MULTIPLE CATEGORIES OF GENERALISED QUINTETS

MARCO GRANDIS AND ROBERT PARÉ

Abstract.

We construct various multiple categories, based on generalised Ehresmann quintets. The main construction is a multiple category whose objects are all the 'lax' multiple categories; the transversal arrows are their strict multiple functors while the arrows in a positive direction are multiple functors of a 'mixed laxity', varying from the lax ones (in direction 1) to the colax ones (in direction ∞).

Introduction

This paper is about strict, weak and lax multiple categories, a higher dimensional extension of double categories that we have studied in the articles [GP6 - GP10]. The first two of them are about the 3-dimensional case, where *intercategories* (a kind of lax multiple category) cover and combine diverse structures like duoidal categories, Gray categories, Verity double bicategories and monoidal double categories.

An infinite dimensional multiple category has objects, i-directed arrows in each direction $i \in \mathbb{N}$, ij-cells of dimension two for all $i < j$, and so on. The *transversal* direction $i = 0$ always has a categorical composition (strictly associative and unitary), while the composition in a *geometric* direction $i > 0$ can be weak, i.e. associative and unitary up to invertible transversal comparisons. The transversal composition has a strict interchange with all the geometric ones, but the latter have ij-interchangers, which are assumed to be invertible in the case of weak multiple categories; more generally, chiral multiple categories and intercategories have directed ij-interchangers, for $i < j$. An n-dimensional multiple category has indices in the ordinal $\mathbf{n} = \{0, 1, ..., n-1\}$. The definitions can be found in [GP6] for the 3-dimensional case, and in [GP8] for the general one; they are briefly sketched in Section 1.1, below.

Many examples of infinite-dimensional weak multiple categories studied in the previous papers are *of cubical type*: loosely speaking, all the positive directions are equivalent. For instance this happens for all weak multiple categories $\mathsf{Span}(\mathbf{C})$ and $\mathsf{Cosp}(\mathbf{C})$, of 'cubical' spans and cospans over a category $\mathbf{C}$ [GP8]. Even many chiral examples, like the chiral triple category $\mathsf{SC}(\mathbf{C})$ of spans and cospans over $\mathbf{C}$ (and its infinite dimensional extensions) only have two kinds of arrows in positive directions: either spans or cospans.

Here we construct various (strict) multiple categories of highly non-cubical character, where the arrows in each direction are of different kinds.

The starting point is Ehresmann's double category of quintets $\mathbb{Q}(\mathbf{C})$ over a 2-category

2000 Mathematics Subject Classification: 18D05, 55U10.
Key words and phrases: multiple category, weak double category, cubical set, quintet.

C, whose horizontal and vertical maps are the maps of **C**, with double cells defined by 2-cells of **C**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\swarrow\varphi"] & B \arrow[d, "v"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \qquad\qquad \varphi\colon vf \to gu\colon A \to D. \tag{1}$$

Their horizontal and vertical compositions are obvious. This construction can be easily extended to a multiple category $\mathsf{Q}(\mathbf{C})$ of *higher quintets of* **C** (see Section 1.2). It is of cubical type, but gives a mould in which we can cast various multiple categories 'of generalised quintets', no longer of cubical type.

Outline. In Section 1, after constructing the multiple category $\mathsf{Q}(\mathbf{C})$ *of higher quintets* over a 2-category **C**, we define the notion of a multiple category of generalised quintets, or - more particularly - of quintet type (Section 1.4).

Section 2 contains our main result: the construction of the multiple category Cmc of chiral multiple categories, indexed by the ordinal $\boldsymbol{\omega} + \mathbf{1} = \{0, 1, ..., \infty\}$. Here the transversal arrows are strict multiple functors while, in direction p (for $1 \leqslant p \leqslant \infty$), the *p-morphisms* are 'multiple functors of mixed laxity', including the lax ones (in direction 1) and the colax ones (in direction ∞). Similar frameworks are concerned with intercategories, and the n-dimensional case.

The next two sections construct a multiple category $\mathsf{GQ}(\mathcal{C})$ *of generalised quintets* over a sequence $\mathcal{C}$ of 2-categories and 2-functors. In particular, a 2-category **C** gives a multiple category $\mathsf{Adj}(\mathbf{C})$ where an i-directed arrow is a chain $u_0 \dashv u_1 ... \dashv u_i$ of consecutive adjunctions in **C**, and a multiple category $\mathsf{Bnd}(\mathbf{C})$ where an i-directed arrow is a bundle $u = (u_0, ..., u_i)$ of parallel arrows of **C**.

Finally in Section 5 we construct a triple category of pseudo algebras, for a 2-monad; it is again of quintet type. The Theorems 5.7 and 5.8 show how algebras and normal pseudo algebras of graphs of categories are related to strict and weak double categories.

Literature. Strict double and multiple categories were introduced and studied by C. Ehresmann and A.C. Ehresmann [Eh, BE, EE1, EE2, EE3]. Strict cubical categories can be seen as a particular case of multiple categories (as shown in [GP8]); their links with strict ω-categories are made clear in [BrM, ABS]. The theory of weak double categories (or pseudo double categories) is analysed in our series [GP1 - GP4], and in other papers like [DP1, DP2, DPP1, DPP2, Fi, FGK1, FGK2, Ga, GP11, Ko1, Ko2, Ni, P1, P2]. For weak cubical categories see [G1, G2, G3, GP5].

Notation. We mainly follow the notation of [GP8 - GP10]. The symbol $\subset$ denotes weak inclusion. Categories and 2-categories are generally denoted as $\mathbf{A}, \mathbf{B}$...; weak double categories as $\mathbb{A}, \mathbb{B}$...; weak or lax multiple categories as A, B... More specific points are recalled below, in Section 1.

Acknowledgements. This work was partially supported by GNSAGA, a research group of INDAM (Istituto Nazionale di Alta Matematica), Italy.

1. Higher quintets

After constructing a multiple category $\mathsf{Q}(\mathbf{C})$ *of (higher) quintets* over a 2-category $\mathbf{C}$, we define multiple categories of generalised quintets, and - more particularly - of quintet type.

In a 2-category the vertical composition of 2-cells is written as $\varphi \otimes \psi$, in diagrammatic order; the whisker composition of arrows and cells by juxtaposition or dots.

1.1. Notation. The definitions of weak and chiral multiple categories can be found in [GP8], or - briefly reviewed - in [GP9], Section 1. Here we only give a sketch of them, while recalling the notation we are using.

The two-valued index α (or β) varies in the set $2 = \{0, 1\}$, also written as $\{-, +\}$.

A *multi-index* $\mathbf{i}$ is a finite subset of $\mathbb{N}$, possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it is understood that $\mathbf{i}$ is finite; writing $\mathbf{i} = \{i_1, ..., i_n\}$ it is understood that $\mathbf{i}$ has n distinct elements, written in the natural order $i_1 < i_2 < ... < i_n$; the integer $n \geqslant 0$ is called the *dimension* of $\mathbf{i}$. We write:

$$\begin{aligned} &\mathbf{i}j = j\mathbf{i} = \mathbf{i} \cup \{j\} && (\text{for } j \in \mathbb{N} \setminus \mathbf{i}), \\ &\mathbf{i}|j = \mathbf{i} \setminus \{j\} && (\text{for } j \in \mathbf{i}). \end{aligned} \tag{2}$$

For a weak multiple category A, the set of $\mathbf{i}$-cells $A_\mathbf{i}$ is written as A_*, A_i, A_{ij} when $\mathbf{i}$ is $\varnothing$, $\{i\}$ or $\{i, j\}$ respectively. Faces and degeneracies, satisfying the *multiple relations*, are denoted as

$$\partial_j^\alpha \colon X_\mathbf{i} \to X_{\mathbf{i}|j}, \qquad e_j \colon X_{\mathbf{i}|j} \to X_\mathbf{i} \qquad\qquad (\text{for } \alpha = \pm,\ j \in \mathbf{i}). \tag{3}$$

The *transversal direction* $i = 0$ is set apart from the positive, or *geometric*, directions. For a *positive multi-index* $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, the *augmented multi-index* $0\mathbf{i} = \{0, i_1, ..., i_n\}$ has dimension $n + 1$, but both $\mathbf{i}$ and $0\mathbf{i}$ have *degree* n. An $\mathbf{i}$-cell $x \in A_\mathbf{i}$ of A is also called an $\mathbf{i}$-*cube*, while a $0\mathbf{i}$-cell $f \in A_{0\mathbf{i}}$ is viewed as an $\mathbf{i}$-*map* $f \colon x \to_0 y$, where $x = \partial_0^- f$ and $y = \partial_0^+ f$. Composition in direction 0 is categorical (and generally realised by ordinary composition of mappings); it is written as $gf = f +_0 g$, with identities $1_x = \mathrm{id}(x) = e_0(x)$.

The *transversal category* $\mathrm{tv}_\mathbf{i}(\mathsf{A})$ consists of the $\mathbf{i}$-cubes and $\mathbf{i}$-maps of A, with transversal composition and identities. Their family forms a multiple object in $\mathbf{Cat}$, indexed by the positive multi-indices.

