

Computing Inhomogeneous Gröbner Bases

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Abstract

In this paper we describe how an idea centered on the concept of *self-saturation* allows several improvements in the computation of Gröbner bases via Buchberger's Algorithm. In a nutshell, the idea is to extend the advantages of computing with homogeneous polynomials or vectors to the general case. When the input data are not homogeneous, we use a technique described in Section 2: the main tool is the procedure of a *self-saturating Buchberger's Algorithm*, and the main result is described in Theorem 22. Another strictly related topic is treated in Section 3 where a mathematical foundation is given to the *sugar trick* which is nowadays widely used in most of the implementations of Buchberger's Algorithm. A special emphasis is given in Section 4 to the case of a single grading, and Section 5 exhibits some timings and indicators showing the practical merits of our approach.

Key words: Gröbner bases, Buchberger's Algorithm

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Introduction

Starting from the sixties, when implementations of Buchberger's famous algorithm for computing Gröbner bases became practically feasible, several attempts were made at improving the range of its application and the efficiency of its performance. On the theoretical side, an important extension was its application to submodules of free modules over the polynomial ring which implies the usage of vectors of polynomials (see (12)).

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On the practical side it has been clear that mainly three of its steps can be optimized. They are the minimalization of the set of critical pairs (see for instance (4) and (7)), the optimization of the reduction procedure (see for instance (2) and (15)), and the sorting used to process the critical pairs during the algorithm. The last aspect is less important when the input polynomials or vectors are homogeneous and the algorithm proceeds with an *increasing degree* strategy. But what happens if the input data are not homogeneous?

A first answer to this question was given in the late eighties. It prescribed to homogenize the input data, run the algorithm, and then dehomogenize the computed Gröbner basis. This strategy is indeed quite simple and in many cases works fine. Its big advantage is that critical pairs are sorted by increasing degree and after a degree is completed the algorithm never goes back to it. The disadvantage is that often it computes too large a set of polynomials or vectors.

Quite soon (we are speaking of the early nineties) a new tool entered the game, the *sugar strategy* (see (10) and Section 3). In a nutshell, the idea was to keep the data non homogeneous, but process the critical pairs as if they were coming from true homogeneous data. This goal is achieved with the help of a manipulated degree called *sugar* which replaces the true degree. Although a complete theoretical background was not laid out, the sugar strategy revealed its strength and the idea gained popularity. Not much later an implementation improving the ordering of the critical pairs was introduced in the computer algebra system Bergman (see (14)).

Recently, inspired by the new development of CoCoA which will lead to the long awaited CoCoA 5, we decided to explore some features of Buchberger's Algorithm in great detail. The main purpose was to give a solid theoretical background to both the sugar strategy and the strategy of selection of critical pairs. We believe that we achieved both goals, so let us explain how. After recalling more or less well-known facts about the homogenization process in Section 1, we move quickly to the construction of what is called the Self-Saturating Buchberger's Algorithm.

To do that, in Section 2 we prove some properties of the saturation (see Proposition 10), define new notions such as $\bar{\sigma}$ -*SatGBasis* and σ -*DehomBasis*, and prove Theorem 14 where all these notions are fully compared. With the aid of this result we define and study several variants of Buchberger's Algorithm, the Weak Self-Saturating Buchberger's Algorithm and the Self-Saturating Buchberger's Algorithm, and finally prove the desired main result, Theorem 20. It simply says that the computation of a Gröbner basis when the input data are inhomogeneous, can be performed by running any Weak Self-Saturating Buchberger's Algorithm. The inspiration to achieve this goal came not only from the above mentioned paper (14), but also from the paper (1) where similar

strategies were described for the efficient computation of toric ideals.

It is also noteworthy to mention the fact that the variants of the Weak Self-Saturating Buchberger's Algorithm include the usual Buchberger's Algorithm as well as the algorithm obtained by homogenizing the input data, run the algorithm, and then dehomogenizing the computed Gröbner basis.

Section 3 is devoted to give a solid foundation to the sugar strategy which, as we said, is already used in several computer algebra systems. To describe it in joking mode we could say that the idea is to make a recipe by adding some *sugar* to the degree of the inhomogeneous vectors and make them *sweeter* in Buchberger's Algorithm. The main result is Proposition 26 which describes the behavior of the sugar during the execution of every variant of Buchberger's Algorithm introduced in the previous section. With this result we can combine the tools of Section 2 with the sugar strategy.

Section 4 treats the case of a single grading and shows how in that situation better results can be achieved (see Theorem 28 and its corollaries). The current implementation in CoCoA deals only with the case of the single gradings, shortly to be extended to the general case, and the final Section 5 shows its excellent behavior on a selected bunch of examples.

Of course we are aware of many algorithms which optimize the computation of some Gröbner bases simply by going around the problem. Among many others we could recall the Gröbner walk algorithm, the FGLM algorithm. But we want to make it clear that our goal here is to optimize Buchberger's Algorithm, not to find alternative strategies to compute Gröbner bases.

As a side remark we observe that every Self-Saturating Buchberger's Algorithm is fully compatible with the SlimGB strategies developed in (2) and with the Hilbert driven algorithms (see (13) and (6)). The integration and interplay of these approaches will be the subject of future work. Finally, the readers should know that the basic terminology is taken from the two books (11), (12).

1 Preliminaries

We assume the basic terminology and facts explained in (12), Section 4.3 and Tutorial 49. Some of them are explicitly recalled for the sake of completeness, hence most of the section contains either well-known facts or easy generalizations of well-known facts.

1.1 Homogenization in a polynomial ring

In this subsection we generalize the natural concept of homogenization to the multigraded case.

We let K be a field and $P = K[x_1, \dots, x_n]$ a polynomial ring. Then we take a matrix $W \in \text{Mat}_{m,n}(\mathbb{Z})$ of rank $m \geq 1$ and new indeterminates y_1, \dots, y_m called **homogenizing indeterminates** where y_i is the homogenizing indeterminate with respect to the i^{th} row of W . Moreover, we equip the polynomial ring $\overline{P} = K[y_1, \dots, y_m, x_1, \dots, x_n]$ with the grading defined by the matrix $\overline{W} = (I_m \mid W)$, where I_m denotes the identity matrix of size m .

Given m -tuples of integers $v_j = (a_{1j}, \dots, a_{mj})$, $j = 1, \dots, s$, we consider the tuple (c_1, \dots, c_m) where $c_k = \max\{a_{k1}, \dots, a_{ks}\}$, $k = 1, \dots, m$, and call it $\text{Top}(v_1, \dots, v_s)$.

Definition 1 Let $f \in P \setminus \{0\}$ and $F \in \overline{P}$.

