

# Hadamard states from data at the de Sitter conformal boundary

*joint work w. András Vasy*

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# INTRODUCTION

Let  $(M, g)$  globally hyperbolic spacetime. **Quantum fields:**

$$(\square_g - m^2)\hat{\psi}(x) = 0, \quad [\hat{\psi}(x), \hat{\psi}(y)] = iG(x, y),$$

where  $G = P_+^{-1} - P_-^{-1}$  – difference of adv./ret. propagator.

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To construct and study  $\hat{\psi}(x)$  one needs:

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$$\text{Sol}(\square_g - m^2) = G\mathcal{C}_c^\infty(M).$$

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- ▶ Splitting of solution space into **particles / anti-particles**:

- ▶ Choice of quasi-free **state**  $\iff \Lambda^\pm : C_c^\infty(M) \rightarrow C^\infty(M)$  s.t.

$$\Lambda^+ - \Lambda^- = iG, \quad \Lambda^\pm \geq 0$$

- ▶ **Hadamard states**:  $\text{WF}'(\Lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm$

# INTRODUCTION

*Distinguished* Hadamard states possible if asymptotic symmetries.

E.g., on **asymptotically de Sitter** spacetimes one expects  
‘asymptotically Bunch-Davies’ Hadamard state.

## ❖ Conformal scattering

- ✓ conformal wave eq., asymptotically flat [Moretti ’08]
- ✓ generalization to cosmological spacetimes, Schwarzschild [Dappiaggi, Moretti, Pinamonti ’09-’11], Schwarzschild-de Sitter [Brum, Joras ’14]
- ❖ proof of purity of states very recent [Gérard, W. ’16]

## ❖ Standard scattering

- ✓ asymptotically static spacetimes, massive case [Gérard, W. ’16]
- ✓ stability of Hadamard condition under adiabatic limits [Dappiaggi, Drago; Drago, Gérard, ’16]

## ❖ Geometric scattering

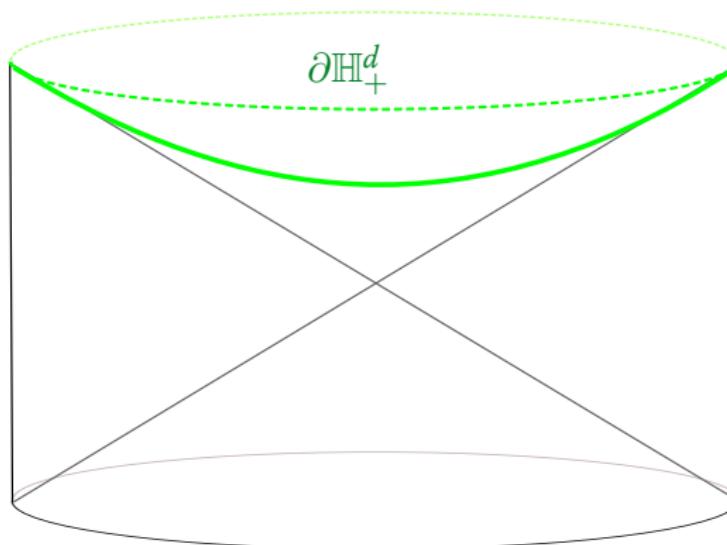
- ✓ asymptotically Minkowski, massless [Vasy, W. ’16]
- ✓ asymptotically de Sitter (global chart), massive [Vasy, W. ’16]

❖ method: extend  $\text{Sol}(\square_g - m^2)$  across the conformal horizon!  
(this talk)

# HYPERBOLIC SPACE

In Minkowski space  $\mathbb{R}^{1,d}$ ,  $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$ ,

$$\mathbb{H}_+^d = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = 1, z_0 > 0\}$$



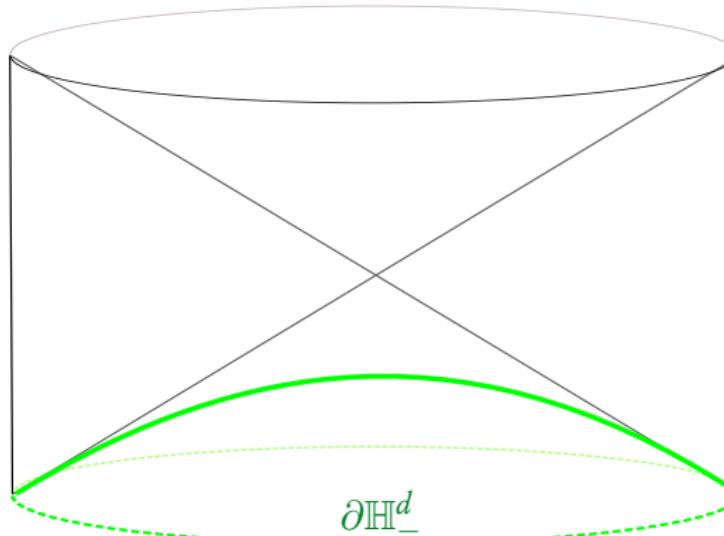
Plane waves  $\phi_{\mathbb{H}_\pm^d, \xi} = |\xi \cdot z|^{i\nu - (d-1)/2} \upharpoonright_{\mathbb{H}_\pm^d}$  (for  $-\Delta_{\mathbb{H}_\pm^d} + \sigma^2 + (d-1)^2/4$ )