Composition of $\mathbf{i}$-cubes and $\mathbf{i}$-maps in a *positive* direction $i \in \mathbf{i}$ (often realised by pullbacks, pushouts, tensor products, etc.) is written in additive notation

$$\begin{aligned} &x +_i y && (\partial_i^+ x = \partial_i^- y), \\ &f +_i g \colon x +_i y \to x' +_i y' && (f \colon x \to x',\ g \colon y \to y',\ \partial_i^+ f = \partial_i^- g). \end{aligned} \tag{4}$$

The transversal composition has a strict interchange with each of the positive operations. The latter are *categorical* and satisfy the *interchange law* up to transversally-invertible comparisons (for $0 < i < j$, see [GP8], Section 3.2)

$$
\begin{aligned}
&\lambda_i x\colon (e_i\partial_i^- x) +_i x \to_0 x && (\textit{left } i\textit{-unitor}),\\
&\rho_i x\colon x +_i (e_i\partial_i^+ x) \to_0 x && (\textit{right } i\textit{-unitor}),\\
&\kappa_i(x,y,z)\colon x +_i (y +_i z) \to_0 (x +_i y) +_i z && (i\textit{-associator}),\\
&\chi_{ij}(x,y,z,u)\colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u) && \\
& && (ij\textit{-interchanger}).
\end{aligned}
\tag{5}
$$

The comparisons are natural with respect to transversal maps; λ_i, ρ_i and κ_i are special in direction i (i.e. their i-faces are transversal identities) while χ_{ij} is special in both directions i, j; all of them commute with ∂_k^α for $k \neq i$ (or $k \neq i, j$ in the last case). Finally the comparisons must satisfy various conditions of coherence, listed in [GP8], Sections 3.3 and 3.4.

More generally for a *chiral multiple category* A the ij-interchangers χ_{ij} are not assumed to be invertible (see [GP8], Section 3.7).

Even more generally, in an *intercategory* we also have ij-interchangers $\mu_{ij}, \delta_{ij}, \tau_{ij}$ involving the units; this extension is studied in [GP6, GP7] for the 3-dimensional case, the really important one. The infinite dimensional case is introduced in [GP8], Section 5, but lacks examples of interests and has a marginal position in [GP8 - GP10], as well as here.

Lax multiple functors and their transversal transformations are defined in [GP8], Section 3.9.

1.2. A FRAMEWORK OF HIGHER QUINTETS. We start from a 2-category $\mathbf{C}$ and construct a multiple category $\mathsf{M} = \mathsf{Q}(\mathbf{C})$ *of (higher) quintets* over $\mathbf{C}$, extending the double category $\mathbb{Q}(\mathbf{C})$ of quintets introduced by C. Ehresmann (see [GP1], Section 1.3).

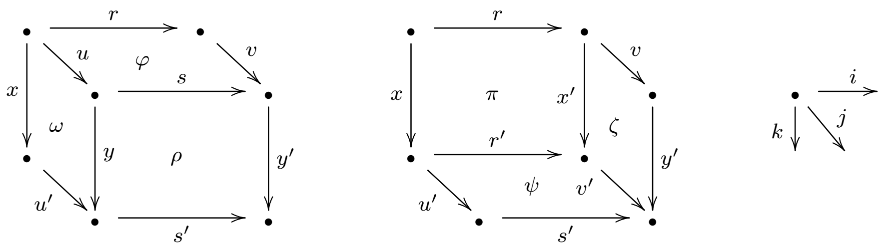
(a) The objects of M are those of $\mathbf{C}$; in every direction $i \geqslant 0$, an i-cell $f\colon X \to_i Y$ is a $\mathbf{C}$-morphism. They form the category $\mathrm{cat}_i(\mathsf{M})$ underlying $\mathbf{C}$.

(b) In dimension 2, an ij-cell (for $0 \leqslant i < j$) is a *quintet* of $\mathbf{C}$, consisting of four morphisms and a 2-cell φ

$$
\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\swarrow\varphi"] & B \arrow[d, "v"] \\
C \arrow[r, "s"'] & D
\end{tikzcd}
\qquad \varphi\colon vr \to su\colon A \to D. \qquad
\begin{tikzcd}
\bullet \arrow[r, "i"] \arrow[d, "j"'] & {} \\
{} &
\end{tikzcd}
\tag{6}
$$

These cells have two obvious composition laws, in directions i and j, and form a double category $\mathrm{dbl}_{ij}(\mathsf{M})$; it is the same as the double category $\mathbb{Q}(\mathbf{C})$, 'displayed' in directions i and j.

(c) In dimension 3 an ijk-cell (for $0 \leqslant i < j < k$) is a 'commutative cube' Π of quintets



(7)

More precisely, we have six quintets

$$\begin{array}{lll} \varphi\colon vr \to su, & \psi\colon v'r' \to s'u' & \text{(two } ij\text{-cells, the faces } \partial_k^\alpha\Pi), \\ \pi\colon x'r \to r'x, & \rho\colon y's \to s'y & \text{(two } ik\text{-cells, the faces } \partial_j^\alpha\Pi), \\ \omega\colon yu \to u'x, & \zeta\colon y'v \to v'x' & \text{(two } jk\text{-cells, the faces } \partial_i^\alpha\Pi). \end{array}$$

which must satisfy the following commutativity relation, in $\mathbf{C}$

$$y'\varphi \otimes \rho u \otimes s'\omega = \zeta r \otimes v'\pi \otimes \psi x\colon y'vr \to s'u'x. \tag{8}$$

The compositions in directions i, j, k amount to compositions of faces in the double categories $\mathrm{dbl}_{ij}(\mathsf{M})$, $\mathrm{dbl}_{ik}(\mathsf{M})$ and $\mathrm{dbl}_{jk}(\mathsf{M})$. We have thus a triple category $\mathrm{trp}_{ijk}(\mathsf{M})$.

(d) Every cell of dimension $n > 3$ is an n-dimensional cube whose 2-dimensional faces are quintets, under the condition that each 3-dimensional face in direction ijk be an ijk-cell, as defined above.

1.3. Coskeletal dimension. We want to express point (d) above saying that the multiple category $\mathsf{Q}(\mathbf{C})$ has *coskeletal dimension* 3, at most. Loosely speaking, this means that each cell of higher dimension is 'bijectively determined' by its boundary.

To formalise this property, we say that an *n-dimensional $\mathbb{N}$-multiple category* A is the 'same' as a multiple category, but based on multi-indices $\mathbf{i} = \{i_1, ..., i_k\} \subset \mathbb{N}$ of dimension $k \leqslant n$.

There is an obvious truncation functor

$$\mathrm{Dtrc}_n\colon \mathbf{Mlt} \to \mathrm{Dim}_n\mathbf{Mlt}, \tag{9}$$

defined on the category of (small) multiple categories (and strict multiple functors), with values in the category of n-dimensional $\mathbb{N}$-multiple categories. Its right adjoint

$$\mathrm{Dcosk}_n\colon \mathrm{Dim}_n\mathbf{Mlt} \to \mathbf{Mlt}, \tag{10}$$

extends an n-dimensional $\mathbb{N}$-multiple category A by adding cells of every higher dimension, with their faces, operations and degeneracies.

The higher cells are inductively defined as follows. If we have already defined the cells up to dimension $k \geqslant n$, an $\mathbf{i}$-cell of dimension $k+1$ is a *face-consistent* $\mathbf{i}$-*family* $x = (x_i^\alpha)$ of k-dimensional cells, where

$$\begin{array}{ll} x_i^\alpha \ \text{ is an } \mathbf{i}|i\text{-cell} & (\text{for } i \in \mathbf{i} \text{ and } \alpha = \pm), \\ \partial_j^\beta x_i^\alpha = \partial_i^\alpha x_j^\beta & (\text{for } i \neq j \text{ in } \mathbf{i} \text{ and } \alpha, \beta = \pm). \end{array} \tag{11}$$

Of course the new faces are defined letting $\partial_i^\alpha(x) = x_i^\alpha$. Then the new operations and degeneracies are determined by the fact that they must be consistent with faces, i.e. satisfy the following conditions (for $j \neq i$):

$$\begin{gathered} \partial_i^-(x +_i y) = \partial_i^-(x), \qquad \partial_i^+(x +_i y) = \partial_i^+(y), \\ \partial_j^\alpha(x +_i y) = \partial_j^\alpha(x) +_i \partial_j^\alpha(y), \\ \partial_i^\alpha(e_i x) = x, \qquad\qquad \partial_j^\alpha(e_i x) = e_i \partial_j^\alpha(x). \end{gathered} \tag{12}$$

Now we say that the multiple category A has *coskeletal dimension* $\leqslant n$ if it is isomorphic to $\mathrm{Dcosk}_n \mathrm{Dtrc}_n(\mathsf{A})$, so that all its cells of higher dimension can be reconstructed from those of dimension up to n, as described above. The *coskeletal dimension* of A is the least such natural number n.

Similar definitions can be given for weak or lax multiple categories.

The coskeletal dimension of the multiple category $\mathsf{Q}(\mathbf{C})$ is 3 at most, and can be less. For instance, if $\mathbf{C}$ is a preordered category (viewed as a locally preordered 2-category), then each cube of quintets (7) commutes, and is 'bijectively determined' by its boundary, so that the coskeletal dimension of $\mathsf{Q}(\mathbf{C})$ is at most 2; moreover, if the preorder of $\mathbf{C}$ is codiscrete, even a quintet is 'bijectively determined' by its boundary, and the coskeletal dimension is at most 1; finally, if $\mathbf{C}$ is a codiscrete category with codiscrete preorder, the coskeletal dimension of $\mathsf{Q}(\mathbf{C})$ is 0.