- (1) Write $f = c_1 t_1 + \dots + c_s t_s$ with $c_1, \dots, c_s \in K \setminus \{0\}$ and distinct terms $t_1, \dots, t_s \in \mathbb{T}^n$. Then the tuple $\text{Top}(\deg_W(t_1), \dots, \deg_W(t_s))$ is called the **top degree** of f with respect to the grading given by W and is denoted by $\text{TopDeg}_W(f)$.
- (2) For every $j = 1, \dots, s$, we let $\deg_W(t_j) = (\tau_{1j}, \dots, \tau_{mj}) \in \mathbb{Z}^m$ and let $(\mu_1, \dots, \mu_m) = \text{TopDeg}_W(f)$. The **homogenization** of f with respect to the grading given by W is the polynomial

$$f^{\text{hom}} = \sum_{j=1}^s c_j t_j y_1^{\mu_1 - \tau_{1j}} \dots y_m^{\mu_m - \tau_{mj}} \in \overline{P}$$

For the zero polynomial, we set $0^{\text{hom}} = 0$.

- (3) The polynomial $F^{\text{deh}} = F(1, \dots, 1, x_1, \dots, x_n) \in P$ is called the **dehomogenization** of F with respect to y_1, \dots, y_m .

Given an ordering τ on \mathbb{T}^n , the monoid of power-products in P , we want to extend it to \mathbb{T}^{m+n} , the monoid of power-products of the *homogenization ring* \overline{P} .

Definition 2 We consider a monoid ordering τ on \mathbb{T}^n , and the relation $\overline{\tau}^W$ on \mathbb{T}^{m+n} which is defined by the following rule. Given two terms $t_1, t_2 \in \mathbb{T}^{m+n}$, we say that $t_1 >_{\overline{\tau}^W} t_2$ if either

$$\deg_{\overline{W}}(t_1) > \deg_{\overline{W}}(t_2)$$

or

$$\deg_{\overline{W}}(t_1) = \deg_{\overline{W}}(t_2) \quad \text{and} \quad t_1^{\text{deh}} >_{\tau} t_2^{\text{deh}}$$

We call $\bar{\tau}^W$ the **extension** of τ by W . If it is clear which grading we are considering, we shall simply denote it by $\bar{\tau}$.

We recall that the grading represented by the matrix W is said to be **positive** if each column of W has some non-zero entry and the first non-zero entry is positive.

Proposition 3 *Let τ be a monoid ordering on \mathbb{T}^n and $\bar{\tau}$ its extension by W .*

- (1) *The relation $\bar{\tau}$ is a $\deg_{\overline{W}}$ -compatible monoid ordering on \mathbb{T}^{m+n} .*
- (2) *If W is positive, the relation $\bar{\tau}$ is a term ordering on \mathbb{T}^{m+n} .*
- (3) *Let $F \in \overline{P}$ be a non-zero homogeneous polynomial. Then there exist $s_1, \dots, s_m \in \mathbb{N}$ such that $\text{LT}_{\bar{\tau}}(F) = y_1^{s_1} \cdots y_m^{s_m} \cdot \text{LT}_{\sigma}(F^{\text{deh}})$.*
- (4) *If τ is of the form $\tau = \text{Ord}(V)$ for a non-singular matrix $V \in \text{Mat}_n(\mathbb{Z})$, then we have $\bar{\tau} = \text{Ord}\begin{pmatrix} I_m & W \\ 0 & V \end{pmatrix}$.*

Proof. For the proof see (12), Proposition 4.3.14 and Lemma 4.3.16. \square

Remark 4 If τ is \deg_W -compatible then $\tau = \text{Ord}(V)$ where $V = \begin{pmatrix} W \\ V' \end{pmatrix}$. Therefore we have

$$\bar{\tau} = \text{Ord}\begin{pmatrix} I_m & W \\ 0 & V' \end{pmatrix} = \text{Ord}\begin{pmatrix} I_m & W \\ -I_m & 0 \\ 0 & V' \end{pmatrix}$$

If $m = 1$ it follows that $\bar{\tau}$ is of y_1 -DegRev type (see (12), Section 4.4) with respect to $\deg_{\overline{W}}$. In particular, if $m = 1$ and $\tau = \text{DegRevLex}$ where **Deg** denotes the standard grading on P it is more common to write $\overline{P} = K[x_1, \dots, x_n, y]$ with the homogenizing indeterminate *at the end*, then we have $\bar{\tau} = \text{DegRevLex}$ where **Deg** denotes the standard grading on \overline{P} .

Definition 5 Let I be an ideal in P and J an ideal in \overline{P} .

- (1) The ideal $I^{\text{hom}} = (f^{\text{hom}} \mid f \in I)$ in \overline{P} is called the **homogenization** of I with respect to the grading given by W .
- (2) The set $J^{\text{deh}} = \{F^{\text{deh}} \mid F \in J\}$ in P is an ideal called the **dehomogenization** of J with respect to y_1, \dots, y_m .

1.2 Homogenization in a free P -module

In this subsection we generalize the multihomogenization procedure to the case of free modules.

Let r be a positive integer, let F denote the free P -module P^r and let e_1, \dots, e_r be the vectors of the canonical basis of F . Then let $\delta_1, \dots, \delta_r \in \mathbb{Z}^m$ and let \bar{F} be the graded free \bar{P} -module $\bar{F} = \bigoplus_{i=1}^r \bar{P}(-\delta_i)$ where the degrees of e_1, \dots, e_r are $\delta_1, \dots, \delta_r$ respectively. We denote by $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$ the monomodule made by the terms $t \cdot e_i \in F$ with $t \in \mathbb{T}^n$, and by $\mathbb{T}^{m+n}\langle e_1, \dots, e_r \rangle$ the monomodule made by the terms $t \cdot e_i \in \bar{F}$ with $t \in \mathbb{T}^{m+n}$. Henceforth, when we consider module orderings on $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$ we always mean module orderings which are compatible with a monoid ordering on \mathbb{T}^n (see (11) Definition 1.4.17). The following definition is a natural generalization of Definition 2.

Definition 6 We consider a module ordering σ on $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$ and the relation $\bar{\sigma}^W$ on $\mathbb{T}^{m+n}\langle e_1, \dots, e_r \rangle$ which is defined by the following rule. Given $t_1 e_i, t_2 e_j \in \mathbb{T}^{m+n}\langle e_1, \dots, e_r \rangle$, we say that $t_1 e_i >_{\bar{\sigma}^W} t_2 e_j$ if either

$$\deg_{\bar{W}}(t_1 e_i) > \deg_{\bar{W}}(t_2 e_j)$$

or

$$\deg_{\bar{W}}(t_1 e_i) = \deg_{\bar{W}}(t_2 e_j) \quad \text{and} \quad t_1^{\text{deh}} e_i >_{\sigma} t_2^{\text{deh}} e_j$$

We call $\bar{\sigma}^W$ the **extension** of σ by W . If it is clear which grading we are considering, we shall simply denote it by $\bar{\sigma}$.

Proposition 7 Let σ be a module ordering on $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$, and let $\bar{\sigma}$ be its extension by W .

