Explicit Formulae for Power Utility Maximization Problems

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Abstract

We consider the problem of expected power utility maximization from terminal wealth in diffusion market models under partial information. Explicit expressions for the value-process and for the optimal strategy are obtained for some interesting special cases. In particular, a closed form solution is given in terms of a PDE with terminal condition for Markovian models. An illustration of the optimal strategy is provided by means of some numerical simulations.

Keywords: Backward stochastic differential equation; power utility maximization problem; basis risk model; stochastic correlation

1 Introduction

We deal with the issue of finding the optimal investment strategy, when trading takes place on a finite interval $[0, T]$ and the quality of the investment is measured by the expected power utility of the related terminal wealth. Moreover, the strategy is based on the investor's observations, which may not include all market information.

Utility maximization problems with partial information have been studied extensively and under various setups, to name but a few works dealing with this subject we mention [2, 3, 4, 5, 6, 9, 15, 16, 21, 24, 26, 29].

We focus on two Brownian models in which we are able to solve the dynamic control problem related to the portfolio optimization explicitly. These models can be seen as particular cases of the semimartingale model treated in [6] admitting an explicit solution. Indeed, in such cases the solution of the backward stochastic differential equation (BSDE) characterizing the value process related to the problem at issue and given in [6] admits an explicit expression. Moreover, for some specific examples also the optimal strategy can be written explicitly and simulated numerically.

We consider the power function $u(x) = \frac{x^p}{p}$, defined on $x > 0$ with $p < 0$ and constant relative risk aversion parameter $1 - p$, and solve the problem of maximizing the expected value of power utility of the value of the portfolio at the final time T. In mathematical terms, we consider the problem

$$\text{maximize} \quad E\left[\frac{(X_T^{x,\pi})^p}{p}\right] \quad \text{over all} \quad \pi \in \Pi, \tag{P1}$$

where the value of the portfolio in T, $X_T^{x,\pi}$, depends on x and π, respectively, the initial capital and the proportion of wealth invested in the risky asset. The proportion π is chosen in a certain class Π (to be defined in the sequel) of self-financing strategies adapted to the observed filtration. The process of returns associated to the risky asset is denoted by R_t and the usual assumption that there exists a bank account which pays no interests is made. Furthermore, without loss of generality, we set $x = 1$. It is not difficult to see that $X_t^{1,\pi} = X_t^{\pi} = 1 + \int_0^t X_{u^-}^{\pi} \pi_u dR_u$ describes the wealth process corresponding to the self-financing strategy π. Since X_t^{π} is the solution of a Doléans equation and $p < 0$, Problem (P1) can be stated in exponential form

$$\text{minimize} \quad E\left[\mathcal{E}_T^p(\pi \cdot R)\right] \quad \text{over all} \quad \pi \in \Pi, \tag{P2}$$

where $\mathcal{E}_t(X)$ indicates the Doléans-Dade exponential of X.

In Sections 2 and 3, two different Brownian settings are characterized by choosing the dynamics of the returns R_t and Problem (P2) is solved specifying the filtration representing the flow of observable information.

In Section 2, we consider a stochastic volatility model as in [7, 10, 22], where the coefficients of the dynamics of returns R_t depend on an observable factor η_t. We suppose that the two Brownian motions driving respectively R_t and the stochastic factor η_t have a stochastic correlation that depends only on the stochastic factor. A similar assumption is met and justified from an economic point of view in an example of [8]. One can think η_t as the price of a not traded asset (e.g. a volatility or a consumer price index) or of an asset tradable in principle but not traded in practice because of restrictions of some kind such as liquidity, legal issues, etc.
Furthermore, we suppose the agent have gaps in the observation of the returns and we compute the optimal strategy relying on the observable factor.
We characterize the solution of the problem through a Brownian BSDE with quadratic growth. If the correlation is constant, we obtain the expression of both the value process, by directly solving the BSDE, and of the optimal strategy. The proof heavily relies on the correlation being constant and it is enough that ρ_t is a deterministic function of time for it to fail.
When ρ_t is stochastic and not constant in time, we can still express the solution of the optimization problem at a fixed time t in a form which preserves the same structure obtained for constant ρ. In this case, the goal is reached by deriving lower and upper bounds for the value process and then by interpolation. This strategy is in the same spirit of [8], a major difference being that we derive lower and upper bounds by using BSDEs.
In Section 2.3 we deal with information sufficiency: we compare our results to the one obtained for a more general diffusion model in full information and discuss the conditions on the model under which the two power utility maximization problems yield the same solution. In particular, it turns out that the conditions we consider in the partial information setting are sufficient for getting the same result as in the full information case.
In the specific case of Markovian coefficients and constant ρ, we obtain the explicit formulae of the optimal strategy in terms of solutions of PDEs. The key point is the establishment of the connection between the BSDE for the value process and the classical Bellman equation for the value function related to the same problem. The PDE characterization allows by classical numerical results to simulate strategies and value functions. Moreover, we illustrate numerically risk aversion asymptotics of the optimal strategies proved in [6] and [23].

Section 3 is devoted to an application to the so called disorder problem (see, e.g., [29] and the literature therein). The dynamics of the asset returns (with constant volatility σ) is given by

$$dR_t = \mu I_{(t \geq \tau)} dt + \sigma dW_t,$$

i.e. the trend in the returns process changes value from 0 to some constant $\mu \neq 0$ at a random time τ with a given priori distribution. The observed filtration is the one generated by R_t, indicated by $\mathcal{F}^R$. The solution of Problem (P2) is given in terms of the a posteriori distribution $p_t = P(\tau \leq t | \mathcal{F}^R_t)$. In particular, the value function related to (P2) is proved to be the solution of a linear PDE. We conclude the section providing some numerical illustrations.

2 Itô model

We consider a continuous market model composed of a non risky asset whose price is supposed to be equal to 1 and of a risky asset. The dynamics of the risky asset returns is represented by an Itô process and is denoted by R_t, while η_t represents an observable factor. The dynamics of R_t and η_t

$$dR_t = \mu_t dt + \sigma_t dW^1_t, \tag{1}$$

$$d\eta_t = b_t dt + a_t dW_t \tag{2}$$

depend on two correlated Brownian motions W^1_t and W_t with stochastic correlation $\rho_t \in [-1, 1]$, $\rho_t dt = d\langle W^1, W\rangle_t$ defined on a complete probability space (Ω, F, P) equipped with a filtration $\mathcal{F}$. We suppose $\mathcal{F}$ is the $P-$augmented filtration generated by two independent Brownian motions W^1_t and W^0_t, i.e. $\mathcal{F} = (\mathcal{F}^{W^1,W^0}_t, t \in [0, T])$ and $F = \mathcal{F}^{W^1,W^0}_T$.
Thus, we can write the Brownian motion W as

$$W_t = \int_0^t \rho_s dW^1_s + \int_0^t \sqrt{1 - \rho^2_s} dW^0_s,$$

with ρ_t denoting the $\mathcal{F}_t-$adapted instantaneous correlation between W^1_t and W_t.

To emphasize the information available to the agent we denote by $\mathcal{G}$ the flow of observable events and rewrite the optimization problem (P2),

$$\text{minimize} \quad E\left[\mathcal{E}^p_T(\pi \cdot R)\right] \quad \text{over all} \quad \pi \in \Pi(\mathcal{G}). \tag{P3}$$

where the class of strategies depends on $\mathcal{G}$.

Then, we study the problem with $\mathcal{G} = \mathcal{F}$ to be termed “full information” case. After that, we compare the two situations and find necessary and sufficient conditions on the model under which the two problems give the same solution. We refer to this part of the section as “sufficiency of information”. We can roughly summarize our plan as follows: we solve the optimization Problem (P3) when the strategy is chosen in a proper set respectively of $\mathcal{F}^\eta$ (in the former case) and, of $\mathcal{F}$ (in the latter case) adapted processes, then we find the best strategy in closed form and compare the obtained results.

We start by considering $\mathcal{G} = \mathcal{F}^\eta$, which means the agent relies only on the observable factor η_t and not on the returns R_t. This could seem restrictive but it is an intermediate step before studying the more realistic situation of delays and gaps in the observation of the return, which is the object of our ongoing research.
We refer to the case $\mathcal{G} = \mathcal{F}^\eta$ as “partial information” and we solve Problem (P3) under suitable hypotheses (conditions 1)–5) below) on the model (1)-(2).