In [GP8], Section 2.7, we have considered a different truncation functor $\mathrm{trc}_n \colon \mathbf{Mlt} \to \mathbf{Mlt_n}$, taking values in the category of $\mathbf{n}$-multiple categories, having multi-indices $\mathbf{i} \subset \mathbf{n} = \{0, ..., n-1\}$. This functor and its right adjoint cosk_n would give another notion of dimension, which seems to be of little interest.

1.4. Multiple categories of generalised quintets. A *multiple category of generalised quintets* will be a multiple category A equipped with a 'forgetful' multiple functor $U \colon \mathsf{A} \to \mathsf{Q}(\mathbf{C})$ with values in the multiple category of quintets over some 2-category $\mathbf{C}$.

More particularly we say that A is *of quintet type* over $\mathbf{C}$ when U satisfies the following condition of '2-dimensional faithfulness':

(i) each cell x of A of dimension 2 or higher is determined by its boundary and by the cell $U(x)$ of $\mathsf{Q}(\mathbf{C})$.

In the general case above, U is just expected to capture properties of A, motivating our interest. Much in the same way as we may be interested in 'forgetful' functors of ordinary categories, even if not faithful.

1.5. Multiple categories of cubical quintets. Extending our construction of the multiple category $\mathsf{Q}(\mathbf{C})$ of quintets over a 2-category $\mathbf{C}$, one can start from a (strict) n-category $\mathbf{C}$ and form a multiple category $\mathsf{Q}^n(\mathbf{C})$, of coskeletal dimension $\leqslant n+1$, so that $\mathsf{Q}^2(\mathbf{C})$ is the previous case of Section 1.2 while $\mathsf{Q}^1(\mathbf{C}) = \mathsf{Cub}(\mathbf{C})$ is the multiple category of commutative cubes over a category (see [GP8], Section 3.5(a)).

We only sketch the construction of $\mathsf{Q}^3(\mathbf{C})$, for a 3-category $\mathbf{C}$. For all cells of dimension 0, 1 and 2 we proceed as in Section 1.2, points (a), (b); then we go on as follows.

(c′) In dimension 3 an ijk-cell Π (for $0 \leqslant i < j < k$) is a cubical diagram of 2-dimensional cells (its faces), as in diagram (7), inhabited by a 3-cell of $\mathbf{C}$ (still written as Π, by abuse of notation)

$$\Pi\colon y'\varphi \otimes \rho u \otimes s'\omega \to \zeta r \otimes v'\pi \otimes \psi x\colon y'vr \to s'u'x. \tag{13}$$

The compositions in directions i, j, k are defined by 'pasting inhabited cubes' in $\mathbf{C}$; we get a triple category $\mathrm{trp}_{ijk}(\mathsf{Q}^3(\mathbf{C}))$.

(d′) A cell of dimension 4 is a face-consistent family of 3-dimensional cells that forms a 'commutative 4-dimensional cube'. Every cell of dimension $n > 4$ is an n-dimensional cube whose 3-dimensional faces are as in (c′), under the condition that each 4-dimensional face be a commutative 4-cube.

2. The multiple category of chiral multiple categories

In [GP10], Section 2, we have constructed the strict *double* category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories, with lax and colax multiple functors, and suitable double cells 'of quintet type'. This construction was the basis of our definition of colax/lax adjunctions between chiral multiple categories.

In [GP6], Section 6, we have constructed the strict *triple* category ICat of 3-dimensional intercategories, whose arrows are 'mixed-laxity functors': the lax triple functors in direction 1, the colax ones in direction 3 and the intermediate *colax-lax morphisms* in direction 2.

We extend now these structures, forming a (strict) multiple category Cmc of chiral multiple categories, indexed by the ordinal $\boldsymbol{\omega} + \mathbf{1} = \{0, 1, ..., \infty\}$. The transversal arrows, or 0-*morphisms*, are strict multiple functors. In direction p (for $1 \leqslant p \leqslant \infty$), the p-*morphisms* are 'functors of mixed laxity', varying from the lax functors (in direction 1) to the colax ones (in direction ∞).

As to notation, a chiral multiple category A is a multiple object of ordinary categories $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ indexed by positive multi-indices $\mathbf{i} = \{i, j, k...\} \subset \mathbb{N}^*$. On the other hand Cmc will be indexed by 'extended' positive multi-indices $\mathbf{p} = \{p, q, r...\} \subset \{1, 2, ..., \infty\}$.

Let us note that, if we restrict Cmc to the weak multiple categories of cubical type (see [GP8]), we still have a multiple category of *non*-cubical type, with different kinds of arrows in each direction.

2.1. Mixed-laxity functors. In degree 0, the objects of Cmc are the (small) chiral multiple categories, and the transversal arrows (or 0-morphisms) are the strict multiple functors $F\colon \mathsf{A} \to_0 \mathsf{B}$.

In degree 1 and direction p (for $1 \leqslant p \leqslant \infty$), a *p-morphism* $R\colon \mathsf{A} \to_p \mathsf{B}$ between chiral multiple categories will be a *mixed-laxity functor* which is colax in all positive directions $i < p$ and lax in all directions $i \geqslant p$. In particular, this is a lax functor for $p = 1$ and a colax functor for $p = \infty$.

Basically, R has components $R_{\mathbf{i}} = \mathrm{tv}_{\mathbf{i}}(R)\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{B})$, for all positive multi-indices $\mathbf{i}$, that are ordinary functors and commute with faces: $\partial_i^\alpha.R_{\mathbf{i}} = R_{\mathbf{i}|i}.\partial_i^\alpha$ (for $i \in \mathbf{i}$).

Moreover R is equipped with i-special *comparison* $\mathbf{i}$-maps $\underline{R}_i$ (for $t \in A_{\mathbf{i}|i}$ and x, y i-consecutive in $A_{\mathbf{i}}$), either in the lax direction for $i \geqslant p$

$$\underline{R}_i(t)\colon e_iR(t) \to_0 R(e_it), \qquad \underline{R}_i(x,y)\colon R(x) +_i R(y) \to_0 R(x +_i y), \tag{14}$$

or in the colax direction for $0 < i < p$

$$\underline{R}_i(t)\colon R(e_it) \to_0 e_iR(t), \qquad \underline{R}_i(x,y)\colon R(x +_i y) \to_0 R(x) +_i R(y). \tag{15}$$

All these comparisons must commute with faces (for $j \neq i$ in $\mathbf{i}$)

$$\partial_j^\alpha \underline{R}_i(t) = \underline{R}_i(\partial_j^\alpha t), \qquad \partial_j^\alpha \underline{R}_i(x,y) = \underline{R}_i(\partial_j^\alpha x, \partial_j^\alpha y). \tag{16}$$

Moreover they have to satisfy the axioms of naturality and coherence (see [GP8], Section 3.9), either in the lax form (lmf.1-4) for $i \geqslant p$, or in the transversally dual form - say (cmf.1-4) - for $i < p$.

Finally there is an axiom of coherence with the interchanger χ_{ij} $(0 < i < j)$ which has three forms (where (a) corresponds to (lmf.5), (c) corresponds to its dual (cmf.5) and (b) is an intermediate case):

(a) for $p \leqslant i < j$ (so that R is i- and j-lax), we have commutative diagrams of transversal maps:

$$\begin{tikzcd}
(Rx +_i Ry) +_j (Rz +_i Ru) \arrow[r, "\chi_{ij}R"] \arrow[d, "\underline{R}_i +_j \underline{R}_i"'] & (Rx +_j Rz) +_i (Ry +_j Ru) \arrow[d, "\underline{R}_j +_i \underline{R}_j"] \\
R(x +_i y) +_j R(z +_i u) \arrow[d, "\underline{R}_j"'] & R(x +_j y) +_i R(z +_j u) \arrow[d, "\underline{R}_i"] \\
R((x +_i y) +_j (z +_i u)) \arrow[r, "R\chi_{ij}"'] & R((x +_j z) +_i (y +_j u))
\end{tikzcd} \tag{17}$$

(b) for $0 < i < p \leqslant j$ (so that R is i-colax and j-lax), we have commutative diagrams:

$$\begin{tikzcd}[column sep=small]
 & (Rx +_i Ry) +_j (Rz +_i Ru) \arrow[rr, "\chi_{ij}R"] & & (Rx +_j Rz) +_i (Ry +_j Ru) \arrow[dr, "\underline{R}_j +_i \underline{R}_j"] & \\
R(x +_i y) +_j R(z +_i u) \arrow[ur, "\underline{R}_i +_j \underline{R}_i"] \arrow[dr, "\underline{R}_j"'] & & & & R(x +_j y) +_i R(z +_j u) \\
 & R((x +_i y) +_j (z +_i u)) \arrow[rr, "R\chi_{ij}"'] & & R((x +_j z) +_i (y +_j u)) \arrow[ur, "\underline{R}_i"'] &
\end{tikzcd} \tag{18}$$

(c) for $0 < i < j < p$ (so that R is i- and$j$-colax), we have commutative diagrams as in (17), with all vertical arrows reversed.

The composition of p-morphisms is easily defined: their comparisons are separately composed.