- (1) The relation $\bar{\sigma}$ is a $\deg_{\bar{W}}$ -compatible module ordering on $\mathbb{T}^{m+n}\langle e_1, \dots, e_r \rangle$.
- (2) If W is positive, then $\bar{\sigma}$ is a module term ordering on $\mathbb{T}^{m+n}\langle e_1, \dots, e_r \rangle$.
- (3) Let $U \in \bar{F}$ be a homogeneous non-zero vector. Then there exist non negative integers s_1, \dots, s_m such that $\text{LT}_{\bar{\sigma}}(U) = y_1^{s_1} \cdots y_m^{s_m} \cdot \text{LT}_{\sigma}(U^{\text{deh}})$.

Proof. It is an easy generalization of Proposition 3. \square

Analogously to Definition 1 and Definition 5 one defines the homogenization and dehomogenization of vectors and submodules. With the following proposition we recall some easy results about homogenization and dehomogenization we will need to prove Theorem 14.

Proposition 8 *Let M be a submodule of F and let N be a graded submodule of \overline{F} .*

- (1) *We have $(M^{\text{hom}})^{\text{deh}} = M$.*
- (2) *If $M = \langle v_1, \dots, v_s \rangle$ then $M^{\text{hom}} = \langle v_1^{\text{hom}}, \dots, v_s^{\text{hom}} \rangle_{:\overline{F}} (y_1 \cdots y_m)^\infty$*
- (3) *If $N = \langle V_1, \dots, V_t \rangle$ where V_1, \dots, V_t are homogeneous vectors, then $N^{\text{deh}} = \langle V_1^{\text{deh}}, \dots, V_t^{\text{deh}} \rangle$.*

Proof. It is an obvious generalization of (12), Corollaries 4.3.5.a and 4.3.8. \square

2 Self-Saturating Buchberger's Algorithm

This section starts with some properties of the saturation and continues with the proof of the main facts (see Theorem 14) which will eventually lead to the algorithm for computing inhomogeneous Gröbner bases. After recalling the definition of a remainder, we write the body of Buchberger's Algorithm to help the reader spotting the differences when we describe some of its variants (see Theorem 20). The section ends with the main Theorem 22. We keep the notation introduced before, in particular, we let σ be a module ordering on $\mathbb{T}^n \langle e_1, \dots, e_r \rangle$, and let $\overline{\sigma}$ be its extension by W .

2.1 Saturation

We start the subsection with an important definition.

Definition 9 Let $U \in \overline{F}$ be a homogeneous vector. We denote $(U^{\text{deh}})^{\text{hom}}$ by U^{sat} and we call it the **saturation** of U . Let N be a graded submodule of \overline{F} . We denote $(N^{\text{deh}})^{\text{hom}}$ by N^{sat} and we call it the **saturation** of N .

The saturation studied here is a special case of the saturation as described in (11) Section 3.5.B, as it is shown in the following Proposition 10.(4).

We are going to illustrate some fundamental properties of the saturation. We recall that Proposition 7.(3) shows the existence of $s_1, \dots, s_m \in \mathbb{N}$ such that the formula $\text{LT}_{\overline{\sigma}}(U) = y_1^{s_1} \cdots y_m^{s_m} \cdot \text{LT}_{\sigma}(U^{\text{deh}})$ holds true.

Proposition 10 (Properties of the Saturation) *Let $U \in \overline{F}$ be a homogeneous non-zero vector, let N be a graded submodule of \overline{F} , and let M be a submodule of F .*

- (1) *There exist $r_1, \dots, r_m \in \mathbb{N}$ such that $\text{LT}_{\overline{\sigma}}(U) = y_1^{r_1} \cdots y_m^{r_m} \cdot \text{LT}_{\overline{\sigma}}(U^{\text{sat}})$.*

- (2) Comparing (1) with the formula $\text{LT}_{\bar{\sigma}}(U) = y_1^{s_1} \cdots y_m^{s_m} \cdot \text{LT}_{\sigma}(U^{\text{deh}})$, we have $r_i \leq s_i$ for $i = 1, \dots, m$.
- (3) We have $(\text{LT}_{\bar{\sigma}}(U))^{\text{sat}} = \text{LT}_{\sigma}(U^{\text{deh}})$.
- (4) We have $N^{\text{sat}} = N :_{\bar{F}} (y_1 \cdots y_m)^{\infty}$.
- (5) We have $N^{\text{sat}} = \langle V^{\text{sat}} \mid V \in N, V \text{ homogeneous} \rangle$.
- (6) We have $(N^{\text{sat}})^{\text{deh}} = N^{\text{deh}}$.
- (7) If $M = \langle v_1, \dots, v_t \rangle$ then $M^{\text{hom}} = \langle v_1^{\text{hom}}, \dots, v_t^{\text{hom}} \rangle^{\text{sat}}$.

Proof. Condition (1) follows from the definition. To prove (2) we denote by (*) the formula $\text{LT}_{\bar{\sigma}}(U) = y_1^{s_1} \cdots y_m^{s_m} \cdot \text{LT}_{\sigma}(U^{\text{deh}})$ in Proposition 7.(3). We observe that $(U^{\text{sat}})^{\text{deh}} = U^{\text{deh}}$, hence, if we apply (*) to U^{sat} we get the equality $\text{LT}_{\bar{\sigma}}(U^{\text{sat}}) = y_1^{s'_1} \cdots y_m^{s'_m} \cdot \text{LT}_{\sigma}(U^{\text{deh}})$ for suitable natural numbers s'_1, \dots, s'_m . Using (1) and (*) we get $r_i + s'_i = s_i$ for $i = 1, \dots, m$. Condition (3) follows from condition (2). Next we prove (4). Let V_1, \dots, V_t be homogeneous vectors which generate N . Using Proposition 8, we deduce the following equality $N^{\text{sat}} = \langle V_1^{\text{sat}}, \dots, V_t^{\text{sat}} \rangle :_{\bar{F}} (y_1 \cdots y_m)^{\infty}$. It remains to show that $N :_{\bar{F}} (y_1 \cdots y_m)^{\infty} = \langle V_1^{\text{sat}}, \dots, V_t^{\text{sat}} \rangle :_{\bar{F}} (y_1 \cdots y_m)^{\infty}$. The inclusion \subseteq is a consequence of the obvious relation $N \subseteq \langle V_1^{\text{sat}}, \dots, V_t^{\text{sat}} \rangle$, while the inclusion \supseteq follows from the observation that $V_i^{\text{sat}} \in N :_{\bar{F}} (y_1 \cdots y_m)^{\infty}$ for $i = 1, \dots, t$. Condition (5) follows from the definition. Clearly (6) follows from (4) and finally, to prove (7) it suffices to combine (4) with Proposition 8.(2). \square

Definition 11 Let N be a graded submodule of \bar{F} and let $V_1, \dots, V_t \in N$ be non-zero homogeneous vectors.

- (1) The set $\{V_1, \dots, V_t\}$ is called a $\bar{\sigma}$ -SatGBasis for N if it is a $\bar{\sigma}$ -Gröbner basis of a graded submodule \tilde{N} of \bar{F} such that $\tilde{N}^{\text{sat}} = N^{\text{sat}}$.
- (2) The set $\{V_1, \dots, V_t\}$ is called a σ -DehomBasis for N if $\{V_1^{\text{deh}}, \dots, V_t^{\text{deh}}\}$ is a σ -Gröbner basis of N^{deh} .