► **Spectral projection:**  $E(x, y) \propto \int \overline{\phi_{\mathbb{H}_\pm^d, \xi}(x)} \phi_{\mathbb{H}_\pm^d, \xi}(y) d\mu(\xi)$

# HYPERBOLIC SPACE

In Minkowski space  $\mathbb{R}^{1,d}$ ,  $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$ ,

$$\mathbb{H}_-^d = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = 1, z_0 < 0\}$$

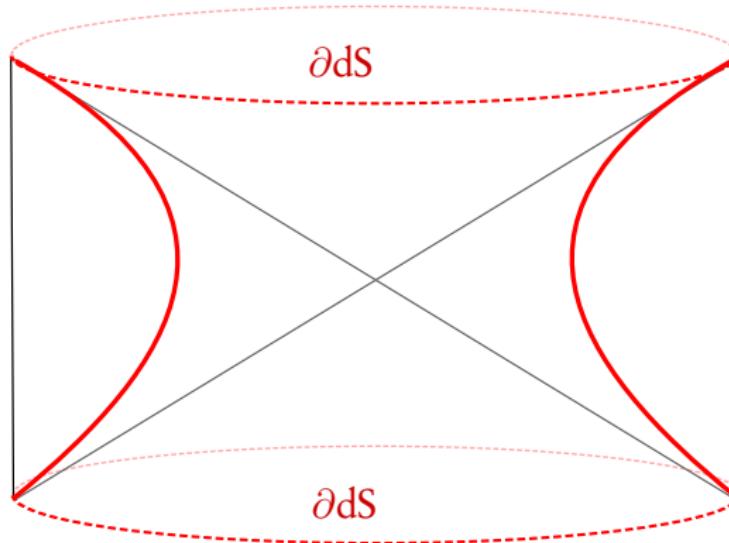


Plane waves  $\phi_{\mathbb{H}_{\pm}^d, \xi} = |\xi \cdot z|^{i\nu - (d-1)/2} \upharpoonright_{\mathbb{H}_{\pm}^d}$  (for  $-\Delta_{\mathbb{H}_{\pm}^d} + \sigma^2 + (d-1)^2/4$ )

► **Spectral projection:**  $E(x, y) \propto \int \overline{\phi_{\mathbb{H}_{\pm}^d, \xi}(x)} \phi_{\mathbb{H}_{\pm}^d, \xi}(y) d\mu(\xi)$

# DE SITTER SPACE

In Minkowski space  $\mathbb{R}^{1,d}$ ,  $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$ ,  
 $dS = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = -1\}$



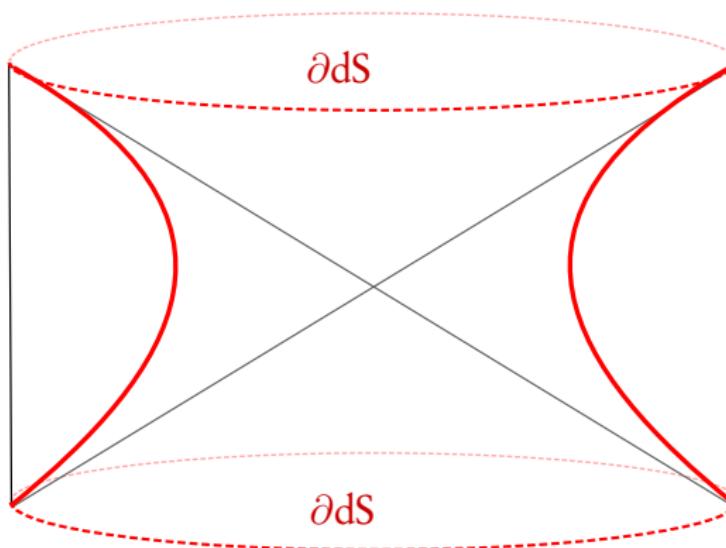
- ▶ Plane waves:  $\phi_{dS,\xi}^{\pm} = (\xi \cdot z \pm i0)^{i\nu-(d-1)/2} \upharpoonright_{dS}$
- ▶ **Bunch-Davies two-point functions:**  

$$\Lambda^{\pm}(x, y) \propto \int \overline{\phi_{dS,\xi}^{\pm}(x)} \phi_{dS,\xi}^{\pm}(y) d\mu(\xi)$$