Finally, a transversal map $(F, G)\colon R \to_0 S$ of p-arrows will be a commutative square

$$\begin{tikzcd}
\bullet \arrow[r, "F"] \arrow[d, "R"', "\bullet" marking] \arrow[dr, phantom, "="] & \bullet \arrow[d, "S", "\bullet" marking] \\
\bullet \arrow[r, "G"'] & \bullet
\end{tikzcd} \qquad SF = GR \qquad \begin{tikzcd}[sep=small] \bullet \arrow[r, "0"] \arrow[d, "p"] & {} \\ {} & \end{tikzcd} \tag{19}$$

with strict functors F, G and p-morphisms R, S. Commutativity means that $SF = GR$ *as* p-morphisms, including comparisons.

(Let us recall that, as already remarked in [GP6], the 'lax-colax' case makes no sense: modifying diagram (17) by reversing all arrows $\underline{R}_j$ would lead to a diagram where no pairs of arrows compose.)

We have thus defined the double category $\mathrm{dbl}_{0p}(\mathsf{Cmc})$ of chiral multiple categories, strict functors and p-morphisms.

2.2. Two-dimensional cubes. To define a pq-cube (for $1 \leqslant p < q \leqslant \infty$) we have to adapt the axioms of transversal transformation (again in [GP8], Section 3.9).

A *pq-cube* $\varphi\colon (U\ {}^{R}_{S}\ V)$ will be a 'generalised quintet' consisting of two p-morphisms R, S, two q-morphisms U, V, together with - roughly speaking - a 'transversal transformation' $\varphi\colon VR \dashrightarrow SU$

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "R", "\bullet" marking] \arrow[d, "U"', "\bullet" marking] \arrow[dr, phantom, "\swarrow\varphi"] & \bullet \arrow[d, "V", "\bullet" marking] \\
\bullet \arrow[r, "S"', "\bullet" marking] & \mathsf{B}
\end{tikzcd} \qquad \varphi\colon VR \dashrightarrow SU. \qquad \begin{tikzcd}[sep=small] \bullet \arrow[r, "p"] \arrow[d, "q"] & {} \\ {} & \end{tikzcd} \tag{20}$$

Again (as in [GP10], Section 2, for the double category $\mathbb{C}\mathrm{mc}$) this is an abuse of notation since there are no composites VR and SU in our structure: the coherence conditions of φ are based on the four morphisms R, S, U, V and all their comparison maps. Precisely, the cell φ consists of a face-consistent family of transversal maps in B

$$\begin{aligned} &\varphi(x) = \varphi_{\mathbf{i}}(x) \colon VR(x) \to_0 SU(x), \qquad &&\text{(for every } \mathbf{i}\text{-cube } x \text{ of } \mathsf{A}),\\ &\partial_i^\alpha.\varphi_{\mathbf{i}} = \varphi_{\mathbf{i}|i}.\partial_i^\alpha &&\text{(for } i \in \mathbf{i}), \end{aligned} \tag{21}$$

so that each component $\varphi_{\mathbf{i}} \colon V_{\mathbf{i}}R_{\mathbf{i}} \to S_{\mathbf{i}}U_{\mathbf{i}} \colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{B})$ is a natural transformation of ordinary functors:

(nat) for all $f \colon x \to_0 y$ in A, we have a commutative diagram of transversal maps in B

$$\begin{tikzcd} VR(x) \arrow[r, "\varphi x"] \arrow[d, "VRf"'] \arrow[dr, phantom, "="] & SU(x) \arrow[d, "SUf"] \\ VR(y) \arrow[r, "\varphi y"'] & SU(y) \end{tikzcd} \tag{22}$$

Moreover φ has to satisfy the following coherence conditions (coh.a), (coh.b), (coh.c) with the comparisons of R, S, U, V, for a degenerated cube $e_i(t)$ (with $t \in A_{\mathbf{i}|i}$) and a composite $z = x +_i y$ in $A_{\mathbf{i}}$.

(coh.a) If $p < q \leqslant i$ (so that R, S, U, V are lax in direction i), we have commutative diagrams:

$$\begin{tikzcd} e_iVR(t) \arrow[r, "e_i(\varphi t)"] \arrow[d, "\underline{V}_i(Rt)"'] & e_iSU(t) \arrow[d, "\underline{S}_i(Ut)"] \\ V(e_iRt) \arrow[d, "V\underline{R}_i(t)"'] & S(e_iUt) \arrow[d, "S\underline{U}_i(t)"] \\ VR(e_it) \arrow[r, "\varphi(e_it)"'] & SU(e_it) \end{tikzcd} \qquad \begin{tikzcd} VRx +_i VRy \arrow[r, "\varphi x +_i \varphi y"] \arrow[d, "\underline{V}_i(Rx,Ry)"'] & SUx +_i SUy \arrow[d, "\underline{S}_i(Ux,Uy)"] \\ V(Rx +_i Ry) \arrow[d, "V\underline{R}_i(x,y)"'] & S(Ux +_i Uy) \arrow[d, "S\underline{U}_i(x,y)"] \\ VR(z) \arrow[r, "\varphi(z)"'] & SU(z) \end{tikzcd} \tag{23}$$

(coh.b) If $p \leqslant i < q$ (so that R, S are lax and U, V are colax in direction i), we have commutative diagrams:

$$\begin{tikzcd} e_iVR(t) \arrow[r, "e_i(\varphi t)"] & e_iSU(t) \arrow[d, "\underline{S}_i(Ut)"] \\ V(e_iRt) \arrow[u, "\underline{V}_i(Rt)"] \arrow[d, "V\underline{R}_i(t)"'] & S(e_iUt) \\ VR(e_it) \arrow[r, "\varphi(e_it)"'] & SU(e_it) \arrow[u, "S\underline{U}_i(t)"'] \end{tikzcd} \qquad \begin{tikzcd} VRx +_i VRy \arrow[r, "\varphi x +_i \varphi y"] & SUx +_i SUy \arrow[d, "\underline{S}_i(Ux,Uy)"] \\ V(Rx +_i Ry) \arrow[u, "\underline{V}_i(Rx,Ry)"] \arrow[d, "V\underline{R}_i(x,y)"'] & S(Ux +_i Uy) \\ VR(z) \arrow[r, "\varphi(z)"'] & SU(z) \arrow[u, "S\underline{U}_i(x,y)"'] \end{tikzcd} \tag{24}$$

(coh.c) If $i < p < q$ (so that R, S, U, V are colax in direction i), we have commutative diagrams as in (23), with all vertical arrows reversed.

The p- and q-composition of these cubes are both defined using componentwise the transversal composition of a chiral multiple category. Namely, for a consistent matrix of pq-cubes and $x \in \mathsf{A}$

$$
\begin{tikzcd}
\bullet \arrow[r, "R"] \arrow[d, "U"'] \arrow[dr, phantom, "\varphi"] & \bullet \arrow[r, "R'"] \arrow[d, "V"] \arrow[dr, phantom, "\psi"] & \bullet \arrow[d, "W"] \\
\bullet \arrow[r, "S"] \arrow[d, "U'"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[r, "S'"] \arrow[d, "V'"] \arrow[dr, phantom, "\tau"] & \bullet \arrow[d, "W'"] \\
\bullet \arrow[r, "T"'] & \bullet \arrow[r, "T'"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, "p"] \arrow[d, "q"'] & {} \\
{} &
\end{tikzcd}
\tag{25}
$$

$$
\begin{aligned}
(\varphi +_p \psi)(x) &= \psi(Rx) +_0 S'(\varphi x) \colon WR'Rx \to S'VRx \to S'SUx, \\
(\varphi +_q \sigma)(x) &= V'(\varphi x) +_0 \sigma(Ux) \colon V'VRx \to V'SUx \to TU'Ux.
\end{aligned}
\tag{26}
$$

We prove below, in Theorem 2.8, that these composition laws are well-defined, i.e. the cells above do satisfy the previous coherence conditions. Moreover, they have been defined via the composition of transversal maps, and therefore are strictly unitary and associative.

Finally, to verify the middle-four interchange law on the four double cells of diagram (25), we compute the composites $(\varphi +_p \psi) +_q (\sigma +_p \tau)$ and $(\varphi +_q \sigma) +_p (\psi +_q \tau)$ on an $\mathbf{i}$-cube x, and we obtain the two transversal maps $W'WR'Rx \to_0 T'TU'Ux$ of the upper or lower path in the following diagram

$$
\begin{tikzcd}
W'WR'Rx \arrow[r, "W'\psi Rx"] & W'S'VRx \arrow[r, "W'S'\varphi x"] \arrow[d, "\tau VRx"'] \arrow[dr, phantom, "="] & W'S'SUx \arrow[d, "\tau SUx"] & \\
& T'V'VRx \arrow[r, "T'V'\varphi x"'] & T'V'SUx \arrow[r, "T'\sigma Ux"'] & T'TU'Ux
\end{tikzcd}
\tag{27}
$$

The square commutes, by naturality of the double cell τ (with respect to the transversal map $\varphi x \colon VR(x) \to_0 SU(x)$), so that the two composites coincide.