Proposition 12 Let $M = \langle v_1, \dots, v_s \rangle$ be a submodule of F , denote by N the module $\langle v_1^{\text{hom}}, \dots, v_s^{\text{hom}} \rangle$, and let $\{V_1, \dots, V_t\}$ be a σ -DehomBasis for N . Then $\{V_1^{\text{deh}}, V_2^{\text{deh}}, \dots, V_t^{\text{deh}}\}$ is a σ -Gröbner basis of M .

Proof. The claim follows from the chain

$$N^{\text{deh}} = \langle v_1^{\text{hom}}, v_2^{\text{hom}}, \dots, v_s^{\text{hom}} \rangle^{\text{deh}} = \langle v_1, v_2, \dots, v_s \rangle = M$$

where the second equality follows from Proposition 8.(3). \square

Lemma 13 Let N be a graded submodule of \bar{F} and let $V_1, \dots, V_t \in N$ be non-zero homogeneous vectors. Then the following conditions are equivalent.

- (1) The set $\{V_1, \dots, V_t\}$ is a σ -DehomBasis for N .
- (2) The set $\{(\text{LT}_{\bar{\sigma}}(V_1))^{\text{sat}}, \dots, (\text{LT}_{\bar{\sigma}}(V_t))^{\text{sat}}\}$ generates $(\text{LT}_{\bar{\sigma}}(N))^{\text{sat}}$.

Proof. By Proposition 10.(3) the set $\{(\text{LT}_{\bar{\sigma}}(V_1))^{\text{sat}}, \dots, (\text{LT}_{\bar{\sigma}}(V_t))^{\text{sat}}\}$ coincides with $\{\text{LT}_{\sigma}(V_1^{\text{deh}}), \dots, \text{LT}_{\sigma}(V_t^{\text{deh}})\}$, and by Proposition 10.(3),(5) we have the equality $(\text{LT}_{\bar{\sigma}}(N))^{\text{sat}} = \langle \text{LT}_{\sigma}(V^{\text{deh}}) \mid V \in N, V \text{ homogeneous} \rangle$. The conclusion follows immediately. \square

Theorem 14 *Let N be a graded submodule of \bar{F} , let $V_1, \dots, V_t \in N$ be non-zero homogeneous vectors, and let us consider the following conditions.*

- (1) *The set $\{V_1, \dots, V_t\}$ is a $\bar{\sigma}$ -Gröbner basis of N .*
- (2) *The set $\{V_1, \dots, V_t\}$ is a $\bar{\sigma}$ -SatGBasis for N .*
- (3) *The set $\{V_1, \dots, V_t\}$ is a σ -DehomBasis for N .*
- (4) *The set $\{V_1^{\text{sat}}, \dots, V_t^{\text{sat}}\}$ generates N^{sat} .*

Then we have the following chain of implications.

$$(1) \implies (2) \implies (3) \implies (4)$$

Proof. The implication $(1) \implies (2)$ is obvious.

To prove $(2) \implies (3)$ let $v \in N^{\text{deh}}$. By Proposition 10.(6) and the assumption, we have $v = V^{\text{deh}}$ with $V \in N$. Then there exists an index i such that $\text{LT}_{\bar{\sigma}}(V_i) \mid \text{LT}_{\bar{\sigma}}(V)$. Consequently $\text{LT}_{\sigma}(V_i^{\text{deh}}) \mid \text{LT}_{\sigma}(V^{\text{deh}})$, and the proof is complete.

To prove $(3) \implies (4)$ we use the equivalent condition of Lemma 13 and proceed by contradiction. Let $U \in N^{\text{sat}}$ be a homogeneous element with minimal $(\text{LT}_{\bar{\sigma}}(U))^{\text{sat}}$ among the elements in N^{sat} and not in $\langle V_1^{\text{sat}}, \dots, V_t^{\text{sat}} \rangle$. We observe that $\text{LT}_{\bar{\sigma}}(U) \in \text{LT}_{\bar{\sigma}}(N^{\text{sat}}) \subseteq (\text{LT}_{\bar{\sigma}}(N))^{\text{sat}}$ and therefore, by assumption, there exists i such that $(\text{LT}_{\bar{\sigma}}(V_i))^{\text{sat}}$ divides $\text{LT}_{\bar{\sigma}}(U)$. We deduce that, for suitable $c \in K$ and $t \in \mathbb{T}^n$ the vector $V = U - ctV_i$ has the properties: $V \in N^{\text{sat}}$; $V \notin \langle V_1^{\text{sat}}, \dots, V_t^{\text{sat}} \rangle$; $\text{LT}_{\bar{\sigma}}(V) <_{\bar{\sigma}} \text{LT}_{\bar{\sigma}}(U)$. By Definition 6, it follows that $(\text{LT}_{\bar{\sigma}}(V))^{\text{sat}} <_{\sigma} (\text{LT}_{\bar{\sigma}}(U))^{\text{sat}}$, a contradiction. \square

In the next example we show that the implications of Theorem 14 cannot be reversed.

Example 15 Let $P = \mathbb{Q}[x, y, z]$, $\sigma = \text{Lex}$. We use a single homogenizing indeterminate which we call h and we write $\bar{P} = \mathbb{Q}[x, y, z, h]$ according to Remark 4; then $\bar{\sigma} = \text{DegLex}$. Let $F_1 = xh^2 - z^3$, $F_2 = x^2h - y^3$, and let N be the ideal of \bar{P} generated by $\{F_1, F_2\}$. If $F_3 = y^3h^3 - z^6$ it is easy to check that $F_3 \in N$ and $(\text{LT}_{\bar{\sigma}}(N))^{\text{sat}} = ((\text{LT}_{\bar{\sigma}}(F_1))^{\text{sat}}, (\text{LT}_{\bar{\sigma}}(F_3))^{\text{sat}}) = (x, y^3)$. Lemma 13 implies that $\{F_1, F_3\}$ is a σ -DehomBasis for N ; however, it is not a $\bar{\sigma}$ -Gröbner basis of any module, therefore $(3) \not\Rightarrow (2)$. Moreover, it is easy to see that $F_1 = F_1^{\text{sat}}$, $F_2 = F_2^{\text{sat}}$, that $(F_1, F_2) = N^{\text{sat}}$, but $\{F_1^{\text{deh}}, F_2^{\text{deh}}\}$ is not a σ -Gröbner basis of N^{deh} . Therefore $(4) \not\Rightarrow (3)$.

Now let $P = \mathbb{Q}[x, y, z]$, $\sigma = \text{DegRevLex}$ and let $\bar{P} = \mathbb{Q}[x, y, z, h]$. In this case we have $\bar{\sigma} = \text{DegRevLex}$ (see Remark 4). Let $F_1 = x^2 - yh$, $F_2 = xy - zh$, let N be the ideal of \bar{P} generated by $\{F_1, F_2\}$, and let $F_3 = y^2h - xzh$, so that $F_3^{\text{sat}} = y^2 - xz$. Then $\{F_1, F_2, F_3\}$ is the reduced Gröbner basis of N , while $\{F_1, F_2, F_3^{\text{sat}}\}$ is the reduced Gröbner basis of a module \tilde{N} such that $\tilde{N}^{\text{sat}} = N^{\text{sat}}$. Therefore $(2) \not\Rightarrow (1)$.