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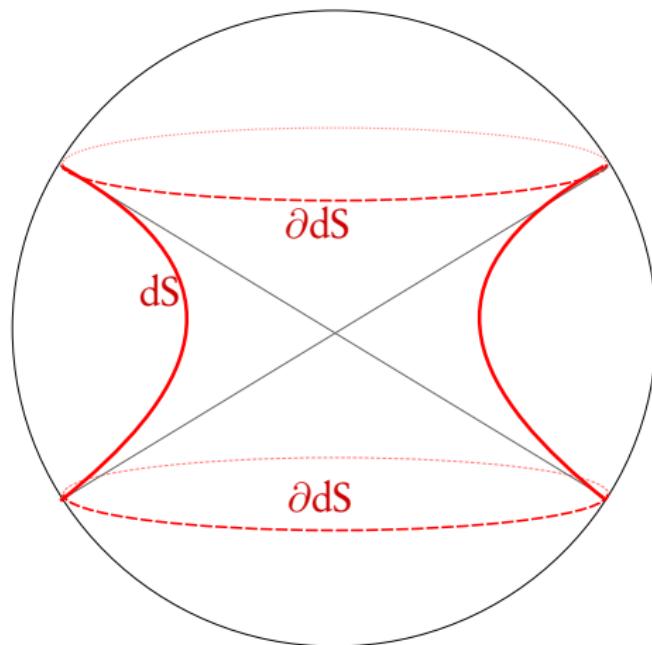
$$dS = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = -1\}$$



- ▶ Plane waves:  $\phi_{dS,\xi}^{\pm} = (\xi \cdot z \pm i0)^{i\nu-(d-1)/2}|_{dS}$
- ?
- How to distinguish  $\phi_{dS,\xi}^+$  vs.  $\phi_{dS,\xi}^-$  asymptotically?

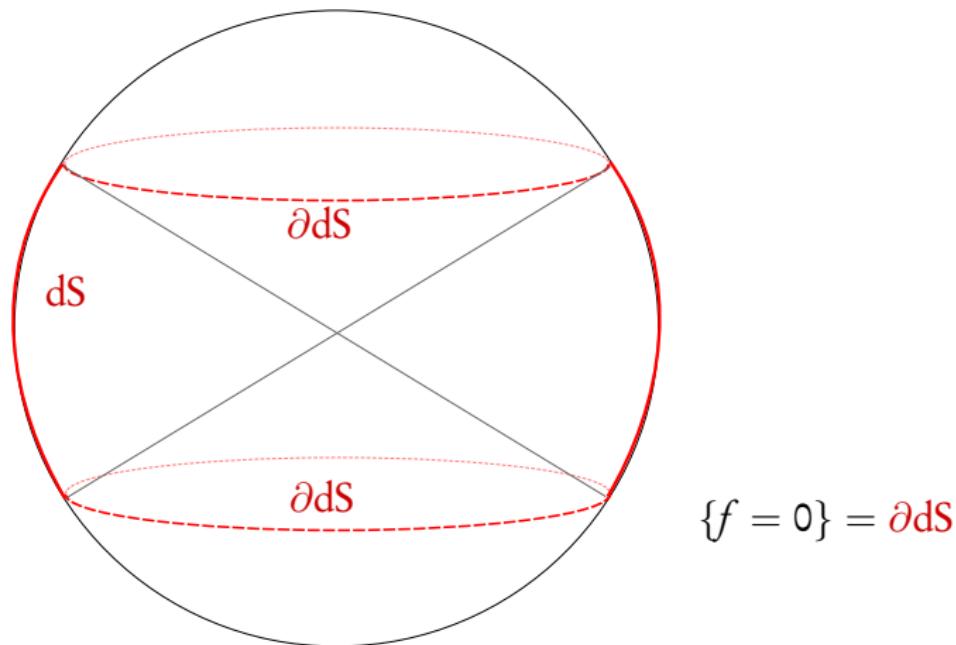
# EXTENDED DE SITTER SPACE

Now in *radially compactified* Minkowski space:



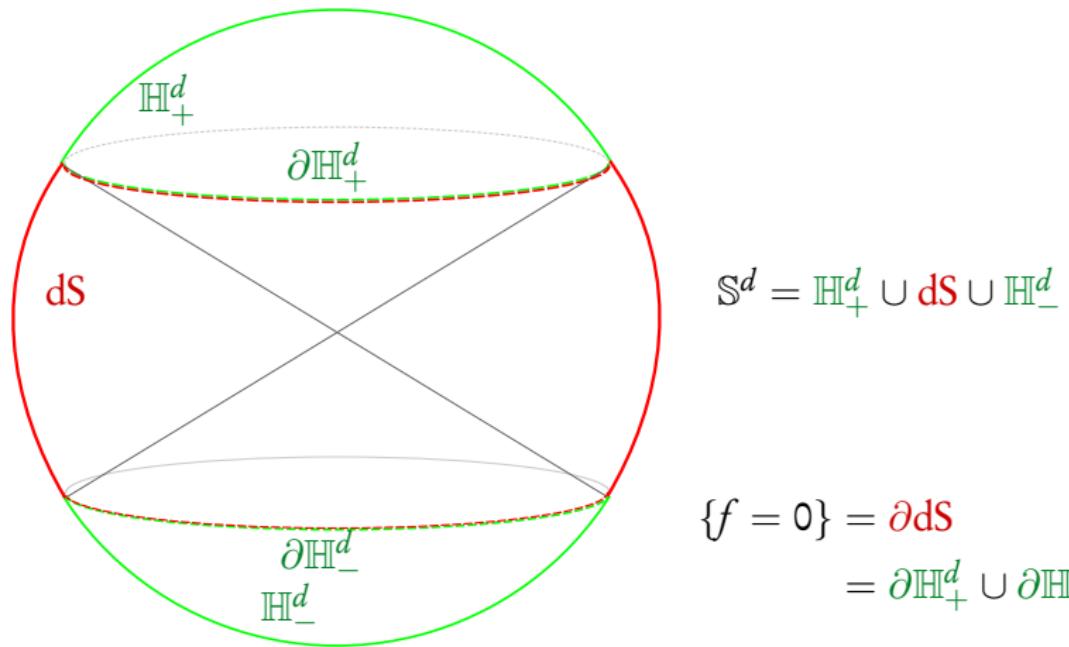
# EXTENDED DE SITTER SPACE

- ▶ Identification  $dS \subset \mathbb{S}^d$ : coord.  $y_{\mathbb{S}} = f y_{dS}, f = \left| \frac{z_0^2 - (z_1^2 + \dots + z_d^2)}{z_0^2 + z_1^2 + \dots + z_d^2} \right|^{\frac{1}{2}}$ .



# EXTENDED DE SITTER SPACE

- ▶ Identification  $\text{dS} \subset \mathbb{S}^d$ : coord.  $y_{\mathbb{S}} = f y_{\text{dS}}, f = \left| \frac{z_0^2 - (z_1^2 + \dots + z_d^2)}{z_0^2 + z_1^2 + \dots + z_d^2} \right|^{\frac{1}{2}}$ .
- ▶ Identification  $\mathbb{H}_{\pm}^d \subset \mathbb{S}^d$ : coord.  $y_{\mathbb{S}} = f y_{\mathbb{H}}$



# EXTENDED DE SITTER SPACE

- ▶ Identification  $\text{dS} \subset \mathbb{S}^d$ : coord.  $y_{\mathbb{S}} = fy_{\text{dS}}, f = \left| \frac{z_0^2 - (z_1^2 + \dots + z_d^2)}{z_0^2 + z_1^2 + \dots + z_d^2} \right|^{\frac{1}{2}}$ .
- ▶ Identification  $\mathbb{H}_{\pm}^d \subset \mathbb{S}^d$ : coord.  $y_{\mathbb{S}} = fy_{\mathbb{H}}$
- ▶  $\mathbb{S}^d = \mathbb{H}_+^d \cup \text{dS} \cup \mathbb{H}_-^d$ ;  $\{f = 0\} = \partial \text{dS} = \partial \mathbb{H}_+^d \cup \partial \mathbb{H}_-^d$
- ▶ **Plane waves**

$$\phi_{\xi}^{\pm} = (\xi \cdot z \pm i0)^{i\nu - (d-1)/2} \upharpoonright_{\mathbb{S}^d} = \begin{cases} f^{i\nu - (d-1)/2} \phi_{\text{dS}, \xi}^{\pm} & \text{on dS} \\ f^{i\nu - (d-1)/2} \phi_{\mathbb{H}, \xi}^{\pm} & \text{on } \mathbb{H}_+^d \end{cases}$$

Setting  $v := -f^2$  on  $\text{dS}$ ,  $v := f^2$  on  $\mathbb{H}_{\pm}^d$ ,  $\phi_{\xi}^{\pm} \sim (v \pm i0)^{-i\nu}!$

- ▶ These are solutions of  $P\phi = 0$ ,  $P = 4v\partial_v^2 + \dots \in \text{Diff}(\mathbb{S}^d)$

$$P = \begin{cases} f^{i\nu - (d-1)/2 - 2} (\square_{\text{dS}} - (\frac{d-1}{2})^2 - \nu^2) f^{-i\nu + (d-1)/2} & \text{on dS,} \\ f^{i\nu - (d-1)/2 - 2} (-\Delta_{\mathbb{H}_{\pm}} + (\frac{d-1}{2})^2 + \nu^2) f^{-i\nu + (d-1)/2} & \text{on } \mathbb{H}_{\pm}^d, \end{cases},$$

# EXTENDED ASYMPTOTICALLY DE SITTER SPACETIMES

- ▶ Same structure extending even asymptotically de Sitter spacetimes  $(M, g)$ :
  - ▶  $g = df^2 - h(f^2, y, dy)$  in  $v < 0$  ( $f^2$  times as. dS metric),  
 $g = df^2 + h_{\pm}(f^2, y, dy)$  in  $v > 0$  ( $f^2$  times as.  $\mathbb{H}_{\pm}^d$  metric)  
 (close to conformal horizon  $\{v = 0\} = \{f = 0\} =: S_+ \cup S_-$ ).
  - ▶ Non-trapping assumption.
- ▶ The Vasy operator

$$P = \begin{cases} f^{i\nu - (d-1)/2 - 2} (\square_{f^2} g - (\frac{d-1}{2})^2 - \nu^2) f^{-i\nu + (d-1)/2} & \text{on } \{v < 0\}, \\ f^{i\nu - (d-1)/2 - 2} (-\Delta_{f^2} g + (\frac{d-1}{2})^2 + \nu^2) f^{-i\nu + (d-1)/2} & \text{on } \{v > 0\}, \end{cases}$$