The class of strategies are chosen to be

$$\Pi(\mathcal{G}) = \{\pi : \mathcal{G} - \text{predictable}, \pi\sigma \cdot W^1 \in BMO(\mathcal{F})\} \tag{3}$$

and

$$V_t = \operatorname*{ess\,inf}_{\pi \in \Pi(\mathcal{G})} E[\mathcal{E}^p_{tT}(\pi \cdot R)|\mathcal{G}_t],$$

is the related value process. We use $\mathcal{E}_{tT}(X)$ to denote the ratio $\frac{\mathcal{E}_T(X)}{\mathcal{E}_t(X)}$.

Remark 2.1. We could use the class of strategies

$$\Pi(\mathcal{G}) = \{\pi : \mathcal{G} - \text{predictable such that } \mathcal{E}^p_{ts}(\pi\sigma \cdot W^1); t \le s \le T \in \text{ class } D\}$$

and prove the part of the theorem related to stochastic ρ exactly as in [8]. As it is apparent from the proof in that paper, a key technical point is based on the BMO property of the strategies, which is recovered through a localization procedure. We require it in the definition of the admissible strategies (3), since we use stochastic control techniques to give a BSDE characterization of the dynamic value of Problem (P3) (see [6]).

2.1 Partial information

Let us focus on $\mathcal{G} = \mathcal{F}^\eta$. We consider model (1)-(2) and assume μ_t, σ_t, a_t and b_t are non anticipative functionals such that

1) $\int_0^T \frac{\mu_t^2}{\sigma_t^2} dt$ is bounded,
2) μ_t, σ_t, a_t and b_t are $\mathcal{F}^\eta-$adapted,
3) $\sigma_t^2 > 0$, $a_t^2 > 0$,
4) equation (2) admits a unique strong solution,
5) ρ_t is $\mathcal{F}^\eta-$adapted.

Note that the dynamics of the returns process R_t is determined by the observable factor η_t and by W_t^1, since all coefficients in the model are $\mathcal{F}^\eta$ non anticipative functionals hence we can write, e.g., $\mu_t = \mu(t, \eta)$. In particular, we observe that, under conditions 1)–5), $\mathcal{F}^\eta = \mathcal{F}^W$ (see [17]) and $\mathcal{F}^{R,\eta} = \mathcal{F}^{W^1,W} \subseteq \mathcal{F}^{W^1,W^0} = \mathcal{F}$.

It is not difficult to see that under conditions 1)–5), Theorem 1 of [6] can be applied yielding the characterization of the value process related to the partial information problem, i.e. $\mathcal{G} = \mathcal{F}^\eta$, as the unique bounded positive solution of a BSDE. That result is adapted here to the Brownian setting. Let us denote by $q = \frac{p}{p-1}$ the exponent conjugate to p.

Theorem 2.1. *Let conditions* $1) - 5)$ *hold true. Then, the value process* V_t *is the unique solution of the BSDE*

$$Y_t = Y_0 + \frac{q}{2} \int_0^t \frac{(\theta_u Y_u + \psi_u \rho_u)^2}{Y_u} du + \int_0^t \psi_u dW_u, \quad Y_T = 1 \tag{4}$$

satisfying the two sided inequality

$$c \le Y_t \le C, \tag{5}$$

where c *and* C *are two constants not depending on* t *such that* $0 < c \le C \le 1$ *and*

$$\theta_t = \frac{\mu_t}{\sigma_t}$$

stands for the market price of risk. Moreover, the optimal strategy is of the form

$$\pi^*_t = \frac{(1-q)}{\sigma_t}\big(\theta_t + \frac{\rho_t \psi_t}{Y_t}\big), \tag{6}$$

where (Y, ψ) *is solution of* (4).

Remark 2.2. The process

$$\widetilde{V}_t = \operatorname*{ess\,inf}_{\pi\in\Pi(\mathcal{F}^\eta)} E[\mathcal{E}^p_{tT}(\pi\cdot\widehat{R})\ \exp\{\frac{p(p-1)}{2}\int_t^T \pi_u^2(1-\widehat{\rho}_u^2)\sigma^2 du\}|\mathcal{F}_t^\eta],$$

where $\widehat{X}_t = E\left[X_t|\mathcal{F}_t^\eta\right]$ denotes for the projection on the observable filtration, represents the value process related to a power utility optimization problem with a multiplicative correction factor.
We observe that assuming $\sigma_t^2 > 0$ and $\mathcal{F}^\eta-$adapted and that $\int_0^T \widehat{\theta}_t^2 dt$ is bounded, $\widetilde{V}_t$ is the unique bounded positive solution of the (4) where in place of the market price of risk θ and the instantaneous correlation ρ, appear respectively $\widehat{\theta}$ and $\widehat{\rho}_t$. This follows from a careful look at the proof of Theorem 1 of [6]. Under conditions 1)–5), it is proved to be equivalent to Problem (P3) (for details, see [6]).

Here and afterwards, with some abuse of notation, we often refer to Y as the solution of the BSDE (4), keeping in mind that the solution is the couple (Y,ψ).
The BSDE characterization of the dynamic value process V_t of Problem (P3) under partial information leads first to find an upper and a lower bound for V_t to be provided in Proposition 2.1. When ρ is deterministic, the two bounds coincide and we find an explicit form of the value process in Corollary 2.1. In contrast, if ρ_t is stochastic by interpolation we derive a formula for V_t (for fixed t) in Theorem 2.2.
Let us define the measure $\widetilde{Q}$ by

$$\frac{d\widetilde{Q}}{dP} = \mathcal{E}_T(-q\,\theta\cdot W^1) \tag{7}$$

and $\widetilde{W}$ denotes the $\widetilde{Q}$-Brownian motion $\widetilde{W}_t = W_t + q\int_0^t \rho_u\theta_u du$ with respect to $\mathcal{F}^\eta$. Indeed, $\widetilde{W}$ is also a $\widetilde{Q}-$Brownian motion with respect to $\mathcal{F}$.

Proposition 2.1. *Assume conditions* 1) − 5) *hold true. Then, the value process* V_t *related to Problem* (P3) *satisfies*

$$\left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\overline{\rho}^2)}{2}\int_t^T\theta_u^2du)}|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q\overline{\rho}^2}} \le V_t \le \left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\underline{\rho}^2)}{2}\int_t^T\theta_u^2du)}|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q\underline{\rho}^2}}, \tag{8}$$

where

$$\overline{\rho} = \sup_{u\ge t}\|\rho_u\|_{L^\infty} \quad \textit{and} \quad \underline{\rho} = \inf_{u\ge t}\|\rho_u\|_{L^\infty}. \tag{9}$$

Proof. Let Y_u be a solution of (4). By Ito's formula, we can write the BSDE satisfied by $Z_u = \ln Y_u$ under the measure $\widetilde{Q}$

$$dZ_u = \frac{q}{2}\theta_u^2 du + \bar{\psi}_u d\widetilde{W}_u - \frac{1}{2}(1-q\rho_u^2)\bar{\psi}_u^2 du, \quad Z_T = 0, \tag{10}$$

with $\bar{\psi}_u = \frac{\psi_u}{Y_u}$. We fix $t\in[0,T]$, and after multiplying the left and right side of (10) by the positive constant $1-q\overline{\rho}^2$, we integrate the BSDE from t to T. Using the terminal condition, we get to

$$\begin{aligned}-\frac{q}{2}\int_t^T(1-q\overline{\rho}^2)\theta_u^2du = &(1-q\overline{\rho}^2)Z_t + \int_t^T\left((1-q\overline{\rho}^2)\bar{\psi}_u d\widetilde{W}_u - \frac{1}{2}(1-q\overline{\rho}^2)^2\bar{\psi}_u^2du\right)\\ &- \frac{q}{2}(1-q\overline{\rho}^2)\int_t^T(\overline{\rho}^2-\rho_u^2)\bar{\psi}_u^2du.\end{aligned}$$

By the previous equation we immediately have

$$e^{-\frac{q}{2}\int_t^T(1-q\bar{\rho}^2)\theta_u^2 du} = Y_t^{1-q\bar{\rho}^2}\mathcal{E}_{tT}\left((1-q\bar{\rho}^2)\bar{\psi}\cdot\widetilde{W}\right)e^{-\frac{q}{2}(1-q\bar{\rho}^2)\int_t^T(\bar{\rho}^2-\rho_u^2)\bar{\psi_u}^2 du}. \tag{11}$$