We are going to use the above results to produce a strategy for computing Gröbner bases. First, we introduce a definition.

Definition 16 Let σ be a module ordering on $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$. Let $G = \{v_1, \dots, v_s\}$ be a set of non-zero elements in F , and let u, v be elements in F . Then u is said to be a **remainder** of v by G if $v \xrightarrow{G} u$ (where \xrightarrow{G} is the rewrite relation defined by G , see (11) Definition 2.2.1) and $\text{LT}_\sigma(v_i) \nmid \text{LT}_\sigma(u)$ for $i = 1, \dots, s$. Analogously, we use the same terminology for \bar{F} . In these cases we write $u = \text{Rem}(v, G)$.

Following the definition, the correct expression would be $u \in \text{Rem}(v, G)$. However, for the sake of simplicity we write $u = \text{Rem}(v, G)$.

2.2 Self Saturation

Now we write a general version of Buchberger's Algorithm. Instead of using the stepwise description given in the books (11) and (12), we prefer to concentrate on the main ingredients. In this way it will be easier for the reader to understand the variations presented below. We recall the notion of S-vector $S(u, v)$ of u, v (see (11), Definition 2.5.1). If $\text{LM}_\sigma(u) = c_u t_u e_i$ and $\text{LM}_\sigma(v) = c_v t_v e_i$, then $S(u, v) = \frac{\text{lcm}(t_u, t_v)}{c_u t_u} u - \frac{\text{lcm}(t_u, t_v)}{c_v t_v} v$. If U, V are homogeneous vectors, some observations on the S-vector $S(U, V)$ are contained in (12), Remark 4.5.3.

We formulate Buchberger's algorithm in a way in which we begin with unprocessed vectors and no pairs, and at each step we can either process a vector (adding pairs) or a pair.

Theorem 17 (Body of Buchberger's Algorithm) *Let u_1, \dots, u_s be non-zero vectors in F (homogeneous non-zero vectors in \bar{F}) and let M be the submodule of F (graded submodule of \bar{F}) generated by $\{u_1, \dots, u_s\}$.*

- (1) **(Initialization)** *Pairs = \emptyset , the pairs; Gens = (u_1, \dots, u_s) , the generators of M ;
 $G = \emptyset$, the σ -Gröbner basis ($\bar{\sigma}$ -Gröbner basis) of M under construction.*
- (2) **(Main loop)** *While Gens $\neq \emptyset$ and Pairs $\neq \emptyset$ do*

- (2a) choose $w \in \text{Gens}$ and remove it from Gens ,
or a pair $(v_i, v_j) \in \text{Pairs}$, remove it from Pairs , and let $w = S(v_i, v_j)$;
- (2b) compute a remainder $v := \text{Rem}(w, G)$;
- (2c) if $v \neq 0$ add v to G and the pairs $\{(v, v_i) \mid v_i \in G\}$ to Pairs .
- (3) **(Output)** Return G .

This is an algorithm which returns a σ -Gröbner basis ($\bar{\sigma}$ -Gröbner basis) of M , whatever choices are made in step (2a) and whatever remainder is computed in step (2b).

Definition 18 Let G be a finite set of homogeneous vectors in \bar{F} and V a homogeneous vector in \bar{F} .

- (1) We call **weak saturating remainder** of V with respect to G a vector obtained in the following way. At each step of the rewrite relation defined by G , the dividend U may be replaced by a homogeneous vector U' of \bar{F} such that $U^{\text{sat}} = (U')^{\text{sat}}$. If the remainder obtained in this way has the property that its leading term is not divisible by any of the leading terms of the elements of G , we denote it by $\text{WeakSatRem}(V, G)$.
- (2) We call **saturating remainder** of V with respect to G , and denote it by $\text{SatRem}(V, G)$, a vector $(\text{Rem}(V, G))^{\text{sat}}$.

Now we describe useful variants of Buchberger's Algorithm.

Definition 19 Let U_1, \dots, U_s be homogeneous vectors in \bar{F} and let N be the graded submodule of \bar{F} generated by $\{U_1, \dots, U_s\}$. If step (2b) in Buchberger's Algorithm is replaced by

(2b') compute $V := \text{WeakSatRem}(W, G)$;

the procedure is called a **Weak Self-Saturating Buchberger's Algorithm (WeakSelfSatBA)**. And, in particular, if it is replaced by the following special case of (2b')

(2b'') compute $V := \text{SatRem}(W, GG)$;

the procedure is called the **Self-Saturating Buchberger's Algorithm (SelfSatBA)**.

A motivation for these names comes from the following result.

Theorem 20 Let U_1, \dots, U_s be homogeneous vectors in \bar{F} and let N be the graded submodule of \bar{F} generated by $\{U_1, \dots, U_s\}$.

- (1) Every *WeakSelfSatBA* applied to $\{U_1, \dots, U_s\}$ computes a σ -DehomBasis for N .
- (2) *SelfSatBA* applied to $\{U_1, \dots, U_s\}$ computes a $\bar{\sigma}$ -SatGBasis for N .

Proof. To prove (1) note that, when we substitute a vector with another with the same saturation, the two vectors have the same dehomogenization. This implies that every reduction $V := \text{WeakSatRem}(W, G)$ mirrors a reduction of W^{deh} by $G^{\text{deh}} = \{U^{\text{deh}} \mid U \in G\}$ with only one possible exception: though V^{deh} might still be reducible by G^{deh} , we may choose not to substitute V with an element with the same saturation (which would allow the “mirror” reduction by G), we go to step (2c) and add V to G . In this case the “mirror” reduction will be later performed as a pair. Note that since $\text{LT}(V)$ is not divisible by any leading term in G this process terminates by Dickson’s Lemma, and the output is a set of vectors $\{V_1, \dots, V_t\}$ such that $\{V_1^{\text{deh}}, \dots, V_t^{\text{deh}}\}$ is a σ -Gröbner basis of $\langle U_1^{\text{deh}}, \dots, U_s^{\text{deh}} \rangle$ which is N^{deh} by Proposition 8.(3). To prove (2) we observe that all the replacements of Rem with SatRem are equivalent to having added some element V^{sat} to Gens and having chosen it in step (2a) just before choosing V which would consequently reduce to 0 via V^{sat} . \square

Example 21 Reconsider $F_1 = xh^2 - z^3$, $F_3 = y^3h^3 - z^6$ in $\overline{P} = \mathbb{Q}[x, y, z, h]$, $\overline{\sigma} = \text{DegLex}$ from Example 15 and run WeakSelfSatBA.