- ▶ Solutions in  $\text{Sol}(P) := \{Pu = 0, \text{WF}(u) \subset N^*\{v = 0\}\}$  can be written as:
$$u = (v + i0)^{-i\nu} a^+ + (v - i0)^{-i\nu} a^- + a, \quad a^+, a^-, a \in \mathcal{C}^\infty(M).$$
- ▶  $P$  fits into *Fredholm framework* of Vasy

# INVERSES OF $P$

## The Vasy operator

$$P = \begin{cases} f^{i\nu-(d-1)/2-2} (\square_{f^2 g} - (\frac{d-1}{2})^2 - \nu^2) f^{-i\nu+(d-1)/2} & \text{on } \{v < 0\}, \\ f^{i\nu-(d-1)/2-2} (\Delta_{f^2 g} + (\frac{d-1}{2})^2 + \nu^2) f^{-i\nu+(d-1)/2} & \text{on } \{v > 0\}, \end{cases},$$

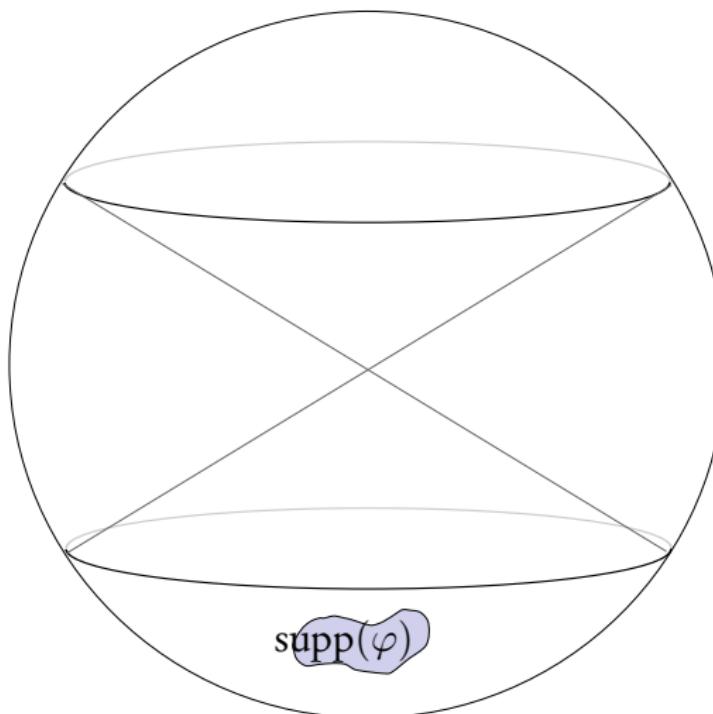
- Solutions in  $\text{Sol}(P) := \{Pu = 0, \text{WF}(u) \subset N^*\{v = 0\}\}$  can be written near  $S_+$  as:

$$u = (v + i0)^{-i\nu} a^+ + (v - i0)^{-i\nu} a^- + a, \quad a^+, a^-, a \in \mathcal{C}^\infty(M).$$

- $P$  fits into *Fredholm framework* of Vasy [Vasy '12-'16]  
 $\Rightarrow P$  has **inverses**  $P_\pm^{-1}$  (as meromorphic functions in  $\nu \in \mathbb{C}$ )
- $P_\pm^{-1}$  *conformally related* related to **ret./adv. propagators** and **meromorphic continuations** of resolvent

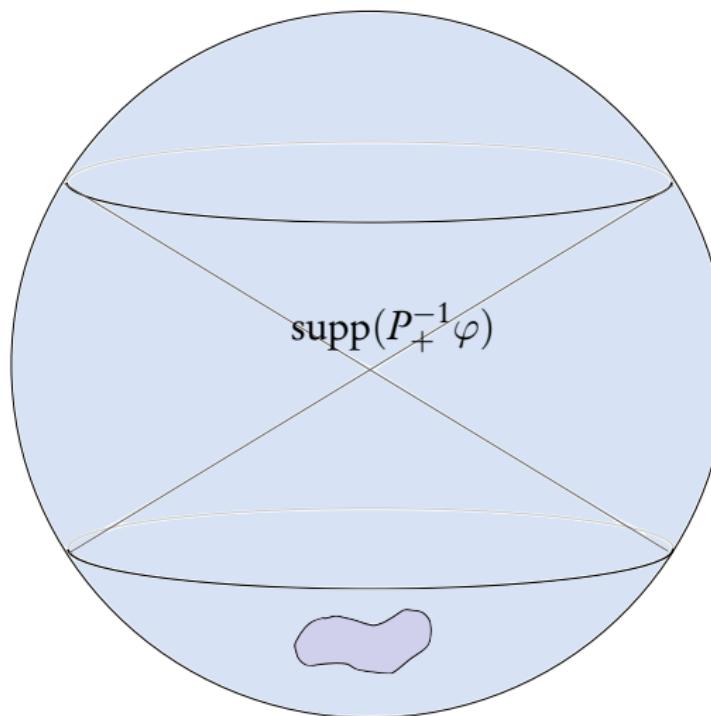
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- ▶ Support properties of  $P_+^{-1}$  [Baskin, Vasy, Wunsch '12]