We observe that with respect to $\widetilde{Q}$ $(1-q\bar{\rho}^2)\bar{\psi}\cdot\widetilde{W}$ is a $BMO(\mathcal{F}^\eta)-$martingale. This follows from (5), recalling that $\bar{\psi}_u = \frac{\psi_u}{Y_u}$, and Lemma 1 in [6] which states that the martingale part of the BSDE (4) is in BMO. $A_t = e^{-\frac{q}{2}(1-q\bar{\rho}^2)\int_0^t(\bar{\rho}^2-\rho_u^2)\bar{\psi}_u^2 du}$ is a bounded decreasing process and, by Theorem 2.3 in [13], $J_t = \mathcal{E}_t\left((1-q\bar{\rho}^2)\bar{\psi}\cdot\widetilde{W}\right)$ is a positive $\widetilde{Q}$ uniformly integrable martingale. Thus the product $J_tA_t(\bar{\rho})$ is a $\widetilde{Q}$ supermartingale. Considering the conditional expectation with respect to $\mathcal{F}_t^\eta$ in (11), we find a lower bound for Y_t, that is

$$Y_t^{1-q\bar{\rho}^2} \geq E^{\widetilde{Q}}\left[e^{-\frac{q}{2}\int_t^T(1-q\bar{\rho}^2)\theta_u^2 du}|\mathcal{F}_t^\eta\right].$$

The upper bound is obtained in a similar manner by using $\underline{\rho}$ instead of $\bar{\rho}$, in that case the product $J_tA_t(\underline{\rho})$ is a $\widetilde{Q}$ (generalized) submartingale and we reach the opposite inequality. □

As a corollary, we immediately obtain the explicit solution of BSDE (4) when ρ is constant, since $\underline{\rho} = \bar{\rho}$.

Corollary 2.1. *Assume conditions* $1)-4)$ *and suppose* ρ *is constant. Then, the value process* V_t *is equal to*

$$V_t = \left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\rho^2)}{2}\int_t^T\theta_u^2 du)}|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q\rho^2}}. \tag{12}$$

Moreover, the optimal strategy π^* *is identified by*

$$\pi_t^* = \frac{(1-q)}{\sigma_t}\left(\theta_t + \frac{\rho h_t}{(1-q\rho^2)(c+\int_0^t h_u d\widetilde{W}_u)}\right), \tag{13}$$

where h_t *is the integrand in the integral representation*

$$e^{-\frac{q(1-q\rho^2)}{2}\int_0^T\theta_t^2 dt} = c + \int_0^T h_t d\widetilde{W}_t. \tag{14}$$

Proof. The expression in (12) follows immediately from (8) since the lower and upper bounds of V_t coincide. Otherwise, one could easily repeat the proof of Proposition 2.1 observing that $J_tA_t(\rho) = J_t$ is a $\widetilde{Q}-$martingale.
To derive the optimal strategy, we use Theorem 2.1, which gives the closed form of the optimal strategy in terms of the solution of the BSDE (4). We want to find an expression for $\frac{\psi}{Y} = \bar{\psi}$ which appears in (6), using the integral representation (14) and (12).

More precisely, from (12) and (14) we can write

$$V_t = e^{\frac{q}{2}\int_0^t\theta_u^2 du}\left(c + \int_0^t h_u d\widetilde{W}_u\right)^{\frac{1}{1-q\rho^2}},$$

hence, $Z_t = \ln V_t$ satisfies the following BSDE

$$dZ_t = \frac{q}{2}\theta_t^2 dt + \frac{1}{1-q\rho^2}\left(\frac{h_t}{(c+\int_0^t h_u d\widetilde{W}_u)}d\widetilde{W}_t - \frac{h_t^2}{2(c+\int_0^t h_u d\widetilde{W}_u)^2}dt\right), \qquad Z_T = 0.$$

Comparing this BSDE with (10), we immediately get

$$\bar{\psi}_t = \frac{h_t}{(1-q\rho^2)(c+\int_0^t h_u d\widetilde{W}_u)}$$

and therefore (13). $\square$

Remark 2.3. If, furthermore μ and σ are constant, then $h = 0$ in (14) and the optimal strategy boils down to $\pi_t^* = \frac{(1-q)\mu}{\sigma^2}$, which keeps a constant proportion of money invested in the risky asset. The same strategy will be optimal when $\rho = 0$, since in that case the information on η will not affect the portfolio optimization problem.

In the opposite case of two perfectly correlated Brownian motions, i.e. $\rho^2 = 1$, (14) coincides with the necessary and sufficient condition for the $p-$optimal martingale measures to coincide with the minimal martingale measure (see Theorem 1 or Corollary 3 in [25]). Thus, our result

$$V_t = \left(E^{\widetilde{Q}}[e^{-\frac{q(1-q)}{2}\int_t^T \theta_u^2 du)}|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q}} \tag{15}$$

confirms the well known relation between portfolio optimization and the dual problem in terms of martingale measures (see, e.g., Proposition 2.3 in [18]). Indeed,

$$V_t = \left(E[\mathcal{E}_{tT}^q(-\theta\cdot W^1)|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q}},$$

where $\mathcal{E}_T(-\theta\cdot W^1)$ is the minimal martingale measure.

Let us now consider the case of a stochastic ρ. We can still express V_t in a form which preserves, for any ω, the same structure it has when ρ is constant. Indeed, adapting Theorem 1 in [8], we find the following result.

Theorem 2.2. *Under the assumptions* $1)-5)$*, there exists a* $\mathcal{F}_t^\eta-$*measurable random variable* $\widetilde{\rho}_t$ *taking values in the interval* $[\underline{\rho},\overline{\rho}]$*, defined in* (9)*, such that*

$$V_t(\omega) = \left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\rho^2)}{2}\int_t^T \theta_u^2 du)}|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q\rho^2}}\Big|_{\rho=\widetilde{\rho}_t(\omega)},$$

for almost all $\omega\in\Omega$.

Proof. We sketch the line of reasoning for sake of completeness (for the complete proof and details, we refer to [8]). Let us define $f(\cdot,\cdot):[\underline{\rho},\overline{\rho}]\times\Omega\to\mathbb{R}$ by

$$f(\rho,\omega) := \left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\rho^2)}{2}\int_t^T \theta_u^2 du}|\mathcal{F}_t^\eta](\omega)\right)^{\frac{1}{1-q\rho^2}}.$$

By condition 1), dominated convergence and Jensen's inequality, the function f admits a version which is continuous and non increasing in ρ for each ω. Using this version and the intermediate value theorem, we conclude that there exist a $\widetilde{\rho}_t$ such that for almost all $\omega\in\Omega$

$$f(\widetilde{\rho}_t(\omega),\omega) = V_t(\omega).$$

It is left to show that $\widetilde{\rho}_t$ is $\mathcal{F}_t^\eta$ measurable and in order to do this one can follow the proof of Theorem 1 in [8]. $\square$

2.2 Full information

Expected utility maximization problems have been studied under various degrees of generality in the case agents have full access to the market information.

It is well known (see, e.g, [14]) that in a general semimartingale model provided there exists an equivalent local martingale measure, the expected power utility maximization from terminal wealth admits a unique solution in the space of predictable $R-$integrable strategies π such that the corresponding wealth process is non negative. When the returns process is a continuous semimartingale, a characterization of the value process related to expected power utility maximization in terms of a BSDE can be found as, e.g., in [18] under quite general assumptions. Along the lines of Theorem 3.1 in [18], we can obtain the same result in our Brownian model (1)-(2).

In this section, we will assume that μ_t, σ_t are $\mathcal{F}^{W^1,W^0}$ progressively measurable processes such that

1) $\int_0^T \frac{\mu_t^2}{\sigma_t^2}dt$ is bounded,

2′) $\sigma_t^2 > 0$.

Note that condition 1), assumed also in Section 2.1, implies the existence of the minimal martingale measure (hence the set of martingale measures is not empty) and, since R_t is continuous, the assumptions of the Theorem 3.1 of [18] are satisfied. By standard dynamic programming arguments, the optimality principle (Proposition 2.1 in [18]) can be proved and we obtain the existence of a solution of the BSDE (16) by directly showing that the value process solves it, without resorting to general existence results from BSDEs' theory.

We refer to [18] for details. However, since we focus on the specific case $p < 0$ and since our choice of admissible strategies is different from that of [18], we include a proof of the following known result. Note that it could be obtained applying Corollary 2 of [6]. Nevertheless we believe that our proof shows that the choice of the set of admissible strategies is not really important for the conclusion of Proposition 2.2 to hold.