- In (2a) we choose $W = F_1$, in (2b’) we get $V = F_1$, and in (2c) we add it to G .
- In (2a) we choose $W = F_3$, in (2b’) we get $V = F_3$, and in (2c) we add it to G and (F_1, F_3) to Pairs.
- In (2a) we choose $W = S(F_1, F_3) = xz^6 - y^3z^3h$, in (2b’) we have these two reduction steps: $W_1 = Wh^2 - F_1z^6 = -y^3z^3h^3 + z^9$; $W_2 = W_1 + F_3z^3 = 0$ and we are done.

The output is $\{F_1, F_3\}$ which is not a $\overline{\sigma}$ -Gröbner basis (see Example 15).

We are ready to state the main result in this section.

Theorem 22 *Let v_1, \dots, v_s be non-zero vectors in F , let M be the submodule generated by the set $\{v_1, \dots, v_s\}$, and let $\{V_1, V_2, \dots, V_t\}$ be the output of any WeakSelfSatBA applied to the set $\{v_1^{\text{hom}}, v_2^{\text{hom}}, \dots, v_s^{\text{hom}}\}$. Then the set $\{V_1^{\text{deh}}, \dots, V_t^{\text{deh}}\}$ is a σ -Gröbner basis of M .*

Proof. Let $N = \langle v_1^{\text{hom}}, v_2^{\text{hom}}, \dots, v_s^{\text{hom}} \rangle$ and let $\{V_1, \dots, V_t\}$ be the output of a WeakSelfSatBA algorithm applied to $\{v_1^{\text{hom}}, v_2^{\text{hom}}, \dots, v_s^{\text{hom}}\}$. Theorem 20 implies that the set $\{V_1, \dots, V_t\}$ is a σ -DehomBasis for N , i.e. that the set $\{V_1^{\text{deh}}, V_2^{\text{deh}}, \dots, V_t^{\text{deh}}\}$ is a σ -Gröbner basis of N^{deh} . The conclusion follows from Proposition 12. \square

3 The Sugar Strategy

If we look at the variants of Buchberger's Algorithm (see Definition 19), we note that they differ from the ordinary algorithm (see Theorem 17) only because they allow the replacement of a vector with another one with the same saturation. Such replacements may create vectors with a different degree, and hence the corresponding critical pairs and reductions have also different degree. We observe that a reduction can also be viewed as a special S-vector as shown in the proof of Proposition 24, so we can concentrate on S-vectors. The idea is that we want to *keep the original degree* every time we actually perform such a replacement. Now it is time to become formal.

Definition 23 Let V, V' be homogeneous vectors in \overline{F} . Then V' is said to be a **companion vector** of V if there exist non-negative integers s_1, \dots, s_m such that $V' = y_1^{s_1} \dots y_m^{s_m} V$.

Proposition-Definition 24 Consider an algorithm either of type WeakSelfSatBA or SelfSatBA. For each homogeneous vector V which is used during the execution of the algorithm, there exists a unique companion vector V^{sw} (here $^{\text{sw}}$ means **sweetened**) which obeys the following rules.

- (1) For every input vector U_1, \dots, U_s we have $U_i^{\text{sw}} = U_i$.
- (2) For every pair of vectors U, V we have $S(U, V)^{\text{sw}} = S(U^{\text{sw}}, V^{\text{sw}})$.
- (3) During the execution of the algorithm, when a vector V is substituted by another vector V' with the property that $(V')^{\text{sat}} = V^{\text{sat}}$, if we have $V^{\text{sw}} = y_1^{a_1} \dots y_m^{a_m} V^{\text{sat}}$, $V' = y_1^{b_1} \dots y_m^{b_m} V^{\text{sat}}$ with suitable non-negative integers $a_1, \dots, a_m, b_1, \dots, b_m$, then we have $(V')^{\text{sw}} = y_1^{c_1} \dots y_m^{c_m} V^{\text{sat}}$ where $(c_1, \dots, c_m) = \text{Top}((a_1, \dots, a_m), (b_1, \dots, b_m))$.

Proof. We need to prove that for each creation of a new vector during the execution of the algorithm, a unique companion vector is defined. Rules (1) and (2) force the uniqueness of the companion vector for the input vectors and the S -vectors. Replacement of a vector with another one with the same saturation is taken care by (3). Every step of reduction is of the type $U - ct'V$ with $c \in K$, $t' \in \mathbb{T}^{m+n}$ and $U - ct'V = S(U, V)$, hence the uniqueness of the companion vector is implied by (2). \square

Definition 25 We denote the degree $\deg_{\overline{W}}(V^{\text{sw}})$ by **sugar**(V), and we denote the degree $\deg_{\overline{W}}(S(V_i, V_j)^{\text{sw}}) = \deg_{\overline{W}}(S(V_i^{\text{sw}}, V_j^{\text{sw}}))$ by **sugar**(V_i, V_j). We say that we use the **sugar strategy** if the choice of the pairs in step (2a) is made starting with the lowest sugar, not the lowest degree.

We observe that the above definitions strictly depend on the operations used along the execution of the algorithm. Elementary properties of the sugar are contained in the following proposition which turns out to be particularly useful for a good implementation.

Proposition 26 *Consider an algorithm either of type WeakSelfSatBA or Self-SatBA and let $U, V \in \overline{F}$ be homogeneous non-zero vectors which are used during its execution.*

- (1) *For every U we have $(U^{\text{sw}})^{\text{sat}} = U^{\text{sat}}$ and $\text{sugar}(U)$ is componentwise greater than or equal to $\deg_{\overline{W}}(U)$.*
- (2) *Suppose that U is reducible by V , let $\text{LT}_{\overline{\sigma}}(U) = t' \text{LT}_{\overline{\sigma}}(V)$, with $t' = y_1^{a_1} \cdots y_m^{a_m} t$, $t \in \mathbb{T}^n$. Let $\text{LC}_{\overline{\sigma}}(U) = c \text{LC}_{\overline{\sigma}}(V)$ and let $A = U - c t' V$ be the result of the reduction. Then we have the equality*

$$\text{sugar}(A) = \text{Top} \left(\text{sugar}(U), \deg_{\overline{W}}(t) + \text{sugar}(V) \right)$$

Proof. Property (1) follows as an immediate consequence of Definition 23, so let us prove property (2). Let $U^{\text{sw}} = y_1^{r_1} \cdots y_m^{r_m} U$, $V^{\text{sw}} = y_1^{s_1} \cdots y_m^{s_m} V$. Then we have

$$\text{LT}_{\overline{\sigma}}(U^{\text{sw}}) = y_1^{r_1} \cdots y_m^{r_m} \text{LT}_{\overline{\sigma}}(U) = y_1^{r_1+a_1} \cdots y_m^{r_m+a_m} t \text{LT}_{\overline{\sigma}}(V) \quad (1)$$

$$\text{LT}_{\overline{\sigma}}(V^{\text{sw}}) = y_1^{s_1} \cdots y_m^{s_m} \text{LT}_{\overline{\sigma}}(V) \quad (2)$$