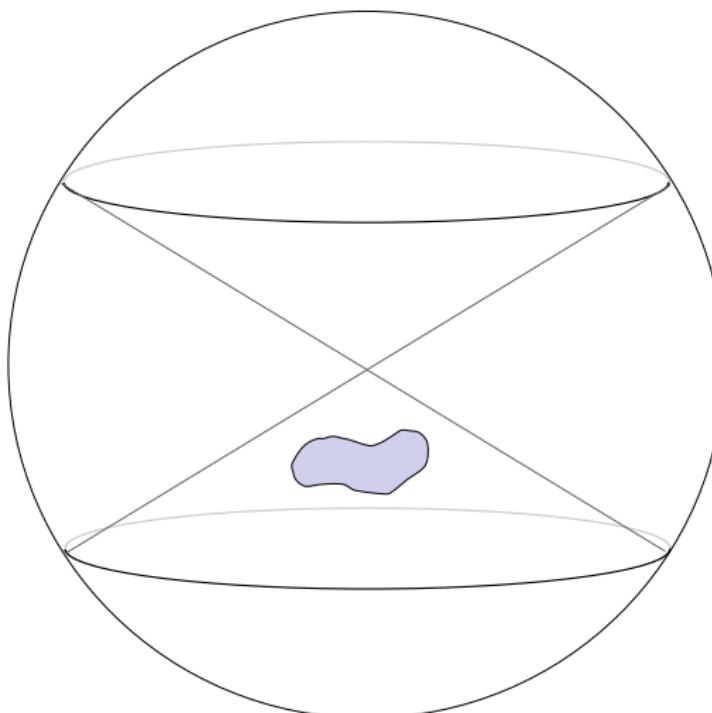
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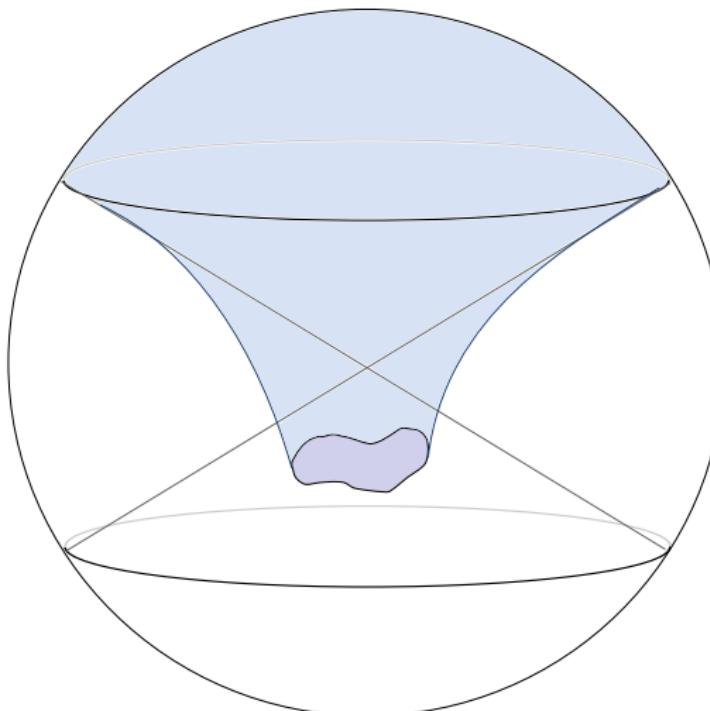
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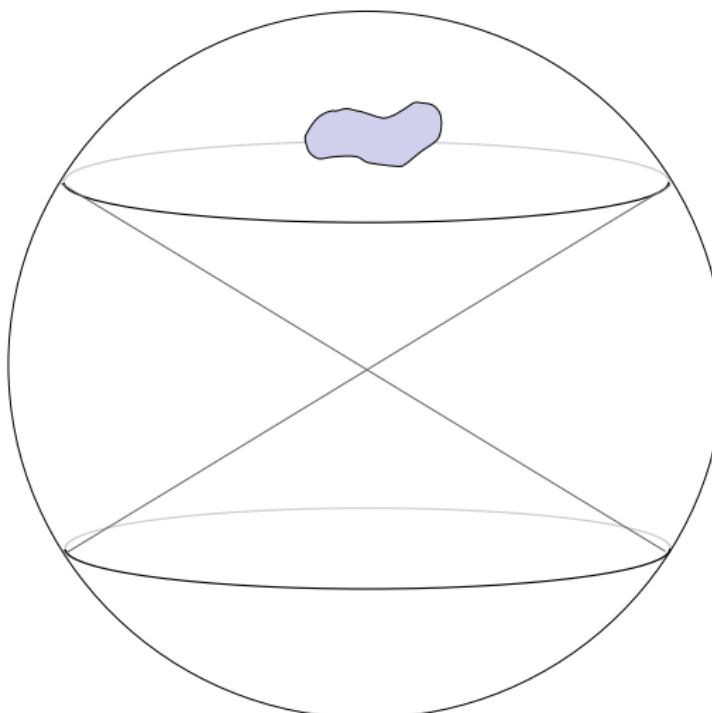
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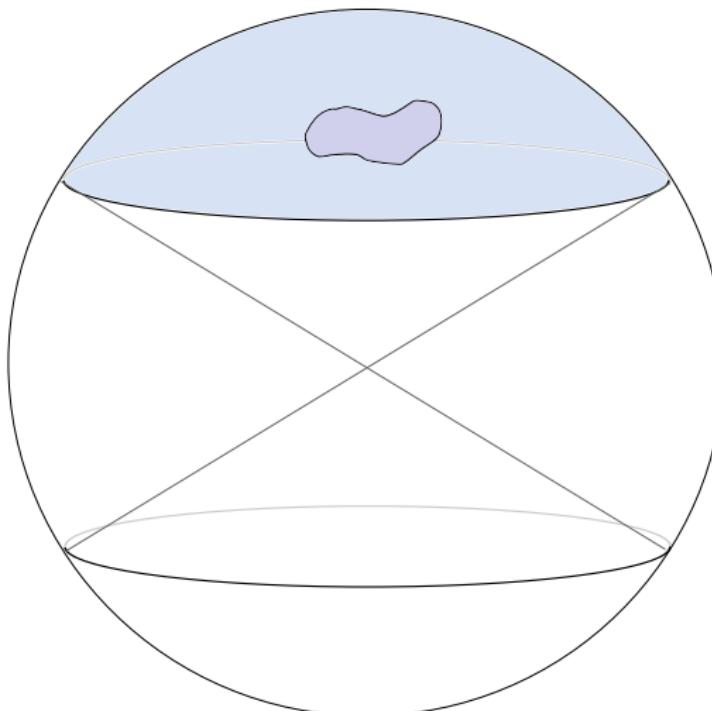
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- $P$  fits into *Fredholm framework* of Vasy  
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- $P_\pm^{-1}$  *conformally related* related to **ret./adv. propagators** and **meromorphic continuations of resolvent**
- **Asymptotic data** of  $u$ :  $\varrho_{S_\pm} u := (a^+, a^-)|_{S_\pm}$   
 Poisson/Møller operator  $\varrho_{S_\pm}^{-1}$  constructed using  $P_\pm^{-1}$

