Why don't we formulate quantum theories on real Hilbert spaces?

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Abstract In principle, the lattice of elementary propositions of a generic quantum system admits a representa- tion in real, complex or quaternionic Hilbert spaces as established by Solér's theorem (1995) closing a long standing problem that can be traced back to von Neumann's mathematical formulation of quantum mechanics. However up to now there are no examples of quantum systems described in real Hilbert spaces. We show that elementary relativistic systems (in Wigner's approach) cannot be described in real Hilbert spaces as a consequence of some peculiarity of continuous unitary projec- tive representations of $SL(2, \mathbb{C})$ related with the theory of polar decomposition of operators. Indeed such a "naive" attempt leads necessarily to an equivalent formulation on a complex Hilbert space. Although this conclusion seems to give a definitive answer to the real-quantum-mechanics issue, it lacks consistency since it does not derive from more general physical hypotheses as the complex one does. Trying a more solid approach we end up with three possibilities: an equivalent description in terms of a Wigner unitary representation in a real, complex or quaternionic Hilbert space. At this point the "naive" result turns out to be a definitely important technical lemma, for it forbids the two extreme possibilities. In conclusion, the real theory is actually complex.