Moreover,

$$\begin{aligned} \text{sugar}(A) &= \text{sugar}(U - c t' V) = \text{sugar}(S(U, V)) = \deg_{\overline{W}}(S(U, V)^{\text{sw}}) \\ &= \deg_{\overline{W}}(S(U^{\text{sw}}, V^{\text{sw}})) = \deg_{\overline{W}} \left(\text{lcm}(\text{LT}_{\overline{\sigma}}(U^{\text{sw}}), \text{LT}_{\overline{\sigma}}(V^{\text{sw}})) \right) \end{aligned}$$

Using formulas (1) and (2) we get

$$\begin{aligned} &\text{lcm}(\text{LT}_{\overline{\sigma}}(U^{\text{sw}}), \text{LT}_{\overline{\sigma}}(V^{\text{sw}})) \\ &= \text{lcm}(y_1^{r_1+a_1} \cdots y_m^{r_m+a_m} t \text{LT}_{\overline{\sigma}}(V), y_1^{s_1} \cdots y_m^{s_m} \text{LT}_{\overline{\sigma}}(V)) \\ &= \text{lcm}(y_1^{r_1+a_1} \cdots y_m^{r_m+a_m} t \text{LT}_{\overline{\sigma}}(V), y_1^{s_1} \cdots y_m^{s_m} t \text{LT}_{\overline{\sigma}}(V)) \end{aligned}$$

Consequently

$$\begin{aligned} \text{sugar}(A) &= \deg_{\overline{W}} \left(\text{lcm}(y_1^{r_1+a_1} \cdots y_m^{r_m+a_m} t \text{LT}_{\overline{\sigma}}(V), y_1^{s_1} \cdots y_m^{s_m} t \text{LT}_{\overline{\sigma}}(V)) \right) \\ &= \text{Top} \left((r_1 + a_1, \dots, r_m + a_m) + \deg_{\overline{W}}(t V), (s_1, \dots, s_m) + \deg_{\overline{W}}(t V) \right) \end{aligned}$$

$$= \text{Top}(\text{sugar}(U), \deg_{\overline{W}}(t) + \text{sugar}(V))$$

where the last equality follows from formulas (1) and (2). \square

Example 27 Consider the polynomial ring $\overline{P} = K[y_1, y_2, x_1, x_2]$ graded by $\overline{W} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Let $U = y_1^2 y_2 x_1^2 - y_1^3 x_2$, $V = y_2 x_1 - x_2$. We observe that U is homogeneous of degree $(4, 1)$ and V is homogeneous of degree $(1, 1)$. With $c = 1$, $t' = y_1^2 x_1$, $t = x_1$ we have the reduction $A = U - y_1^2 x_1 V = y_1^2 x_1 x_2 - y_1^3 x_2$ which is homogeneous of degree $(4, 1)$. Now we consider two cases.

- Case 1 Assume that $U^{\text{sw}} = U$, $V^{\text{sw}} = y_1 V$ so that $\text{sugar}(U) = (4, 1)$ and $\text{sugar}(V) = (2, 1)$. According to Proposition 26.(2), we have $\text{sugar}(A) = \text{Top}((4, 1), \deg_{\overline{W}}(x_1) + (2, 1)) = \text{Top}((4, 1), (1, 0) + (2, 1)) = (4, 1)$.
- Case 2 Assume instead that $U^{\text{sw}} = U$, $V^{\text{sw}} = y_1 y_2 V$ so that we have $\text{sugar}(U) = (4, 1)$ and $\text{sugar}(V) = (2, 2)$. Then $\text{LT}_{\overline{\sigma}}(U^{\text{sw}}) = y_1^2 y_2 x_1^2$, $\text{LT}_{\overline{\sigma}}(V^{\text{sw}}) = y_1 y_2^2 x_1$. Their fundamental syzygy is $(y_2, -y_1 x_1)$ whose degree is $(4, 2)$. This fact is in agreement with Proposition 26.(2) for which $\text{sugar}(A) = \text{Top}((4, 1), (1, 0) + (2, 2)) = (4, 2)$. It is interesting to observe that a rule of the type $\text{sugar}(tV) = \deg(t) + \text{sugar}(V)$ would have lead to $\text{sugar}(y_1^2 x_1 V) = (5, 2)$, wrongly suggesting that the sugar of A should have to be $(5, 2)$.

4 Single Gradings

In this section we restrict our attention to the case of positive \mathbb{N} -gradings i.e. gradings defined by a matrix W consisting of a single row with positive entries. Then we have a single homogenizing indeterminate which will be called just y . A first consideration in this direction was made in Remark 4, but we can say more.

Theorem 28 *Let $W \in \text{Mat}_{1,n}(\mathbb{Z})$ be a matrix with positive entries and let P be graded by W and let σ be a \deg_W -compatible term ordering.*

- (1) *If v is a non-zero vector in F , we have $\text{LT}_{\overline{\sigma}}(v^{\text{hom}}) = \text{LT}_{\sigma}(v)$.*
- (2) *If N is a graded submodule of \overline{F} , and $G = \{V_1, V_2, \dots, V_t\}$ is a homogeneous $\overline{\sigma}$ -Gröbner basis of N , then the set $\{V_1^{\text{sat}}, V_2^{\text{sat}}, \dots, V_t^{\text{sat}}\}$ is a $\overline{\sigma}$ -Gröbner basis of N^{sat} .*

Proof. Claim (1) is clear. To prove claim (2) we let V be a vector in N^{sat} ; we need to show that $\text{LT}_{\overline{\sigma}}(V_i^{\text{sat}}) \mid \text{LT}_{\overline{\sigma}}(V)$ for some $i \in \{1, \dots, t\}$. Proposi-

tion 10.(4) implies that $y^a \cdot V \in N$ for some $a \in \mathbb{N}$. As a consequence $y^a \cdot \text{LT}_{\bar{\sigma}}(V) \in \text{LT}_{\bar{\sigma}}(N)$, hence there exists $V_i \in G$ such that $\text{LT}_{\bar{\sigma}}(V_i) \mid y^a \cdot \text{LT}_{\bar{\sigma}}(V)$. Now, $y \nmid \text{LT}_{\bar{\sigma}}(V_i^{\text{sat}})$ by (1) applied to $v = V_i^{\text{deh}}$, hence $\text{LT}_{\bar{\sigma}}(V_i^{\text{sat}}) \mid \text{LT}_{\bar{\sigma}}(V)$ and this concludes the proof. \square

Corollary 29 *Let N be a graded submodule of \bar{F} , and let $\{V_1, V_2, \dots, V_t\}$ be the output of SelfSatBA applied to a set of homogeneous generators of N .*

- (1) *We have $V_i = V_i^{\text{sat}}$ for $i = 1, \dots, t$.*
- (2) *The set $\{V_1, V_2, \dots, V_t\}$ is a $\bar{\sigma}$ -Gröbner basis of N^{sat} .*