# MAIN RESULT I

Set  $G := P_+^{-1} - P_-^{-1}$ .

Theorem ([Vasy,W.])

For  $\nu$  not a pole of  $P_{\pm}(\nu)^{-1}$ , *isomorphisms*:

$$\frac{\mathcal{C}^\infty(M)}{P\mathcal{C}^\infty(M)} \xrightarrow{G} \text{Sol}(P) \xrightarrow{\restriction_{\{\nu<0\}} \circ f^{i\nu+(d-1)/2}} \text{Sol}(\square - m^2).$$

- ⇒ solutions of  $(\square - m^2)u = 0$  on **asymptotically dS** region have canonical *weighted extensions* to  $M$ .
- ⇒ same conclusion for *quantum fields*.

## MAIN RESULT II

Set  $G := P_+^{-1} - P_-^{-1}$ . Recall  $\varrho_{S_+} u$  — asymptotic data of  $u$  at  $S_+$ .

Theorem ([Vasy,W.])

*Hadamard two-point functions induced from data at conformal boundary  $S_+$ :*

$$\Lambda^+ \propto G^* \varrho_{S_+}^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varrho_{S_+} G, \quad \Lambda^- \propto G^* \varrho_{S_+}^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \varrho_{S_+} G,$$

in particular  $\Lambda^+ - \Lambda^- = iG$

- ▶ gives **Bunch-Davies two-point functions** in exact dS case
- ▶ extends **spectral projection** from **as. hyperbolic** region
- ▶ Hadamard condition by *propagation of singularities* (inc. *radial sets* version)
- ▶  $\Lambda^+ - \Lambda^- = iG$  from “pairing formula”

# MORE ISOMORPHISMS

Let  $H_+$  be one of the two as. hyperbolic regions.

Theorem ([Vasy,W.])

For  $\nu$  not a pole of  $P_\pm(\nu)^{-1}$ , isomorphisms:

$$\left( \frac{\mathcal{C}^\infty(H_+)}{(-\Delta + m^2)\mathcal{C}^\infty(H_+)} \right)^{\oplus 2} \longrightarrow \left( \text{Sol}(-\Delta + m^2) \right)^{\oplus 2} \longrightarrow \text{Sol}(P).$$

$\Rightarrow$  (Linear) fields on as. dS  $\longleftrightarrow$  pair of fields on as.  $\mathbb{H}_+^d$ .