Proposition 2.2. *Assume conditions 1) and 2') hold true. Then the value process V_t^F, related to Problem* (P3) *when $\mathcal{G} = \mathcal{F}$, is characterized as the unique bounded positive solution of the following BSDE*

$$Y_t^F = Y_0^F + \frac{q}{2}\int_0^t \frac{(\theta_u Y_u^F + \psi_u^F)^2}{Y_u^F}du + \int_0^t \psi_u^F dW_u^1 + \int_0^t \psi_u^{\perp F} dW_u^0, \;\; Y_T^F = 1 \tag{16}$$

where W^0 is a $\mathcal{F}-$Brownian motion orthogonal to W^1 and $\mathcal{F} = \mathcal{F}^{W^0,W^1}$, and the optimal strategy is of the form

$$\pi_t^* = \frac{(1-q)}{\sigma_t}(\theta_t + \frac{\psi_t^F}{Y_t^F}), \tag{17}$$

with $(Y_t^F, \psi_t^F, \psi_t^{\perp F})$ solution of (16).

Proof. By [18], we know that there exists a unique bounded solution of the BSDE (16). Indeed, in Theorem 3.1 of [18] the unique bounded solution is provided by

$$\widetilde{V}_t = \operatorname*{ess\,inf}_{\pi \in \widetilde{\Pi}(\mathcal{F})} E[\mathcal{E}_{tT}^p(\pi \cdot R) \mid \mathcal{F}_t],$$

which represents the value process related to Problem (P2) where the set of admissible strategies is

$$\widetilde{\Pi}(\mathcal{F}) = \{\pi : \mathcal{F} - \text{predictable } s.t. \; E(\mathcal{E}_T^p(\pi \cdot R)) < \infty\}. \tag{18}$$

Note that condition 1) implies assumption B^* of Theorem 3.1 and that $\widetilde{V}_t$ is the unique solution bounded from below by a positive constant (by duality arguments) and from above by 1 (since the null strategy belongs to $\widetilde{\Pi}(\mathcal{F})$). Therefore, $0 < c \leq \widetilde{V}_t \leq 1$. Moreover, in [18], it is also shown that the strategy π_t^* defined in (17) is optimal and belongs to the class $\widetilde{\Pi}(\mathcal{F})$.

We want to show that $\widetilde{V}_t = V_t^F$.
To see $\widetilde{V}_t \geq V_t^F$, we first prove that π_t^* belongs to $\Pi(\mathcal{F})$.
Let Y^F be any solution of (16), satisfying $c \leq Y_t^F \leq 1$, where c is a positive constant. By writing Ito's formula for $(Y_t^F)^2$ and using the boundary condition, we show that the martingale part of Y^F belongs to $BMO(\mathcal{F})$ and, in particular, $\psi^F \cdot W^1$ does. Indeed, we have

$$
\begin{aligned}
(Y_T^F)^2 - (Y_\tau^F)^2 =& 1 - (Y_\tau^F)^2 \\
=& q \int_\tau^T (\theta_u Y_u^F + \psi_u^F)^2 du + \int_\tau^T 2Y_u^F (\psi_u^F dW_u^1 + \psi_u^{\perp F} dW_u^0) \\
&+ \int_\tau^T \left((\psi_u^F)^2 + (\psi_u^{\perp F})^2 \right) du,
\end{aligned}
$$

where τ is any $\mathcal{F}-$stopping time. Then

$$
\begin{aligned}
\int_\tau^T \left((\psi_u^F)^2 + (\psi_u^{\perp F})^2 \right) du \leq& 1 - q \int_\tau^T (\theta_u Y_u^F + \psi_u^F)^2 du \\
&- 2 \int_\tau^T Y_u^F (\psi_u^F dW_u^1 + \psi_u^{\perp F} dW_u^0) \\
\leq& 1 - 2 \int_\tau^T Y_u^F (\psi_u^F dW_u^1 + \psi_u^{\perp F} dW_u^0).
\end{aligned}
\tag{19}
$$

Without loss of generality we will assume that $\psi^F \cdot W^1 + \psi^{\perp F} \cdot W^0$ is a square integrable martingale, otherwise one can use localization arguments. By taking the conditional expectation in (19), we find that

$$
E[\int_\tau^T \left((\psi_u^F)^2 + (\psi_u^{\perp F})^2 \right) du | \mathcal{F}_\tau] \leq 1. \tag{20}
$$

Then,

$$
(1-q)^2 E[\int_\tau^T (\theta_u + \frac{\psi_u^F}{Y_u^F})^2 du | \mathcal{F}_\tau] \leq (1-q)^2 2E[\int_\tau^T (\theta_u^2 + \frac{\psi_u^{F2}}{Y_u^{F2}}) du | \mathcal{F}_\tau],
$$

which is bounded by a positive constant because of $Y^F \geq c$, condition 1) and (20). This proves that π_t^* belongs to $\Pi(\mathcal{F})$ and that $V_t^F \leq \widetilde{V}_t$.

Now, we prove the opposite inequality.
We consider the product $\widetilde{V}_t \mathcal{E}_t^p(\pi \cdot R)$ for any $\pi \in \Pi(\mathcal{F})$. By Ito' s formula, after some computations, one can write

$$
\widetilde{V}_t \mathcal{E}_t^p(\pi \cdot R) = \widetilde{V}_0 \mathcal{E}_t \big((p\pi\sigma + \frac{\psi^F}{\widetilde{V}}) \cdot W^1 + \frac{\psi^{\perp F}}{\widetilde{V}} \cdot W^0 \big) e^{\frac{p(p-1)}{2} \int_0^t \left(\pi_u \sigma_u - \frac{1}{(1-p)} (\frac{\mu_u}{\sigma_u} + \frac{\psi_u^F}{\widetilde{V}_u}) \right)^2 du},
$$

where we use $(\widetilde{V}_t, \psi_t^F, {\psi_t^{\perp}}^F)$ to denote the unique solution of (16). Since π belongs to $\Pi(\mathcal{F})$, $\widetilde{V}$ is bounded and by (20), we have that $p\pi\sigma + \frac{\psi^F}{\widetilde{V}} \cdot W^1 + \frac{{\psi^\perp}^F}{\widetilde{V}}$ is a $BMO(\mathcal{F})$ martingale and, by Theorem 2.3 in [13], the exponential martingale

$$\mathcal{E}_t\big((p\pi\sigma + \frac{\psi^F}{V}) \cdot W^1 + \frac{{\psi^\perp}^F}{V} \cdot W^0\big)$$

is uniformly integrable. $\widetilde{V}_t \mathcal{E}_t^p(\pi \cdot R)$ is a (generalized) submartingale, since it is the product of a positive uniformly integrable exponential martingale and a strictly positive increasing process. Therefore,

$$\widetilde{V}_t \le E(\mathcal{E}_{tT}^p(\pi \cdot R) \mid \mathcal{F}_t) \qquad \text{a.s.}$$

and

$$\widetilde{V}_t \le V_t^F \qquad \text{a.s.}.$$

Thus $\widetilde{V}_t$ coincides with V_t^F and, as consequence, V_t^F is the unique bounded positive solution of (16).

□

2.3 Sufficiency of information

In this section we discuss the conditions on the model which guarantee that the value processes related to the two problems with $\mathcal{G} = \mathcal{F}^\eta$ and $\mathcal{G} = \mathcal{F}$, respectively, coincide. The key point is played by the BSDE characterization of the value processes we gave previously.
We refer to [21] for related results in exponential hedging and in the continuous semimartingale setting. We stress that in our Brownian model, as we shall see, these conditions turn out to be concrete hypotheses on the model coefficients.

We recall the model (1)-(2) defined on the complete probability space (Ω, F, P), equipped with the $P-$augmented filtration $\mathcal{F} = (\mathcal{F}_t^{W^1,W^0}, t \in [0,T])$, where W_t^0 and W_t^1 are two independent Brownian motions. Moreover, W_t is a Brownian motion correlated with W^1 with instantaneous stochastic correlation ρ_t. We assume the observed information is given by the flow $\mathcal{F}^\eta$ and, for simplicity, we make the standing assumption $\mathcal{F}^\eta = \mathcal{F}^W$. $\mathcal{F}^\eta = \mathcal{F}^W$ holds, for instance, if $2), 3)$ and 4) are satisfied, but we are not requiring these conditions here.