2.3. Transversal maps of degree two. Given two pq-cubes

$$
\varphi \colon (U \, {}^{R}_{S} \, V), \qquad\qquad \varphi' \colon (U' \, {}^{R'}_{S'} \, V') \tag{28}
$$

a transversal pq-map $(F, G, F', G') \colon \varphi \to_0 \varphi'$ (of degree two and dimension three) is a quadruple of strict functors forming four transversal maps of degree 1

$$
\begin{aligned}
&(F, G) \colon R \to_0 R', \qquad && (F', G') \colon S \to_0 S', \\
&(F, F') \colon U \to_0 U', \qquad && (G, G') \colon V \to_0 V',
\end{aligned}
\tag{29}
$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
\mathsf{A} \arrow[rr, "R"] \arrow[dr, "F"] \arrow[dd, "U"'] & & \bullet \arrow[dr, "G"] \arrow[dl, phantom, "="] & \\
& \bullet \arrow[rr, "R'"] \arrow[dd, "U'"] \arrow[dl, phantom, "="] & & \bullet \arrow[dd, "V'"] \arrow[dll, phantom, "\varphi'"] \\
\bullet \arrow[dr, "F'"'] & & & \\
& \bullet \arrow[rr, "S'"'] & & \mathsf{B}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
\mathsf{A} \arrow[rr, "R"] \arrow[dd, "U"'] & & \bullet \arrow[dr, "G"] \arrow[dd, "V"] \arrow[dll, phantom, "\varphi"] & \\
& & & \bullet \arrow[dd, "V'"] \arrow[dl, phantom, "="] \\
\bullet \arrow[rr, "R'"] \arrow[dr, "F'"'] & & \bullet \arrow[dr, "G'"] \arrow[d, phantom, "="] & \\
& \bullet \arrow[rr, "S'"'] & & \mathsf{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, "0"] & {} \\
{} & {}
\end{tikzcd}
$$

and such that 'the cube commutes', in the sense that, for every $\mathbf{i}$-cube x of A, the following transversal maps of B coincide

$$
G'(\varphi x)\colon G'VR(x) \to G'SU(x), \qquad\qquad \varphi'(Fx)\colon VR'F(x) \to S'U'F(x). \tag{30}
$$

We have thus defined the triple category $\mathrm{trp}_{0pq}(\mathsf{Cmc})$ of chiral multiple categories, with strict functors and p- and q-morphisms (for $0 < p < q \leqslant \infty$). Its indices vary in the pointed ordered set $\{0, p, q\}$.

2.4. Three-dimensional cubes. A *pqr-cube* (for $0 < p < q < r \leqslant \infty$) will be a 'commutative cube' Π determined by its six faces:

- two pq-cubes φ, ψ (the faces $\partial^\alpha_r \Pi$),
- two pr-cubes π, ρ (the faces $\partial^\alpha_q \Pi$),
- two qr-cubes ω, ζ (the faces $\partial^\alpha_p \Pi$),

$$
\begin{tikzcd}[column sep=small, row sep=small]
\mathsf{A} \arrow[rr, "R"] \arrow[dr, "U"] \arrow[dd, "X"'] & & \bullet \arrow[dr, "V"] \arrow[dl, phantom, "\varphi"] & \\
& \bullet \arrow[rr, "S"] \arrow[dd, "Y"] \arrow[dl, phantom, "\omega"] & & \bullet \arrow[dd, "Y'"] \arrow[dll, phantom, "\rho"] \\
\bullet \arrow[dr, "U'"'] & & & \\
& \bullet \arrow[rr, "S'"'] & & \mathsf{B}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
\mathsf{A} \arrow[rr, "R"] \arrow[dd, "X"'] \arrow[ddrr, phantom, "\pi"] & & \bullet \arrow[dr, "V"] \arrow[dd, "X'"] & \\
& & & \bullet \arrow[dd, "Y'"] \arrow[dl, phantom, "\zeta"] \\
\bullet \arrow[rr, "R'"] \arrow[dr, "U'"'] & & \bullet \arrow[dr, "V'"] \arrow[d, phantom, "\psi"] & \\
& \bullet \arrow[rr, "S'"'] & & \mathsf{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "p"] \arrow[d, "r"'] \arrow[dr, "q"] & {} \\
{} & {}
\end{tikzcd}
\tag{31}
$$

The commutativity condition means that, for every $\mathbf{i}$-cube x of A, the following composed transversal arrows in B coincide

$$
\begin{aligned}
&S'\omega x.\rho Ux.Y'\varphi x\colon Y'VR(x) \to Y'SU(x) \to S'YU(x) \to S'U'X(x) \\
&\psi Xx.V'\pi x.\zeta Rx : Y'VR(x) = V'X'R(x) = V'R'X(x) = S'U'X(x).
\end{aligned}
\tag{32}
$$

These cubes are composed in direction p, q, or r, by pasting cubes (with the operations of 2-dimensional cubes). Again, these operations are associative, unitary and satisfy the middle-four interchange by pairs.

2.5. Higher items. A transversal pqr-map $F\colon \Pi \to_0 \Pi'$ between pqr-cubes is determined by its boundary, a face-consistent family of eight transversal maps of degree two (and dimension three)

$$
\partial^\alpha_j F\colon \partial^\alpha_j \Pi \to_0 \partial^\alpha_j \Pi' \qquad\qquad (j \in \{p, q, r\}), \tag{33}
$$

under no other conditions. Their operations are computed on such faces.

We have thus defined a quadruple category of chiral multiple categories, with strict functors and p-, q-, r-morphisms (for extended positive integers $p < q < r$). The indices vary in the pointed ordered set $\{0, p, q, r\}$.

Finally, we have the multiple category Cmc (indexed by the ordinal $\boldsymbol{\omega}+\mathbf{1}$), where each cell of dimension $\geqslant 4$ (starting with the transversal maps of degree 3 considered above and the cubes of dimension 4, not yet considered) is determined by a face-consistent family of all its iterated faces of dimension 3.

In the truncated case we have the (n+1)-dimensional multiple category $\mathsf{Cmc}_\mathbf{n}$ of (small) chiral $\mathbf{n}$-multiple categories, where the objects are indexed by the ordinal $\mathbf{n} = \{0, ..., n{-}1\}$, while $\mathsf{Cmc}_\mathbf{n}$ is indexed by $\mathbf{n}+\mathbf{1}$ (the previous ∞ being replaced by n). But one should note that $\mathsf{Cmc}_\mathbf{n}$ is *not* an ordinary truncation of Cmc, as its objects too are truncated.

Cmc is a substructure of the - similarly defined - multiple category Inc of small infinite dimensional intercategories, and $\mathsf{Cmc}_\mathbf{n}$ is a substructure of the $(n{+}1)$-dimensional multiple category $\mathsf{Inc}_\mathbf{n}$ of small $\mathbf{n}$-intercategories.

2.6. Comments. These multiple categories are related to various double or triple categories previously constructed.

(a) A chiral 1-multiple category is just a category, and $\mathsf{Cmc}_\mathbf{1}$ is the double category of small categories, with commutative squares of functors as double cells.

(b) A chiral 2-multiple category is a weak double category. Let us recall that we have already studied (in [GP2], Section 2) the double category $\mathbb{D}\mathrm{bl}$ of weak double categories, with lax and colax functors - where double adjunctions live. Later $\mathbb{D}\mathrm{bl}$ has been extended to a triple category $\mathsf{S}\mathbb{D}\mathrm{bl}$ of weak double categories, with strict, lax and colax functors [GP8], Section 1); in the latter all 2-dimensional cells are inhabited by possibly non-trivial transformations, while in Cmc_2 the 01- and 02-cells are 'commutative squares', inhabited by identities. Thus $\mathsf{Cmc}_\mathbf{2}$ extends $\mathbb{D}\mathrm{bl}$ but is a triple subcategory of $\mathsf{S}\mathbb{D}\mathrm{bl}$.

(c) Multiple adjunctions live in the double category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories, with lax and colax multiple functors ([GP10], Section 2). This can be extended to a triple category $\mathsf{S}\mathbb{C}\mathrm{mc}$ of chiral multiple categories, with strict, lax and colax functors, where again all 2-dimensional cells are inhabited by possibly non-trivial transformations. Then $\mathsf{S}\mathbb{C}\mathrm{mc}$ contains the triple category obtained from Cmc by restricting to the multi-indices $\mathbf{i} \subset \{0, 1, \infty\}$. But there seems to be no reasonable way of extending Cmc with non-trivial quintets on $0p$-cells, for $1 < p < \infty$.

(d) The quadruple category $\mathsf{Inc}_\mathbf{3}$ of 3-dimensional intercategories is an extension of the triple category ICat of [GP6], Section 6, obtained by adding strict functors in the transversal direction and 'commutative transversal cells'.

2.7. The quintet type. There is a multiple functor

$$\mathrm{tv}\colon \mathsf{Cmc} \to \mathsf{Q}(\mathrm{Mlt}(\mathbf{Cat})), \tag{34}$$

with values in the multiple category of quintets over the 2-category of multiple objects in **Cat** (indexed by $\mathbb{N}^*$), which sends:

- a chiral multiple category A to the multiple object $\mathrm{tv}(\mathsf{A}) = (\mathrm{tv}_{\mathbf{i}}\mathsf{A})_{\mathbf{i}\subset\mathbb{N}^*}$ of its transversal categories,
- a p-morphism $F\colon \mathsf{A} \to_p \mathsf{B}$ to the multiple morphism $\mathrm{tv}(F) = (F_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}\mathsf{A} \to \mathrm{tv}_{\mathbf{i}}\mathsf{B})_{\mathbf{i}}$ formed by its transversal components, the ordinary functors $F_{\mathbf{i}}$ (for $p \geqslant 0$),
- a pq-cell $\varphi\colon VR \dashrightarrow SU$ (as in (20) or also in (19)) to the multiple transformation $\mathrm{tv}(\varphi) = (\varphi_{\mathbf{i}}\colon V_{\mathbf{i}}R_{\mathbf{i}} \to S_{\mathbf{i}}U_{\mathbf{i}})_{\mathbf{i}}$ formed by its transversal components, the ordinary natural transformations $\varphi_{\mathbf{i}}$ (for $0 \leqslant p < q$),
- each higher cell Φ of Cmc to the corresponding cell of the codomain, by acting on all the 2-dimensional faces of Φ.