# SUMMARY & OUTLOOK

The new:

- ✓ Canonical Hadamard states from asymptotic  $S_+$  or  $S_-$  data
  - 💡 Closely related to Calderón projectors
  - ✖ Similar results for wave equation on asymptotically Minkowski spacetimes.
- ✓ Extension of *non-interacting* QFT across the conformal horizon

Some questions:

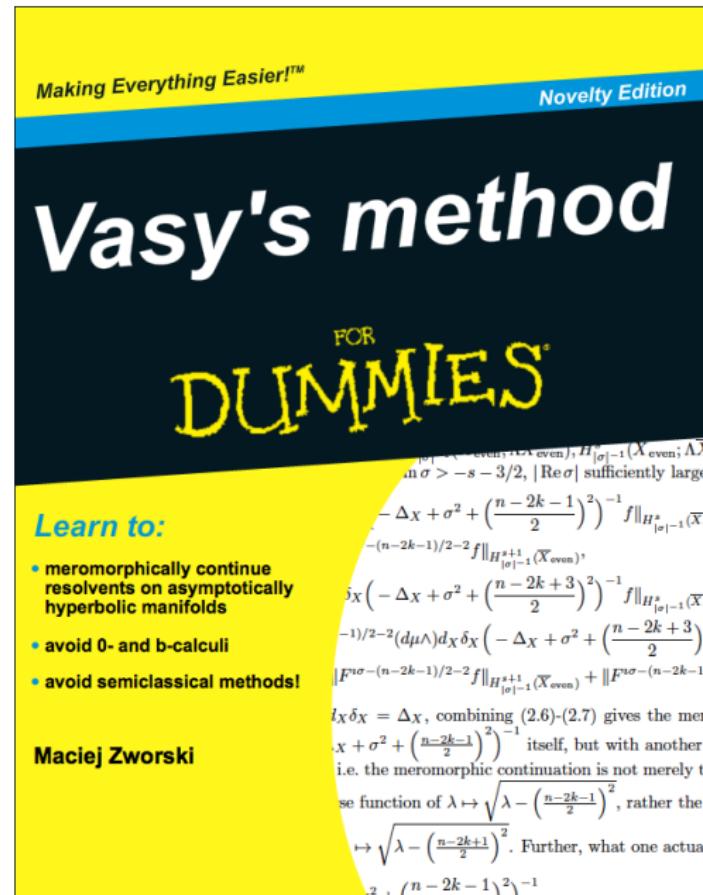
- ❓ Consequences for Strominger's dS/CFT correspondence?
- ❓ Extension across the boundary for *non-interacting (non-linear) theories*?

*Thank you for your attention!*

# ADVERTISEMENT

- Advantageous even purely from perspective of **as. hyperbolic spaces**, see [Vasy '13] and:

+ Conjectures on Vasy's operator by [Lebeau, Zworski '16]



# APPENDIX — RADIAL ESTIMATES

A toy example:

- ▶  $M = \mathbb{R}^2$ , coordinates  $(x, y) \in \mathbb{R}^2$ , dual coordinates  $(\xi, \eta)$
- ▶  $Pu = xu$  — multiplication operator
- ▶ characteristic set  $\mathcal{N} = \{x = 0\}$ , Hamiltonian v. field  $-\partial_\xi$
- ▶ **radial set**  $N^*\{x = 0\} = \{x = 0, \eta = 0\}$  with components

$$\mathcal{R}^+ \cup \mathcal{R}^- := \{x = 0, \eta = 0, \xi > 0\} \cup \{x = 0, \eta = 0, \xi < 0\}$$

- ▶ near  $\mathcal{R}^\pm$  change of coordinates on  $T^*M \setminus o$ ,  $\theta = \eta/\xi$ ,  $\tilde{\rho} = \pm\xi^{-1}$  gives Hamiltonian v. field proportional to

$$\tilde{\rho}\partial_{\tilde{\rho}} + \theta\partial_\theta + x\partial_x.$$

Bicharacteristics flow from **source at  $\mathcal{R}^-$**  to **sink at  $\mathcal{R}^+$** !

- ▶ Two inverses  $(x \pm i0)^{-1}$  correspond to **high regularity at  $\mathcal{R}^\mp$**  and **low regularity at  $\mathcal{R}^\pm$** .

# APPENDIX — RADIAL ESTIMATES

## Theorem

Let  $(M, g)$  be a Lorentzian scattering space. Let  $P$  be the rescaled wave operator, let us denote by  $\mathcal{R}_i$  any of the components of the radial sets, and let  $u \in H_b^{-\infty, l}(M)$ .

1. If  $m < \frac{1}{2} - l$  and  $m$  is nonincreasing along the bicharacteristic flow in the direction approaching  $\mathcal{R}_i$ , then

$$\mathrm{WF}_b^{m,l}(u) \cap \mathcal{R}_i = \emptyset \text{ if } \mathrm{WF}_b^{m-1,l}(Pu) \cap \mathcal{R}_i = \emptyset$$

and provided that  $(U \setminus \mathcal{R}_i) \cap \mathrm{WF}_b^{m,l}(u) = \emptyset$  for some neighborhood  $U \subset \Sigma \cap {}^bS^*M$  of  $\mathcal{R}_i$ .

2. If  $m_0 > \frac{1}{2} - l$ ,  $m \geq m_0$  and  $m$  is nonincreasing along the bicharacteristic flow in the direction going out from  $\mathcal{R}_i$  then

$$\mathrm{WF}_b^{m,l}(u) \cap \mathcal{R}_i = \emptyset \text{ if } (\mathrm{WF}_b^{m_0,l}(u) \cup \mathrm{WF}_b^{m-1,l}(Pu)) \cap \mathcal{R}_i = \emptyset.$$