Definition 2.1. We will say that the filtration $\mathcal{G}$ is sufficient for the optimization problem

$$\text{minimize} \quad E\left[\mathcal{E}_T^p(\pi \cdot R)\right] \quad \text{over all} \quad \pi \in \Pi(\mathcal{F}), \tag{P4}$$

with

$$\Pi(\mathcal{F}) = \{\pi : \mathcal{F} - \text{predictable}, \pi\sigma \cdot W^1 \in BMO(\mathcal{F})\},$$

if $V_0(\mathcal{G}) = V_0^F$ where (as before) V_t^F denotes the value process related to (P4) and

$$V_t(\mathcal{G}) = \operatorname*{ess\,inf}_{\pi \in \Pi(\mathcal{G})} E[\mathcal{E}_{tT}^p(\pi \cdot R) | \mathcal{G}_t].$$

Recall that $\Pi(\mathcal{G})$ is the subset of strategies in $\Pi(\mathcal{F})$ which are $\mathcal{G}-$predictable and note that $V_t(\mathcal{F}^\eta) = V_t$ and $V_t(\mathcal{F}) = V_t^F$.

Remark 2.4. If the information given by the flow $\mathcal{F}^\eta$ is sufficient, i.e.

$$V_0 = V_0^F,$$

then, the optimal strategy in the full information problem π^{*F} will be in $\Pi(\mathcal{F}^\eta)$ and

$$V_t^F = E\left[\mathcal{E}_{tT}^p(\pi^{*F}\cdot R)|\mathcal{F}_t\right].$$

Thus, we will have that the value V_t will be given by

$$V_t = E\left[\mathcal{E}_{tT}^p(\pi^{*F}\cdot R)|\mathcal{F}_t^\eta\right] = E\left[V_t^F|\mathcal{F}_t^\eta\right].$$

Moreover, if the returns process R is observable ($\mathcal{F}^\eta-$adapted), then

$$E\left[\mathcal{E}_{tT}^p(\pi^{*F}\cdot R)|\mathcal{F}_t\right] = E\left[\mathcal{E}_{tT}^p(\pi^{*F}\cdot R)|\mathcal{F}_t^\eta\right],$$

where the last equality is due to the fact that every $\mathcal{F}^\eta-$martingale is a $\mathcal{F}-$local martingale. Therefore $V_t = V_t^F$.

Theorem 2.3. *Suppose conditions* 1) *and* 2′) *are satisfied, and σ is $\mathcal{F}^\eta-$adapted. $V_t = V_t^F$ if and only if V_t^F satisfies the following BSDE*

$$Y_t = Y_0 + \frac{q}{2}\int_0^t \frac{(\widehat{\theta}_u Y_u + \psi_u\widehat{\rho}_u)^2}{Y_u}du + \int_0^t \psi_u dW_u, \quad Y_T = 1, \tag{21}$$

where the process ψ is $\mathcal{F}^\eta-$predictable and $\widehat{\theta}_t = E\left[\theta_t|\mathcal{F}_t^\eta\right]$ and $\widehat{\rho}_t = E\left[\rho_t|\mathcal{F}_t^\eta\right]$ are the projection of θ_t and ρ_t on $\mathcal{F}_t^\eta$.

Proof. Under conditions 1) and 2′), by Proposition 2.2, V_t^F is the unique (bounded strictly positive) solution of the BSDE (16) and the optimal strategy is (17).

Let $V_t^F = V_t$. Since $V_0^F = V_0$, the optimal strategy π^{F^*} is $\mathcal{F}^\eta-$predictable. Since σ is $\mathcal{F}^\eta-$adapted, from (17) we see that

$$\theta_t + \frac{\psi_t^F}{V_t^F} = \widehat{\theta}_t + \frac{\widehat{\psi}_t^F}{V_t^F}. \tag{22}$$

V^F is $\mathcal{F}^\eta-$adapted and, by (22) and (16) also the part of bounded variation is $\mathcal{F}^\eta-$adapted. Thus, the martingale part of V^F is $\mathcal{F}^\eta-$adapted and hence it is a $\mathcal{F}^\eta-$martingale. Under the standing assumption $\mathcal{F}^\eta = \mathcal{F}^W$, by the martingale representation theorem, we can write

$$\int_0^t \psi_u^F dW_u^1 + \int_0^t \psi_u^{\perp F} dW_u^0 = \int_0^t \psi_u dW_u, \tag{23}$$

where ψ is a $\mathcal{F}^\eta-$predictable process.
Taking the covariation of the left and right-hand side in (23) with respect to W^1 we obtain

$$\int_0^t \psi_u^F du = \int_0^t \psi_u\rho_u du.$$

Clearly $\widehat{\psi}_t^F = \psi_t\widehat{\rho}_t$, which plugged together with (23) and (22) into (16), shows that V_t^F is solution of the BSDE (21), which is considerably simpler than (16).

Viceversa, if V_t^F is solution of the BSDE (21), V_t^F is $\mathcal{F}^\eta-$adapted by Remark 2.2.

Since all the $\mathcal{F}^\eta-$martingale are $\mathcal{F}-$local martingale, the canonical decomposition given by the BSDE (21) coincides with the one given by the BSDE (16) (see, e.g., (9.27) in [12]). Comparing the parts of bounded variation it follows that

$$(\widehat{\theta}_t V_t^F + \psi_t \widehat{\rho}_t)^2 = (\theta_t V_t^F + \psi_t \rho_t)^2.$$

This gives $\theta_t V_t^F + \psi_t \rho_t$ and therefore the optimal strategy (17) are $\mathcal{F}^\eta-$adapted. Therefore,

$$V_t^F = E[\mathcal{E}_{tT}^p(\pi^{*F} \cdot R)|\mathcal{F}_t] = E[\mathcal{E}_{tT}^p(\pi^{*F} \cdot R)|\mathcal{F}_t^\eta],$$

the second equality since V_t^F is $\mathcal{F}^\eta-$adapted.

$\pi^{*F} \in \Pi(\mathcal{F}^\eta)$ we have

$$E[\mathcal{E}_{tT}^p(\pi^{*F} \cdot R)|\mathcal{F}_t^\eta] \geq \operatorname*{ess\,inf}_{\pi \in \Pi(\mathcal{F}^\eta)} E[\mathcal{E}_{tT}^p(\pi \cdot R)|\mathcal{F}_t^\eta] = V_t.$$

On the other end, $V_t^F \leq V_t$, therefore the equality $V_t^F = V_t$ follows.

□

Corollary 2.2. *If all the conditions of Theorem 2.1 are satisfied, then $V_t = V_t^F$.*

From Theorem 2.1, the process V_t is the unique bounded strictly positive solution of equation (21). Note that under 1)–5), $\widehat{\theta}_t = \theta_t$ and $\widehat{\rho}_t = \rho_t$.
Using $W_t = \int_0^t \rho_s dW_s^1 + \int_0^t \sqrt{1-\rho_s^2} dW_s^0$ in (21), we see that from the solution (Y_t, ψ_t) of the BSDE (21), we can define the triplet $Y_t^F = Y_t$, $\psi_t^F = \rho_t \psi_t$ and $\psi_t^{\perp F} = \sqrt{1-\rho_t^2}\psi_t$ and that the triplet $(Y_t^F, \psi_t^F, \psi_t^{\perp F})$ solves the BSDE (16).
Thus, by Proposition 2.2, $Y_t^F = V_t^F$ and so $V_t^F = V_t$.

Remark 2.5. When the correlation ρ is constant, conditions 1)–4) guarantee the filtration $\mathcal{F}^\eta-$ is sufficient for the problem P4. Indeed, as a consequence of Corollary 2.1, we have that if conditions 1)–4) hold and ρ constant, then $V_t^F = V_t$ and takes the form (12).

Indeed, one can prove that V_t satisfies (21) with

$$\psi_t = \frac{e^{\frac{q}{2}\int_0^t \theta_u^2 du}}{1-q\rho^2} \left(c + \int_0^t h_u d\widetilde{W}_u\right)^{\frac{q\rho^2}{1-q\rho^2}} h_t, \tag{24}$$

by Corollary 2.1, and then, as in Corollary 2.2, that $V_t = V_t^F$.

2.4 Markovian model

In this section, we assume conditions 1) − 4) and the correlation ρ to be constant. Thus, by Corollary 2.2, the problems in partial and full information are equivalent. Assuming Markovian coefficients in the model, we can rewrite the explicit formulas of the value process (12) and the optimal strategy (14) obtained in Corollary 2.1 in terms of the solution of a PDE.

We introduce the function

$$K(t,x) = E^{\widetilde{Q}}[e^{-\frac{q(1-q\rho^2)}{2}\int_t^T \theta^2(u,\eta_u)du} \mid \eta_t = x], \tag{25}$$

with $\widetilde{Q}$ defined in (7), so that the value process V_t is equal to $K(t,\eta_t)^{\frac{1}{1-q\rho^2}}$.