This multiple functor makes Cmc into a multiple category of quintet type, in the sense of Section 1. Moreover Cmc has coskeletal dimension 3: note that the 3-dimensional cubes of (31) are determined by their boundary, *under* a commutativity condition which is not automatically satisfied.

2.8. Theorem. *In* Cmc *the composition law* $\varphi +_p \psi$ *of pq-cubes is well-defined by the formula* (27)

$$(\varphi +_p \psi)(x) = \psi(Rx) +_0 S'(\varphi x)\colon WR'Rx \to S'VRx \to S'SUx, \tag{35}$$

in the sense that this family of transversal maps does satisfy the conditions (coh.a) - (coh.c) of 2.2.

Proof. The argument is an extension of a similar one for the double category $\mathbb{D}\mathrm{bl}$ in [GP2], Section 2, or for the double category $\mathbb{C}\mathrm{mc}$ in [GP10], Section 2, taking into account the mixed laxity of the present 'functors'. We prove the three coherence axioms with respect to a composed cube $z = x +_i y$ in $A_{\mathbf{i}}$; one would work in a similar way for a degenerate cube $e_i(t)$, with $t \in A_{\mathbf{i}|i}$.

First we prove (coh.a), letting $p < q \leqslant i$, so that all our functors R, R', S, S', U, V, W are lax in direction i. This amounts to the commutativity of the outer diagram below, formed of transversal maps (the index i being omitted in $+_i$ and in all comparisons $\underline{R}_i, \underline{R}'_i$, etc.)

$$\begin{tikzcd}
WR'Rz \arrow[r, "\psi Rz"] & S'VRz \arrow[r, "S'\varphi z"] & S'SUz \\
WR'(Rx+Ry) \arrow[u, "WR'\underline{R}"] \arrow[r, "\psi(Rx+Ry)"] & S'V(Rx+Ry) \arrow[u, "S'V\underline{R}"'] & S'S(Ux+Uy) \arrow[u, "S'S\underline{U}"'] \\
W(R'Rx+R'Ry) \arrow[u, "W\underline{R}'R"] & S'(VRx+VRy) \arrow[u, "S'\underline{V}R"'] \arrow[r, "S'(\varphi x+\varphi y)"] & S'(SUx+SUy) \arrow[u, "S'\underline{S}U"'] \\
WR'Rx+WR'Ry \arrow[u, "\underline{W}R'R"'] \arrow[r, "\psi Rx+\psi Ry"'] & S'VRx+S'VRy \arrow[u, "\underline{S}'VR"'] \arrow[r, "S'\varphi x+S'\varphi y"'] & S'SUx+S'SUy \arrow[u, "\underline{S}'SU"']
\end{tikzcd}$$

Indeed, the two hexagons commute by (coh.a), applied to φ and ψ, respectively. The upper rectangle commutes by naturality of ψ on $\underline{R}_i(x, y)$. The lower rectangle commutes by axiom (lmf.2) [GP8], Section 3.9, on the lax functor S', with respect to the transversal **i**-maps $\varphi x\colon VR(x) \to_0 SU(x)$ and $\varphi y\colon VR(y) \to_0 SU(y)$

$$S'(\varphi x +_i \varphi y).\underline{S}'_i(VR(x), VR(y)) = \underline{S}'_i(SU(x), SU(y)).(S'(\varphi x) +_i S'(\varphi y)). \qquad (36)$$

The proof of (coh.c) is transversally dual to the previous one. To prove (coh.b) we let $p \leqslant i < q$, so that R, R', S, S' are lax, while U, V, W are colax in direction i. We reverse the comparisons $\underline{U}_i, \underline{V}_i, \underline{W}_i$ in the diagram above

$$\begin{tikzcd}
WR'Rz \arrow[r, "\psi Rz"] & S'VRz \arrow[r, "S'\varphi z"] & S'SUz \arrow[d, "S'S\underline{U}"] \\
WR'(Rx+Ry) \arrow[u, "WR'\underline{R}"] \arrow[r, "\psi(Rx+Ry)"] & S'V(Rx+Ry) \arrow[u, "S'V\underline{R}"'] \arrow[d, "S'\underline{V}R"] & S'S(Ux+Uy) \\
W(R'Rx+R'Ry) \arrow[u, "W\underline{R}'R"] \arrow[d, "\underline{W}R'R"] & S'(VRx+VRy) \arrow[r, "S'(\varphi x+\varphi y)"] & S'(SUx+SUy) \arrow[u, "S'\underline{S}U"'] \\
WR'Rx+WR'Ry \arrow[r, "\psi Rx+\psi Ry"'] & S'VRx+S'VRy \arrow[u, "\underline{S}'VR"'] \arrow[r, "S'\varphi x+S'\varphi y"'] & S'SUx+S'SUy \arrow[u, "\underline{S}'SU"']
\end{tikzcd}$$

and note that the two hexagons commute, by (coh.b) on φ and ψ, while the rectangles are unchanged.

■

3. Generalised quintets for chains of adjunctions

We now construct a non-cubical multiple category of generalised quintets, out of a sequence of 2-categories and 2-functors. Then we deduce a non-cubical multiple category $\mathsf{Adj}(\mathbf{C})$ where an i-directed arrow is a *chain of adjunctions* $u_0 \dashv u_1 \dashv ... \dashv u_i$ of length i, in a fixed 2-category $\mathbf{C}$.

Examples of unbounded chains of adjunctions can be found in [Bo].

3.1. A MULTIPLE CATEGORY DERIVED FROM A SEQUENCE OF 2-CATEGORIES. We start from a sequence $\mathcal{C}$ of 2-functors of 2-categories, *which are supposed to be the identity on a common set of objects* S

$$(\mathcal{C}) \qquad ... \longrightarrow \mathbf{C}_i \xrightarrow{U_i} ... \longrightarrow \mathbf{C}_2 \xrightarrow{U_2} \mathbf{C}_1 \xrightarrow{U_1} \mathbf{C}_0 \qquad (37)$$

$$U_{ji} = U_{i+1}...U_j\colon \mathbf{C}_j \to \mathbf{C}_i \qquad\qquad (0 \leqslant i < j),$$

and we construct a multiple category $\mathsf{M} = \mathsf{GQ}(\mathcal{C})$ of *generalised quintets* over the sequence $\mathcal{C}$, or *quintets modulo* U. The 2-functors U_{ji} are called the *forgetful functors* of $\mathcal{C}$.

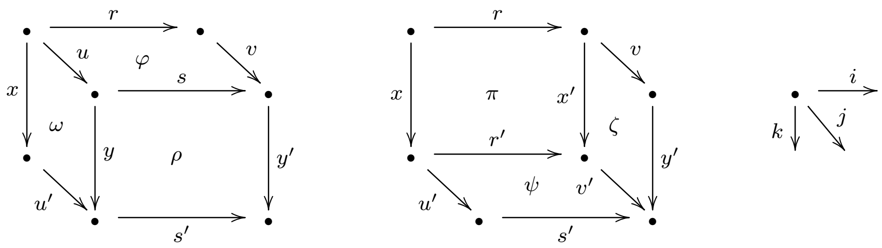
(a) The set of objects of M is S. In dimension 1, an i-cell $u\colon X \to_i Y$ is a $\mathbf{C}_i$-morphism ($i \geqslant 0$). They form the category $\mathrm{cat}_i(\mathsf{M})$ underlying $\mathbf{C}_i$.

(b) In dimension 2 an ij-cell (for $0 \leqslant i < j$) is a *U-quintet*, or *quintet modulo* $U_{ji}\colon \mathbf{C}_j \to \mathbf{C}_i$. It amounts to: two $\mathbf{C}_i$-morphisms r, s (its j-faces), two $\mathbf{C}_j$-morphisms u, v (its i-faces) and a 2-cell φ in $\mathbf{C}_i$

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "v"] \\ C \arrow[r, "s"'] & D \end{tikzcd} \qquad \varphi\colon (U_{ji}v)r \to s(U_{ji}u)\colon A \to D. \qquad \begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d, "j"] & {} \\ {} \end{tikzcd} \tag{38}$$

Such an ij-cell has an obvious underlying cell $|\varphi|$ in the double category $\mathbb{Q}(\mathbf{C}_i)$ of quintets over $\mathbf{C}_i$. U-quintets inherit from the latter two composition laws in directions i and j, and form a double category $\mathrm{dbl}_{ij}(\mathsf{M})$ with a cellwise-faithful double functor $\mathrm{dbl}_{ij}(\mathsf{M}) \to \mathbb{Q}(\mathbf{C}_i)$.

