Pi^{(-)}_{\mathrm{i}\varepsilon}.$$

Wick rotation (ultrastatic/static)

For $0 \leq \theta \leq \pi$ we introduce

$$g_\vartheta := -\mathrm{e}^{-2\mathrm{i}\vartheta}\mathrm{d}t \otimes \mathrm{d}t + g_\Sigma, \qquad V_\vartheta := \mathrm{e}^{-\mathrm{i}\vartheta} V$$

and the corresponding Wick-rotated Klein–Gordon operator

$$P_\vartheta := -\mathrm{e}^{2\mathrm{i}\vartheta}|g|^{-\frac{1}{2}}(\mathrm{i}\partial_t - V_\vartheta)|g|^{\frac{1}{2}}(\mathrm{i}\partial_t - V_\vartheta) - \Delta_{\vec{A}} + m^2$$

This has a Wick-rotated Feynman propagator

$$E^{\mathrm{F}}_\vartheta(t,s) = \theta(t-s)\mathrm{e}^{-\mathrm{i}(t-s)B_\vartheta}\Pi^{(+)} - \theta(s-t)\mathrm{e}^{-\mathrm{i}(t-s)B_\vartheta}\Pi^{(-)}$$

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Theorem

If P is static,

$$G^{\mathrm{F}} = \operatorname*{s-lim}_{\theta \searrow 0} P_\theta^{-1}$$

in the sense of operators from $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$ for $s > \frac{1}{2}$.

Non-static spacetimes

Aim: Generalize the previous results to non-static spacetimes.

Some problems:

- (Unique) self-adjoint extension of P?
- Existence of resolvent limit?
- Which projection $\Pi^{(\pm)}$?

Asymptotic complementarity

Transport the spectral projectors using $U(t,s)$:

$$\Pi_s^{(\pm)}(t) := U(t,s)\Pi_s^{(\pm)}U(s,t)$$

$$\Pi_{\pm\infty}^{(\pm)}(t) := \lim_{s\to\pm\infty} \Pi_s^{(\pm)}(t)$$

Proposition

Suppose that $\operatorname{Ran}\Pi_{-\infty}^{(+)}(t)$ and $\operatorname{Ran}\Pi_{+\infty}^{(-)}(t)$ are complementary. Then

$$R(t) := \mathbb{1} + \left(\Pi_{-\infty}^{(+)}(t) - \Pi_{+\infty}^{(+)}(t)\right)^2$$

is invertible and

$$\Pi^{(\pm)}(t) := \Pi_{-\infty}^{(\pm)}(t)R(t)^{-1}\Pi_{+\infty}^{(\pm)}(t)$$

are complementary projections for each t.

‘Canonical’ propagators

The canonical non-classical propagators are defined as

$$E^{(\pm)}(t,s) := \pm U(t,s)\Pi^{(\pm)}(s)$$
$$E^{\mathrm{F}}(t,s) := \theta(t-s)E^{(+)}(t,s) + \theta(s-t)E^{(-)}(t,s)$$

Denote by G^{F} the corresponding Feynman proapgator for P.

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Denote by G^{F} the corresponding Feynman proapgator for P.

Conjecture

If asymptotic complementarity holds, P is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(M)$ and

$$G^{\mathrm{F}} = \operatorname*{s-lim}_{\varepsilon \searrow 0} (P - \mathrm{i}\varepsilon)^{-1}$$

in the sense of operators from $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$ for $s > \frac{1}{2}$.

First results

Theorem

Suppose that P is non-static only in a finite time-interval and that asymptotic complementarity holds.
Then there exists a pseudo-resolvent $G^{\mathrm{F}}_{\pm \mathrm{i}\varepsilon}$, $\varepsilon > 0$, such that

$$G^{\mathrm{F}} = \operatorname*{s-lim}_{\varepsilon \searrow 0} G^{\mathrm{F}}_{\pm \mathrm{i}\varepsilon}$$

in the sense of operators from $\langle t \rangle^{-s} L^2(M)$ to $\langle t \rangle^{s} L^2(M)$ for $s > \frac{1}{2}$.
The pseudo-resolvent $G^{\mathrm{F}}_{\pm \mathrm{i}\varepsilon}$ defines a distinguished self-adjoint extension of P.

Literature

- **Feynman Propagators on Static Spacetimes**
  J. Dereziński and D. Siemssen: arXiv:1608.06441 [math-ph]
- Two more articles in preparation

Related:

- **Covariant "In–Out" Formalism for Creation by External Fields**
  H. Rumpf and H. K. Urbantke: Annals of Physics 114, 332–355 (1978)
- **The Feynman Propagator on Perturbations of Minkowski Space**
  J. Gell-Redman, N. Haber and A. Vasy: CMP 342, 333–384 (2016)
- **The massive Feynman propagator on asymptotically Minkowski spacetimes**
  C. Gérard and M. Wrochna: arXiv:1609.00192 [math-ph]
- **Hadamard property of the in and out states for Klein-Gordon fields on asymptotically static spacetimes**
  C. Gérard and M. Wrochna: arXiv:1609.00190 [math-ph]