Proposition 2.3. *Assume conditions* $1) - 4)$ *and* ρ *constant. Then the value process takes the form* $V_t = (K(t, \eta_t))^{\frac{1}{1-q\rho^2}}$, *where* $K(t, x)$ *is the solution of the PDE*

$$K_t(t,x) + K_x(b(t,x) - q\rho\theta a(t,x)) + \frac{1}{2}a^2(t,x)K_{xx}(t,x) - K(t,x)(\frac{q}{2}(1-q\rho^2)\theta^2(t,x)) = 0, \quad (26)$$

with terminal condition $K(T, x) = 1$ *and* K_t, K_x *and* K_{xx} *denoting partial derivatives of* K. *Moreover, the optimal strategy* π_t^* *is identified by*

$$\pi_t^* = \frac{(1-q)}{\sigma(t,\eta_t)}\left(\theta(t,\eta_t) + \frac{K_x(t,\eta_t)}{(1-q\rho^2)K(t,\eta_t)}\rho a(t,\eta_t)\right),$$

where $K(t, x)$ *satisfies the PDE* (26).

Proof. We recall from Proposition 2.1 and (10), that $Z_t = \ln V_t$ satisfies the following BSDE

$$dZ_t = -\frac{1}{2}(\bar{\psi}_t^2(1-q\rho^2) - q\theta_t^2)dt + \bar{\psi}_t d\widetilde{W}_t$$

and $Z_T = 0$, where $\bar{\psi}_t = \frac{\psi_t}{Y_t}$ and $\widetilde{W}_t = W_t + q\rho\int_0^t \theta_u du$.
Then, integrating over the interval $[0, T]$ and multiplying by $(1 - q\rho^2)$, we obtain

$$(1-q\rho^2)Z_0 + (1-q\rho^2)\int_0^T \bar{\psi}_u d\widetilde{W}_u - \frac{(1-q\rho^2)^2}{2}\int_0^T \bar{\psi}_u^2 du = -\frac{q}{2}(1-q\rho^2)\int_0^T \theta_u^2(u,\eta_u)du$$

and by taking the exponential on both sides we have

$$c\mathcal{E}_T((1-q\rho^2)\bar{\psi}\cdot\widetilde{W}) = e^{-\frac{q}{2}(1-q\rho^2)\int_0^T \theta_u^2(u,\eta_u)du}, \quad (27)$$

where $c = e^{Z_0(1-q\rho^2)}$.
Now, (25) and (27) imply that

$$K(t,\eta_t) = ce^{\frac{q(1-q\rho^2)}{2}\int_0^t \theta^2(u,\eta_u)du}\mathcal{E}_t((1-q\rho^2)\bar{\psi}\cdot\widetilde{W}).$$

Hence, by the product rule, we have that $K(t, \eta_t)$ solves the following backward equation

$$dK(t,\eta_t) = K(t,\eta_t)(\frac{q(1-q\rho^2)}{2}\theta^2(t,\eta_t)dt + ((1-q\rho^2)\bar{\psi}_t d\widetilde{W}_t)), \quad K(T,\eta_T) = 1. \quad (28)$$

On the other hand, since the dynamics of η_t under $\tilde{Q}$ is

$$d\eta_t = b(t,\eta_t)dt + a(t,\eta_t)(d\widetilde{W}_t - q\rho\theta_t(t,\eta_t)dt),$$

by applying Ito's formula to $K(t, \eta_t)$, we can write

$$\begin{aligned} dK(t,\eta_t) =& K_x(t,\eta_t)(b(t,\eta_t)dt + a(t,\eta_t)(d\widetilde{W}_t - \rho q\theta(t,\eta_t)dt)) + K_t(t,\eta_t)dt \quad (29) \\ &+ \frac{1}{2}K_{xx}(t,\eta_t)a^2(t,x)dt. \end{aligned}$$

Comparing (28) with (29) and equalizing the martingale part and finite variation part we obtain that

$$\bar{\psi}_t(t,\eta_t) = \frac{K_x(t,\eta_t)a(t,\eta_t)}{(1-q\rho^2)K(t,\eta_t)},$$

and the optimal strategy π^* takes on the following form

$$\pi^*(t,\eta_t) = \frac{(1-q)}{\sigma(t,\eta_t)}\left(\theta(t,\eta_t) + \frac{K_x(t,\eta_t)}{(1-q\rho^2)K(t,\eta_t)}\rho a(t,\eta_t)\right),$$

where $K(t,x)$ satisfies the PDE

$$K(t,x)\frac{q(1-q\rho^2)}{2}\theta^2(t,x) = K_x(t,x)(b(t,x) - a(t,x)\rho q\theta(t,\eta_t)) + K_t(t,x) + \frac{1}{2}K_{xx}(t,x)a^2(t,x),$$

for each $t \in (0,T)$ and $K(T,x) = 1$.
By a classical result, under regularity conditions on the coefficients (see, e.g., [28]), there exists $K \in C^{1,2}([0,T]\times\mathbb{R})$, satisfying (26), with terminal condition $K(T,x) = 1$.

□

Remark 2.6. Note that the process h_t appearing in (14) can be written as

$$h(t,\eta_t) = e^{\frac{q}{2}(1-q\rho^2)\int_0^t \theta^2(u,\eta_u)du} K_x(t,\eta_t)a(t,\eta_t).$$

Example 2.1. In [1] and [11], the authors study insurance related derivatives based on some nontradable underlings, which are correlated with tradable assets. They calculate exponential utility-based indifference prices by using a representation in terms of solutions of BSDE with quadratic growth generators. In particular, in [1] they provide simple sufficient conditions for general BSDE to satisfy a Markov property and to be differentiable with respect to the initial condition of the forward equation with quadratic nonlinearity. The problem of numerical approximation for BSDE and the convergence of numerical approximation schemes is treated in [11]. For such systems of stochastic equations path regularity of the solution processes is instrumental.

Similarly to [1], we proved that the problem of p-power utility maximization leads to study a BSDE with quadratic growth.

For the Markovian model (1)-(2) studied in Section 2.4, we characterized the optimal strategy in terms of the PDE which we will use to simulate it in concrete examples. The advantage of using PDEs (instead of the BSDEs) is due to well known convergence of the discretized solution for numerical methods.
We consider the following example of the Markovian model (1)-(2):

$$\begin{aligned} dR_t =& 0.1\eta_t dt + 0.04 dW_t^1, \\ d\eta_t =& \eta_t dt + \eta_t dW_t. \end{aligned}$$

We choose $T = 1$, a constant correlation coefficient ρ and $p = -0.25$ (or equivalently $q = 0.2$).

In order to obtain the optimal strategy, it is sufficient to simulate the dynamic of η and solve the PDE equation by standard numerical techniques. In Figure 1 different paths of the optimal strategy related to different values of the correlation coefficient ρ are displayed.

FIGURE 1 AROUND HERE

According to the simulation, it seems the investor should invest up to 4 times his wealth on the risky asset at the beginning. This is clearly due to the influence of the initial data. Moreover, the simulation better performs as the time goes to maturity.

Since we have an explicit formula for the optimal strategy and a quite simple way to simulate it at our disposal, we keep ρ fixed and study the behavior of the strategy for different values of the risk aversion coefficient. The optimal strategy for the $p-$power utility optimization, for $p < 0$, is denoted by $\pi^*(p)$; $\pi^*(-\infty)$ and $\pi^*(0)$ stand for the extremal cases. It is well known that those extremal cases correspond to the optimal strategies related to the exponential and logarithmic utility maximization problems, respectively. See the Appendix and [6] for details.

Table 1 displays Monte Carlo estimates of the norm $\|((1-p)\pi^*(p) - \pi^*(-\infty)) \cdot \rho\sigma W\|_{\mathcal{H}^2}$ as the value of p varies. The simulation of $\pi^*(-\infty)$ has been obtained in the same manner by using the PDE in Eq. (21) in [21]. We can observe that as p tends to $-\infty$ the norm goes to 0.

p	**q**	$\|((1-p)\pi^*(p) - \pi^*(-\infty)) \cdot \rho\sigma W\|_{\mathcal{H}^2}$
-0.11	0.10	0.063
-0.43	0.30	0.036
-1.00	0.50	0.19
-2.33	0.70	0.06
-99.00	0.99	0.00

Table 1: Monte Carlo estimates of the norm as p varies $(p \to -\infty)$

This result is not surprising. Indeed, it is known from [6, 23], that the optimal strategies of the power utility problem converge (in $\mathcal{H}^2$) to that of the exponential one as $p \to -\infty$.