(c) In dimension 3 an ijk-cell (for $0 \leqslant i < j < k$) is a cube of U-quintets whose image in $\mathbf{C}_i$ commutes



(39)

More precisely, we have six U-quintets determined by 2-cells in $\mathbf{C}_i$ or $\mathbf{C}_j$

- ij-cells: $\varphi\colon (U_{ji}v)r \to s(U_{ji}u)$, $\quad \psi\colon (U_{ji}v')r' \to s'(U_{ji}u')$ (2-cells in $\mathbf{C}_i$),
- ik-cells: $\pi\colon (U_{ki}x')r \to r'(U_{ki}x)$, $\quad \rho\colon (U_{ki}y')s \to s'(U_{ki}y)$ (2-cells in $\mathbf{C}_i$),
- jk-cells: $\omega\colon (U_{kj}y)u \to u'(U_{kj}x)$, $\quad \zeta\colon (U_{kj}y')v \to v'(U_{kj}x')$ (2-cells in $\mathbf{C}_j$).

They must satisfy the following commutativity relation, in $\mathbf{C}_i$

$$Uy'.\varphi \otimes \rho.Uu \otimes s'.U\omega = U\zeta.r \otimes Uv'.\pi \otimes \psi.Ux\colon Uy'.Uv.r \to s'.Uu'.Ux,$$

where U stands for U_{ji} or U_{ki} when it operates on $\mathbf{C}_j$ or $\mathbf{C}_k$, respectively.

The compositions in directions i, j, k amount to compositions of faces in the double categories $\mathrm{dbl}_{ij}(\mathsf{M})$, $\mathrm{dbl}_{jk}(\mathsf{M})$ and $\mathrm{dbl}_{ik}(\mathsf{M})$. We have thus a triple category $\mathrm{trp}_{ijk}(\mathsf{M})$.

(d) Every cell of dimension $n > 3$ is an n-dimensional cube whose 2-dimensional faces are U-quintets, under the condition that each 3-dimensional face in direction ijk be an ijk-cell, as defined above.

3.2. Comments. As a particular case, if we start from a 2-category $\mathbf{C}$ and build the sequence $\mathcal{C}$ of its identities, this procedure gives the multiple category $\mathsf{Q}(\mathbf{C})$ *of quintets over* $\mathbf{C}$.

In general $\mathsf{GQ}(\mathcal{C})$ is a multiple category of generalised quintets, with respect to the obvious forgetful multiple functor $U\colon \mathsf{GQ}(\mathcal{C}) \to \mathsf{Q}(\mathbf{C}_0)$; its coskeletal dimension is at most 3.

If we assume that all the forgetful functors U_i are faithful, this multiple functor satisfies condition 1.4(i), and $\mathsf{GQ}(\mathcal{C})$ becomes a multiple category of quintet type, over the 2-category $\mathbf{C}_0$.

3.3. Chains of adjunctions. The previous construction allows us to build a multiple category $\mathsf{Adj}(\mathbf{C})$ of *chains of adjunctions* over a 2-category $\mathbf{C}$.

We begin by forming the 2-category $\mathbf{C}_i = \mathrm{Adj}_i(\mathbf{C})$ of i-chains of adjunctions in $\mathbf{C}$, for $i \geqslant 0$.

For $i = 0$ we just let $\mathrm{Adj}_0(\mathbf{C}) = \mathbf{C}$. In general, an object of $\mathrm{Adj}_i(\mathbf{C})$ is an object of $\mathbf{C}$, and a morphism

$$u = (u_0, ..., u_i; \eta_1, \varepsilon_1, ..., \eta_i, \varepsilon_i)\colon X \to_i Y, \tag{40}$$

is a chain of adjunctions $u_0 \dashv u_1 \dashv ... \dashv u_i$ in $\mathbf{C}$, with

$$\begin{array}{l} u_0\colon X \to Y, \qquad u_1\colon Y \to X, \qquad u_2\colon X \to Y, ... \\ (\eta_1\colon 1_X \to u_1u_0,\ \varepsilon_1\colon u_0u_1 \to 1_Y)\colon u_0 \dashv u_1 \quad (u_0\eta_1 \otimes \varepsilon_1u_0 = 1,\ \eta_1u_1 \otimes u_1\varepsilon_1 = 1), \\ (\eta_2\colon 1_Y \to u_2u_1,\ \varepsilon_2\colon u_1u_2 \to 1_X)\colon u_1 \dashv u_2 \quad (u_1\eta_2 \otimes \varepsilon_2u_1 = 1,\ \eta_2u_2 \otimes u_2\varepsilon_2 = 1), \\ ... \end{array} \tag{41}$$

The chain will also be written as $u = (u_0, u_1, u_2, ...)$, leaving units and counits understood.

The composition of u with a consecutive arrow $v\colon Y \to_i Z$ is just an iterated composition of adjunctions

$$vu = (v_0, v_1, v_2, ...).(u_0, u_1, u_2, ...) = (v_0u_0, u_1v_1, v_2u_2, ...)\colon X \to_i Z, \tag{42}$$

that ends with v_iu_i (resp. u_iv_i) when i is even (resp. odd). The identity of X is a chain of identities.

A 2-cell of $\mathrm{Adj}_i(\mathbf{C})$

$$\varphi = (\varphi_0, ..., \varphi_i)\colon (u_0, ..., u_i; \eta_1, \varepsilon_1, ...\,) \to (v_0, ..., v_i; \eta_1, \varepsilon_1, ...\,)\colon X \to_i Y, \tag{43}$$

$$\begin{tikzcd} u_0 \arrow[r, dash, "\dashv" marking] \arrow[d, "\varphi_0"'] & u_1 \arrow[r, dash, "\dashv" marking] & u_2 \arrow[r, dash, "\dashv" marking] \arrow[d, "\varphi_2"] & ... \\ v_0 \arrow[r, dash, "\dashv" marking] & v_1 \arrow[r, dash, "\dashv" marking] \arrow[u, "\varphi_1"'] & v_2 \arrow[r, dash, "\dashv" marking] & ... \end{tikzcd}$$

amounts to a 2-cell $\varphi_0\colon u_0 \to v_0$ of $\mathbf{C}$, together with its mates in the corresponding adjunctions

$$\begin{aligned}
&\varphi_0\colon u_0 \to v_0\colon X \to Y, && \\
&\varphi_1\colon v_1 \to u_1\colon Y \to X && (\varphi_1 = \eta_1 v_1 \otimes u_1\varphi_0 v_1 \otimes u_1\varepsilon_1), \\
&\varphi_2\colon u_2 \to v_2\colon X \to Y && (\varphi_2 = \eta_2 u_2 \otimes v_2\varphi_1 u_2 \otimes v_2\varepsilon_2), \\
&\ldots &&
\end{aligned} \tag{44}$$

The vertical composition of φ with a 2-cell $\psi = (\psi_0, ..., \psi_i)\colon v \to w\colon X \to_i Y$ amounts to the composite $\varphi_0 \otimes \psi_0$, and can be written as

$$\varphi \otimes \psi = (\varphi_0 \otimes \psi_0, \psi_1 \otimes \varphi_1, \varphi_2 \otimes \psi_2, ...)\colon u \to w\colon X \to_i Y, \tag{45}$$

since mates agree with composition (in a contravariant way, of course). The identity of u is the sequence $(\mathrm{id}(u_0), ..., \mathrm{id}(u_i))$.

The whisker composition of a 2-cell $\varphi = (\varphi_0, \varphi_1, \varphi_2, ...)\colon u \to v\colon X \to_i Y$ with arrows $r\colon X' \to_i X$ and $s\colon Y \to_i Y'$ is

$$s\varphi r = (s_0\varphi_0 r_0, r_1\varphi_1 s_1, s_2\varphi_2 r_2, ...)\colon sur \to svr\colon X' \to_i Y', \tag{46}$$

since mates agree with whisker composition (again in a contravariant way).

The 2-functor $U_{ji}\colon \mathbf{C}_j \to \mathbf{C}_i$ forgets the components of arrows and cells of all indices $i+1, ..., j$.

Finally we define $\mathsf{Adj}(\mathbf{C})$ as the multiple category $\mathsf{GQ}(\mathcal{C})$ of this sequence of 2-categories.

3.4. Truncation. The multiple category $\mathsf{Adj}(\mathbf{C})$ extends the double category $\mathbb{A}\mathrm{dj}(\mathbf{C})$ of morphisms and adjunctions of $\mathbf{C}$ considered in [GP1], Section 3.5 (written down for $\mathbf{C} = \mathbf{Cat}$, but the general case works in the same way).

To show that $\mathbb{A}\mathrm{dj}(\mathbf{C})$ can be identified with the 2-dimensional truncation of $\mathsf{Adj}(\mathbf{C})$, we begin by noting that the 0- and 1-cells of $\mathsf{Adj}(\mathbf{C})$ coincide with the horizontal and vertical arrows of $\mathbb{A}\mathrm{dj}(\mathbf{C})$; we are left with examining the 2-dimensional cells.

In a 01-cell of $\mathsf{Adj}(\mathbf{C})$ the vertical arrows are adjunctions $u = (u_\bullet, u^\bullet, \eta, \varepsilon)$ and $v = (v_\bullet, v^\bullet, \eta', \varepsilon')$ in $\mathbf{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "v"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \qquad \varphi_\bullet\colon v_\bullet f \to g u_\bullet\colon A \to D, \qquad \begin{tikzcd} \bullet \arrow[r, "0"] \arrow[d, "1"] & {} \\ {} & \end{tikzcd} \tag{47}$$

and φ amounts to a single 2-cell $\varphi_\bullet$ of $\mathbf{C}$, as above. But we can equivalently add a mate $\varphi^\bullet\colon fu^\bullet \to v^\bullet g$

$$\varphi^\bullet = \eta' f u^\bullet \otimes v^\bullet \varphi_\bullet u^\bullet \otimes v^\bullet g\varepsilon\colon\ fu^\bullet \to v^\bullet v_\bullet f u^\bullet \to v^\bullet g u_\bullet u^\bullet \to v^\bullet g, \tag{48}$$

and this completes the double cell $\varphi = (\varphi_\bullet, \varphi^\bullet)$ as defined in [GP1].