Proof. Using Theorem 20 we deduce that $\{V_1, V_2, \dots, V_t\}$ is a $\bar{\sigma}$ -Gröbner basis of a graded submodule \tilde{N} of \bar{F} such that $\tilde{N}^{\text{sat}} = N^{\text{sat}}$. On the other hand, by construction SelfSatBA produces as output saturated vectors. Therefore $V_i = V_i^{\text{sat}}$ for $i = 1, \dots, t$. Now we can use the above theorem to deduce that $\{V_1, V_2, \dots, V_t\}$ is a homogeneous $\bar{\sigma}$ -Gröbner basis of $\tilde{N}^{\text{sat}} = N^{\text{sat}}$, and the proof is complete. \square

Corollary 30 *Let v_1, \dots, v_s be non-zero vectors in F , let M be the submodule generated by $\{v_1, \dots, v_s\}$, and let $\{V_1, V_2, \dots, V_t\}$ be the output of SelfSatBA applied to the set $\{v_1^{\text{hom}}, v_2^{\text{hom}}, \dots, v_s^{\text{hom}}\}$. Then $\{V_1, V_2, \dots, V_t\}$ is a $\bar{\sigma}$ -Gröbner basis of M^{hom} .*

Proof. It follows from Corollary 29.(2) and Proposition 10.(7). \square

Example 31 The following example shows that the above theorem and its corollary cannot be extended to \mathbb{N}^m -gradings defined by matrices with $m > 1$. The main reason is that (1) of the above theorem is not true anymore. Let $P = K[x_1, x_2, x_3]$ with

$$\sigma = \text{Ord} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

If we let $\bar{P} = K[y_1, y_2, x_1, x_2, x_3]$, we have

$$\bar{W} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{\sigma} = \text{Ord} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We consider the ideal I generated by $\{x_1x_3 - y_1y_2x_3, y_2x_2^2 - y_1x_1\}$ in \bar{P} . We check that the element $x_2^2x_3 - y_1^2x_3$ is not in I , but the element $y_2(x_2^2x_3 - y_1^2x_3)$ which is equal to $x_3(y_2x_2^2 - y_1x_1) + y_1(x_1x_3 - y_1y_2x_3)$ is in I and therefore the element $x_2^2x_3 - y_1^2x_3$ is in I^{sat} . Consequently, if we let $v = x_2^2 - x_1$, we see that $v^{\text{hom}} = y_2x_2^2 - y_1x_1$ and hence $\text{LT}_{\bar{\sigma}}(v^{\text{hom}}) \neq \text{LT}_{\sigma}(v)$. Moreover $\{x_1x_3 - y_1y_2x_3, y_2x_2^2 - y_1x_1\}$ is the reduced $\bar{\sigma}$ -Gröbner basis of I and both polynomials are saturated, but it cannot be the reduced $\bar{\sigma}$ -Gröbner basis of I^{sat} , since we have just seen that $I \neq I^{\text{sat}}$.

5 Strategies and Timings

In this paper we restrict our investigation and implementation in CoCoA to the case of the single grading. The implementation is just a prototype and it is planned to include its final form in the forthcoming CoCoA 5.

We have already mentioned that a way to compute a Gröbner basis with inhomogeneous data is to homogenize the input data, compute the Gröbner basis and then dehomogenize the result. This strategy is achieved using a Weak Self-Saturating Buchberger's Algorithm where the choice is to never saturate and choose the pair or generator of lowest degree in step (2a).

For degree compatible orderings and inhomogeneous input, the Self-Saturating Buchberger's Algorithm is nothing but the standard Buchberger's Algorithm with sugar. In step (2a) we choose the pair or generator of lowest sugar. The usage of homogeneous data makes the computation of the sugar slightly more complicated. The result is a small overhead.

Even if we said that we *always* saturate, we do not need to saturate after every reduction step, but we saturate only at the end when the vector (or polynomial) is no longer reducible, thus avoiding several operations of saturating.

The file containing the text of the examples discussed here can be found at <http://cocoa.dima.unige.it/research/papers/BigCabRob09.cocoa>

The `c7` example is the classical cyclic 7 system, non homogeneous. Examples `mora9`, `hairer2`, `Butcher` and `Kin1` are well known in the literature. Example `t51` is an implicitization problem. Example `Lex` is a zero dimensional ideal in a polynomial ring with three indeterminates, whose Gröbner basis is computed with respect to `Lex`, while `Elim` is an elimination problem with three polynomials in five indeterminates.

A and H stand for the sugar and homogeneous versions of the standard Buchberger algorithm respectively and *S* for the self-saturating version. For every example we examine some experimental data about the Buchberger's Algorithm performance, namely cardinality of a reduced Gröbner basis (before dehomogenizing in the H and S cases), the number of S-polynomials reduced and the number of pairs considered during a run, plus the time spent during the computation. The timings are for a special version of the CoCoALib-0.99

on a Intel Core2Duo system with 2MB RAM running Linux openSUSE 10.3.

	<u>c7</u>	<u>A</u>	<u>H</u>	<u>S</u>		<u>hairer2</u>	<u>A</u>	<u>H</u>	<u>S</u>
GBLen		209	443	209			72	506	72
PolyRed		2060	2199	2060			560	3149	560
PairsIns		61549	97910	61549			6905	127765	6905
Time		4.52s	3.18s	4.76s			1.26s	16.00s	1.40s
	<u>t51P</u>	<u>A</u>	<u>H</u>	<u>S</u>		<u>mora9</u>	<u>A</u>	<u>H</u>	<u>S</u>
GBLen		6	80	58			2266	3977	2552
PolyRed		76	239	191			22099	43513	25350
PairsIns		242	3160	1715			2657075	7906276	3368371
Time		3.40s	5.23s	0.64s			9.91s	33.05s	12.64s
	<u>Butch</u>	<u>A</u>	<u>H</u>	<u>S</u>		<u>Kin1</u>	<u>A</u>	<u>H</u>	<u>S</u>
GBLen		23	188	28			43	477	135
PolyRed		369	987	516			625	4418	1471
PairsIns		5635	17578	9256			10779	113526	23124
Time		1.85s	3.38s	1.40s			3.34s	10.80s	1.57s
	<u>Lex</u>	<u>A</u>	<u>H</u>	<u>S</u>		<u>Elim</u>	<u>A</u>	<u>H</u>	<u>S</u>
GBLen		4	122	122			99	353	353
PolyRed		409	345	345			845	1488	1488
PairsIns		1465	7381	7381			14330	62128	62128
Time		4.40s	0.59s	0.61s			68.80s	24.45s	24.56s

The first two Gröbner bases are computed with respect to the **DegRevLex** ordering; we notice that the self-saturating algorithm behavior is the same

as the standard algorithm, the only difference being some overhead in the saturating case, due to more complex sugar computations, as expected.

The last six Gröbner bases are computed with respect to lexicographic or elimination orderings; we see that in several cases the saturating algorithm offers an efficient alternative to the standard/homogenizing algorithm.

We also notice that the numbers entered into the tables, with the exception of the timings, offer a very partial indication of the complexity of Gröbner basis computations: only in two out of eight cases the fastest algorithm is the one with the lowest indicators. In fact, the possibility of using a large pool of reducers can lead to faster reductions and hence to an overall better performance of the algorithm itself, as it is clearly seen in most of the H and S versions.

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