Similarly, the norm $\|(\pi^*(p) - \pi^*(0)) \cdot \rho\sigma W\|_{\mathcal{H}^2}$, as $p \to 0$ tends to 0 (see Table 2), confirming the theoretical findings of [23] (see the Appendix), which applies also to our particular model in the partial information setting.

p	**q**	$\|(\pi^*(p) - \pi^*(0)) \cdot \rho\sigma W\|_{\mathcal{H}^2}$
-2.33	0.70	0.024
-1.00	0.50	0.023
-0.43	0.30	0.008
-0.11	0.10	0.001
-0.01	0.01	0.000

Table 2: Monte Carlo estimates of the norm as p varies $(p \to 0)$

3 Application to the disorder problem

In this section power utility maximization is solved by deriving the explicit formula for the optimal strategy in the case the observation come from the asset prices (or by the returns).

The model is related to the so-called disorder problem: the drift of the asset return is supposed to change value (from 0 to a constant μ, $\mu \neq 0$) at a random time, which is not observable. So, the dynamics of the risky asset returns is

$$dR_t = \mu I_{(t \geq \tau)} dt + \sigma dW_t^1, \tag{30}$$

where W^1 and the random variable τ are defined on the complete probability space (Ω, F, P) equipped with the $P-$augmented filtration $\mathcal{F} = (\mathcal{F}_t, t \in [0,T])$, $F = \mathcal{F}_T$. The random variable τ has the following distribution

$$P(\tau = 0) = \bar{p}, \qquad \text{and} \qquad P(\tau > t \mid \tau > 0) = e^{-\gamma t}, \quad \text{for all } t \in [0,T]$$

for some known constants $\bar{p} \in [0,1]$ and $\gamma > 0$. We assume that the Brownian motion W^1 and the random variable τ are independent.
We remark that we observe the process R but we can not see when the random time τ occurs; equivalently, we can not distinguish wether the movements of R are due to the drift or to the diffusion component. The filtration $\mathcal{G}$ is here represented by $\mathcal{F}^R$, so that in this case the returns are $\mathcal{G}-$adapted.
This feature makes the portfolio maximization problem easier to solve: all we need is to rewrite the canonical decomposition of R with respect to the smaller filtration and solve the problem as in the full information case (see, e.g. [22] and [24]).

Denote by $p_t = P(\tau \leq t \mid \mathcal{F}_t^R)$ the a posteriori probability process. The canonical decomposition of R with respect to $\mathcal{F}^R$ is

$$dR_t = \mu p_t dt + \sigma dW_t,$$

where W is a $\mathcal{F}^R-$Brownian motion (the "innovation process", see [17]).

Thus, we consider Problem (P3) with $\mathcal{G} = \mathcal{F}^R$. Note that $\mathcal{F}^W \subseteq \mathcal{F}^R$. The characterization of the value process is

$$Y_t = Y_0 + \frac{q}{2} \int_0^t \frac{(\theta p_u Y_u + \psi_u)^2}{Y_u} du + \int_0^t \psi_u dW_u, \quad Y_T = 1, \tag{31}$$

where $\theta = \frac{\mu}{\sigma}$. Moreover, the optimal strategy is

$$\pi_t^* = \frac{(1-q)}{\sigma} \left(\theta p_t + \frac{\psi_t}{Y_t} \right). \tag{32}$$

This result can be easily deduced, e.g., from [24] or from our Proposition 2.2. Note that $\int_0^T \left(\frac{\mu p_t}{\sigma}\right)^2 dt \leq C$ and σ is a positive constant, so that 1) and 2') are satisfied and that any $\mathcal{F}^R-$martingale admits the representation as a stochastic integral with respect to the innovation process, i.e. $\int_0^t \psi_u dW_u$ (see Theorem 5.17 in [27]).

We keep the notation similar to that of Section 2; for instance, any martingale with respect to the observed filtration is written as a stochastic integral with respect to the Brownian motion W in both cases, but there are some differences. Here the observable filtration is the one generated by the returns R, which are known, whereas in the previous section they were not. Moreover, W (the innovation process) is a Brownian motion only under the observed filtration and not with respect to $\mathcal{F}$, while in Section 2 it was with respect to both.

From [27], it follows that the process p_t satisfies the stochastic differential equation

$$p_t = p_0 + \theta \int_0^t p_u(1-p_u)dW_u + \gamma \int_0^t (1-p_u)du. \tag{33}$$

Moreover, introduce the measure

$$\frac{d\overline{Q}}{dP} = \mathcal{E}_T(-q\theta \int_0^{\cdot} p_u dW_u).$$

In the following proposition we find an explicit expression for the value process and for the optimal strategy.

Proposition 3.1. *The value process of Problem* (P3) *admits the following expression*

$$V_t = E^{\overline{Q}}(e^{-\frac{q(1-q)}{2}\int_t^T \theta^2 p_u^2 du} | \mathcal{F}_t^R)^{\frac{1}{1-q}} \tag{34}$$

and the optimal strategy is equal to

$$\pi_t^* = \frac{(1-q)}{\sigma}\left(\theta p_t + \frac{h_t}{(1-q)(c+\int_0^t h_u dW_u)}\right), \tag{35}$$

where h_t is the integrand of the integral representation

$$e^{-\frac{q(1-q)}{2}\int_0^T \theta^2 p_u^2 du} = c + \int_0^T h_u d\widetilde{W}_u,$$

where $\widetilde{W}_t = W_t + q\theta \int_0^t p_u du$ is a $(\mathcal{F}^R, \overline{Q})-$Brownian motion.

Proof. Let us consider a positive bounded solution of the BSDE (31) Y_t.
By Ito's formula, setting $\bar{\psi} = \frac{\psi}{Y}$, we find that $Z_t = \ln Y_t$ is the solution of

$$Z_t = Z_0 + \frac{1}{2}\int_0^t \left(q\big(\theta p_u + \bar{\psi}_u\big)^2 - \bar{\psi}_u^2\right) du + \int_0^t \bar{\psi}_u dW_u, \;\; Z_T = 0. \tag{36}$$

Hence, using the boundary condition, we have

$$-\frac{q}{2}\int_0^T \theta^2 p_t^2 dt = Z_0 + \frac{(q-1)}{2}\int_0^T \bar{\psi}_t^2 dt + \int_0^T \bar{\psi}_t dW_t + q\theta \int_0^T p_t \bar{\psi}_t dt.$$

Note that by Girsanov theorem $\widetilde{W}_t = W_t + q\theta \int_0^t p_u du$ is a $(\mathcal{F}^R, \overline{Q})-$Brownian motion.
Hence, multiplying by $1-q$ and taking the exponential of both sides of the previous equality we obtain

$$e^{-\frac{q(1-q)}{2}\int_0^T \theta^2 p_u^2 du} = c\mathcal{E}_T((1-q)\bar{\psi} \cdot \widetilde{W}),$$

where $c = e^{(1-q)Z_0}$.
On the one hand, since $\mathcal{E}_t((1-q)\bar{\psi} \cdot \widetilde{W})$ is a $(\mathcal{F}^R, \overline{Q})-$martingale, we have that

$$E^{\overline{Q}}\left(\mathcal{E}_T((1-q)\bar{\psi} \cdot \widetilde{W}) | \mathcal{F}_t^R\right) = \mathcal{E}_t((1-q)\bar{\psi} \cdot \widetilde{W}). \tag{37}$$

On the other hand, by the representation theorem (see [17, Theorem 5.4]) we have

$$E^{\overline{Q}}\left(e^{-\frac{q(1-q)}{2}\int_0^T \theta^2 p_u^2 du} | \mathcal{F}_t^R\right) = c + \int_0^t h_u d\widetilde{W}_u. \tag{38}$$

Thus, taking into account that the Doléans exponential $\mathcal{E}_t((1-q)\bar{\psi}\cdot\widetilde{W})$ is solution of $dX_t = X_{t-}(1-q)\bar{\psi}_t d\widetilde{W}_t$, from (37) and (38) we obtain

$$\bar{\psi}_t = \frac{h_t}{(1-q)(c+\int_0^t h_u d\widetilde{W}_u)}.$$

Now, by (36) we find

$$Z_t - \frac{q}{2}\int_0^t \theta^2 p_u^2 du = Z_0 + \int_0^t \bar{\psi}_u d\widetilde{W}_u - \frac{(1-q)}{2}\int_0^t \bar{\psi}_u^2 du,$$

which, again multiplying by $(1-q)$ and taking the exponential, by (37) implies that

$$e^{(1-q)Z_t} = E^{\overline{Q}}\left(e^{-\frac{q(1-q)}{2}\int_t^T \theta^2 p_u^2 du}|\mathcal{F}_t^R\right).$$

Therefore, V_t is given by (34) and, by (32), the optimal strategy is equal to (35). □

Remark 3.1. The optimal strategy essentially relies on the a posteriori distribution. If we compare this strategy to the optimal one computed observing the filtration $\mathcal{F}$, i.e. knowing τ, in place of p_t we will have the function $I_{t\geq\tau}$. It is clear that the two strategies will be different unless $I_{t\geq\tau}$ is $\mathcal{F}^R-$adapted and this happens only if τ is deterministic since we assumed W^1 and τ independent.

Remark 3.2. Note that the expression (15) in Section 2, which describes the value process corresponding to the particular case $\rho^2 = 1$, resembles (34), where in place of θ_t, $\widetilde{Q}$ and $\mathcal{F}^\eta$ appear θp_t, $\overline{Q}$ and $\mathcal{F}^R$, respectively. Indeed, there are several analogies; in particular in the Itô model when $\rho^2 = 1$ the returns are observable.

Now, we use the fact that the process $(p_t, \mathcal{F}_t^R)$, solution of (33), is a strong Markov process. Indeed, by Proposition 3.1, we can characterize the value process as the solution of a linear PDE and we can express the optimal strategy in terms of the solution of this linear PDE and of its partial derivatives. The following proposition is reminiscent Proposition 2.3, in which we assumed η to be Markovian.

Proposition 3.2. *The value process related to Problem* (P3) *takes on the form* $V_t = (K(t,p_t))^{\frac{1}{1-q}}$, *where* p_t *satisfies* (33) *and* $K(t,x)$ *is* $C^{1,2}([0,T]\times(0,1))$ *and satisfies*

$$-\frac{q(1-q)}{2}\theta^2 x^2 K(t,x) + K_x(t,x)(1-x)(\gamma - q\theta^2 x^2) + K_t(t,x) + \frac{1}{2}K_{xx}(t,x)\theta^2 x^2 (1-x)^2 = 0, \quad (39)$$

with the final condition $K(T,x) = 1$.
Besides, the optimal strategy is equal to

$$\pi_t^* = \frac{(1-q)}{\sigma}\left(\theta p_t + \frac{K_x(t,p_t)\theta p_t(1-p_t)}{(1-q)K(t,p_t)}\right). \quad (40)$$

Proof. Since the process $(p_t, \mathcal{F}_t^R)$, solution of (33), is a strong Markov process, by Proposition 3.1 one has $V_t = (K(t,p_t))^{\frac{1}{1-q}}$, with

$$K(t,x) = E^{\overline{Q}}\left(e^{-\frac{q(1-q)}{2}\theta^2\int_t^T p_u^2 du}|p_t = x\right). \quad (41)$$

By (33), under the measure $\overline{Q}$, the process p_t satisfies the stochastic differential equation

$$p_t = p_0 + \theta \int_0^t p_u(1-p_u)d\widetilde{W}_u + \int_0^t \left(\gamma - q\theta^2 p_u^2\right)(1-p_u)du. \tag{42}$$

Using Ito's formula

$$\begin{aligned} d(K(t,p_t)e^{-\frac{q(1-q)}{2}\theta^2\int_0^t p_u^2 du}) = e^{-\frac{q(1-q)}{2}\theta^2\int_0^t p_u^2 du}\Big(&K_x(t,p_t)\theta p_t(1-p_t)d\widetilde{W}_t \\ &+ \big(-\frac{q(1-q)}{2}\theta^2 p_t^2 K(t,p_t) + (1-p_t)(\gamma - q\theta^2 p_t^2)K_x(t,p_t) \\ &+ K_t(t,p_t) + \frac{1}{2}K_{xx}(t,p_t)\theta^2 p_t^2(1-p_t)^2\big)dt\Big), \end{aligned} \tag{43}$$

where K_t, K_x, K_{xx} are partial derivatives of K.
Since $K(t,p_t)e^{-\frac{q(1-q)}{2}\theta^2\int_0^t p_u^2 du}$ is a $(\mathcal{F}^R, \overline{Q})-$martingale by (38), (41) and (43) we have that $K(t,x)$ satisfies (39) with the final condition $K(T,x) = 1$.
The existence of a solution $K \in C^{1,2}([0,T]\times\mathbb{R})$ of (39) follows from [28].

Hence, since $Z_t = \frac{1}{1-q}\ln K(t,p_t)$ by (34) and (41), equalizing the martingale part of the two expressions, we find

$$\bar{\psi}_t = \frac{K_x(t,p_t)\theta p_t(1-p_t)}{K(t,p_t)(1-q)}. \tag{44}$$

Substituting (44) in (32) we obtain the optimal strategy (40). □

Example 3.1. Similarly to Example 2.1, in order to simulate the optimal strategy we use an algorithm based on the theory of the finite elements to find a solution of the PDE and simulate the process p by (33). In Figure 2, we plot a possible path of the optimal strategy π^* for the power utility maximization with $q = 0.8$ and with the choice of parameters in the disorder problem $\mu = 0.3$ $\sigma = 0.4$ and $\gamma = 0.5$.

FIGURE 2 AROUND HERE

We can observe that the trend depends strongly on the choice of the parameters. Indeed, the stopping time is related to p_t, whose dynamic follows (42) (with $p_0 = 0.1$).

4 Appendix

In [6, 23], the authors proved the convergence of the optimal strategies related to the power utility maximization to the one of the exponential as the power risk aversion goes to infinity. The results are obtained in the continuous semimartingale setting for partial and full information, respectively. In [23], the author also studies the convergence of the optimal strategies to the logarithmic one as the risk aversion tends to 1 (in full information). An analogous result holds in the partial information case. We briefly summarize the theoretical results, which we have referred to in the examples of Section 2.1.

In the exponential utility maximization problem

$$\text{maximize} \quad E\left[-\exp(-X_T^{x,\theta})\right] \quad \text{over all} \quad \theta \in \Pi(\mathcal{G}), \tag{P5}$$

$X_T^{x,\theta}$ denotes the final value of a portfolio starting from the initial capital x and the strategy θ is the dollar amount invested in the asset. The characterization of the optimal strategy θ^* can be found in [21]. In [6], under some technical assumptions that are satisfied if $1)-5)$ hold true, the $p-$optimal strategy $\pi^*(p)$ (in terms of proportion) scaled by the risk aversion coefficient $(1-p)$ is proved to converge to the optimal exponential strategy θ^*, where the convergence is understood in the norm $BMO(\mathcal{F})$ (recall the class $\Pi(\mathcal{G})$). More precisely we refer to (1)-(2), and translating Corollary 3 of [6] to our setting, we have the following

Theorem 4.1. *Let conditions* $1)-5)$ *be satisfied. Then*

$$\| ((1-p)\pi^*(p) - \theta^*) \cdot \widehat{M} \|_{BMO(F)} \to 0 \quad as\, p \to -\infty.$$

where $\widehat{M}$ *is* $\rho\sigma \cdot W$*.*

This kind of convergence may seem surprising at first glance. A nice heuristic argument which explains it through the convergence of an auxiliary sequence of shifted power utilities is in [23].

With reference to the logarithmic utility maximization problem

$$\text{maximize} \quad E\left[\log(X_T^{x,\pi})\right] \quad \text{over all} \quad \pi \in \Pi(\mathcal{G}), \tag{P6}$$

here the strategy represents the proportion of wealth invested in the asset (as for the power utility). Similarly to [6], it can be proved that the value process related to (P6) can be characterized by a BSDE and that the optimal strategy is of the form

$$\pi_t^* = \frac{\mu_t}{\sigma_t}.$$

Moreover, the following convergence result can be shown

Theorem 4.2. *Let conditions* $1)-3)$ *be satisfied. Then*

$$\|(\pi^*(p) - \pi^*) \cdot \widehat{M}\|_{BMO(\mathcal{F})} \to 0, \qquad p \to 0,$$

where $\widehat{M}$ *is* $\rho\sigma \cdot W$*.*

The convergence of the strategy in $BMO(\mathcal{F})$ implies the convergence in $\mathcal{L}^2(\rho\sigma \cdot W)$ and this one is the convergence illustrated in Example 2.1, where we denote θ^* with $\pi^*(-\infty)$ and π^* with $\pi^*(0)$.

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