4. Generalised quintets for arrow bundles

We now study a similar construction of a non-cubical multiple category of generalised quintets, based on a sequence of 2-categories and 2-functors of a different shape.

Then we deduce a non-cubical multiple category $\mathsf{Bnd}(\mathbf{C})$ where an i-directed arrow is a 'bundle' $u = (u_0, ..., u_i)$ of parallel arrows in a fixed 2-category $\mathbf{C}$.

4.1. A SECOND CONSTRUCTION OF GENERALISED QUINTETS. We start now from a sequence $\mathcal{C}$ of 2-functors which *raise the index*, and again are the identity on a common set $S = \mathrm{Ob}\mathbf{C}_i$ of objects

$$(\mathcal{C}) \qquad\qquad \mathbf{C}_0 \xrightarrow{U_0} \mathbf{C}_1 \xrightarrow{U_1} \mathbf{C}_2 \longrightarrow \;...\; \longrightarrow \mathbf{C}_i \xrightarrow{U_i} \;... \tag{49}$$

$$U_{ij} = U_{j-1}...U_i \colon \mathbf{C}_i \to \mathbf{C}_j \qquad\qquad (0 \leqslant i < j),$$

The 2-functors U_{ij} are called the *structural functors* of $\mathcal{C}$. We shall construct a multiple category $\mathsf{M} = \mathsf{GQ}(\mathcal{C})$ of *generalised quintets* over the sequence $\mathcal{C}$, of coskeletal dimension 3. The construction is much the same as in 3.1, taking into account that the structural functors work the other way round.

(a) Again, the set of objects is S and an i-cell $u \colon X \to_i Y$ is a $\mathbf{C}_i$-morphism ($i \geqslant 0$). They form the category $\mathrm{cat}_i(\mathsf{M})$ underlying $\mathbf{C}_i$.

(b) In dimension 2 an ij-cell (for $0 \leqslant i < j$) is a *U-quintet*, or *quintet modulo* $U_{ij} \colon \mathbf{C}_i \to \mathbf{C}_j$. It amounts to: two $\mathbf{C}_i$-morphisms r, s (its j-faces), two $\mathbf{C}_j$-morphisms u, v (its i-faces) and a 2-cell φ in $\mathbf{C}_j$

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "v"] \\ C \arrow[r, "s"'] & D \end{tikzcd} \qquad \varphi \colon v.U_{ij}r \to U_{ij}s.u \colon A \to D. \qquad \begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d, "j"] & {} \\ {} & \end{tikzcd} \tag{50}$$

Such an ij-cell has an obvious underlying cell $|\varphi|$ in the double category $\mathbb{Q}(\mathbf{C}_j)$ of quintets over $\mathbf{C}_j$. U-quintets inherit from the latter two composition laws in directions i and j, and form a double category $\mathrm{dbl}_{ij}(\mathsf{M})$.

(c) In dimension 3 an ijk-cell (for $0 \leqslant i < j < k$) is a cube of U-quintets whose image in $\mathbf{C}_k$ commutes

$$\begin{tikzcd}[column sep=small] \bullet \arrow[rr, "r"] \arrow[dr, "u"] \arrow[dd, "x"'] & \arrow[d, phantom, "\varphi"] & \bullet \arrow[dr, "v"] & \\ & \bullet \arrow[rr, "s"] \arrow[dd, "y"] \arrow[dr, phantom, "\rho"] & & \bullet \arrow[dd, "y'"] \\ \bullet \arrow[dr, "u'"'] \arrow[u, phantom, "\omega"] & & & \\ & \bullet \arrow[rr, "s'"'] & & \bullet \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \bullet \arrow[rr, "r"] \arrow[dd, "x"'] \arrow[ddrr, phantom, "\pi"] & & \bullet \arrow[dr, "v"] \arrow[dd, "x'"] & \\ & & & \bullet \arrow[dd, "y'"] \arrow[dl, phantom, "\zeta"] \\ \bullet \arrow[rr, "r'"] \arrow[dr, "u'"'] \arrow[drr, phantom, "\psi"] & & \bullet \arrow[dr, "v'"] & \\ & \bullet \arrow[rr, "s'"'] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d, "k"'] \arrow[dr, "j"] & {} \\ {} & {} \end{tikzcd} \tag{51}$$

More precisely, we have six U-quintets determined by 2-dimensional cells in $\mathbf{C}_j$ or $\mathbf{C}_k$

$$\begin{array}{llll}
\text{- } ij\text{-cells:} & \varphi\colon v.U_{ij}r \to U_{ij}s.u, & \psi\colon v'.U_{ij}r' \to U_{ij}s'.u' & (\text{2-cells in } \mathbf{C}_j),\\
\text{- } ik\text{-cells:} & \pi\colon x'.U_{ik}r \to U_{ik}r'.x, & \rho\colon r.U_{ik}s \to U_{ik}s'.y & (\text{2-cells in } \mathbf{C}_k),\\
\text{- } jk\text{-cells:} & \omega\colon y.U_{jk}u \to U_{jk}u'.x, & \zeta\colon r.U_{jk}v \to U_{jk}v'.x' & (\text{2-cells in } \mathbf{C}_k).
\end{array}$$

They must satisfy the following commutativity relation, in the 2-category $\mathbf{C}_k$

$$r.U\varphi \otimes \rho.Uu \otimes Us'.\omega = \zeta.Ur \otimes Uv'.\pi \otimes U\psi.x\colon\ r.Uv.Ur \to Us'.Uu'.x,$$

where U stands for U_{ik} or U_{jk} when it operates on $\mathbf{C}_i$ or $\mathbf{C}_j$, respectively.

The compositions in directions i, j, k amount to compositions of faces in the double categories $\mathrm{dbl}_{ij}(\mathsf{M})$, $\mathrm{dbl}_{jk}(\mathsf{M})$ and $\mathrm{dbl}_{ik}(\mathsf{M})$. We have thus a triple category $\mathrm{trp}_{ijk}(\mathsf{M})$.

(d) Every cell of dimension $n > 3$ is an n-dimensional cube whose 2-dimensional faces are U-quintets, under the condition that each 3-dimensional face in direction ijk be an ijk-cell, as defined above.

Again, the coskeletal dimension of the multiple category $\mathsf{M} = \mathsf{GQ}(\mathcal{C})$ is at most 3. If all the functors U_i are embeddings of 2-categories (injectives on objects, morphisms and 2-cells), $\mathsf{GQ}(\mathcal{C})$ can be obtained as a multiple subcategory of $\mathsf{Q}(\mathbf{C}_\infty)$, where $\mathbf{C}_\infty$ is the colimit of diagram (49) - or some larger structure if convenient. $\mathsf{GQ}(\mathcal{C})$ is thus a multiple category of quintet type over the 2-category $\mathbf{C}_\infty$.

4.2. A multiple category of arrow bundles. As an example, we start from a 2-category $\mathbf{C}$ and construct a diagram $\mathcal{C}$ of 2-categories and 2-functors, of type (49).

For every $i \geqslant 0$, $\mathbf{C}_i$ is a 2-category with the same objects of $\mathbf{C}$, and *bundles* of arrows and cells

$$\begin{array}{ll}
u = (u_0, ..., u_i)\colon A \to_i B, & u_0, ..., u_i \in \mathbf{C}(A, B),\\
\varphi = (\varphi_0, ..., \varphi_i)\colon u \to v\colon A \to_i B, & \varphi_0\colon u_0 \to v_0, ..., \varphi_i\colon u_i \to v_i \ \text{ in } \ \mathbf{C},
\end{array} \qquad (52)$$

so that $\mathbf{C}_0 = \mathbf{C}$. Furthermore the 2-functor $U_i\colon \mathbf{C}_i \to \mathbf{C}_{i+1}$ repeats the last arrow or cell of a bundle.

We shall write $U_{ij}(u_0, ..., u_i) = (u_0, ..., u_j)$, leaving as understood that $u_i = ... = u_j$; similarly for 2-cells.

Plainly we are just considering an increasing filtration of the 2-category $\mathbf{C}_\infty$ that has unbounded bundles $(u_h)_{h \geqslant 0}$, $(\varphi_h)_{h \geqslant 0}$ of parallel arrows and cells of $\mathbf{C}$ (eventually constant, if we want).

Therefore the multiple category $\mathsf{Bnd}(\mathbf{C}) = \mathsf{GQ}(\mathcal{C})$ *of arrow bundles over* $\mathbf{C}$ is a multiple subcategory of $\mathsf{Q}(\mathbf{C}_\infty)$. It can be described as follows.

(a) The set of objects is $\mathrm{Ob}\mathbf{C}$ and an i-cell $u = (u_0, ..., u_i)\colon A \to_i B$ is a bundle of $i+1$ parallel arrows of $\mathbf{C}$. They form the category $\mathrm{cat}_i(\mathsf{Bnd}(\mathbf{C})) = \mathbf{C}_i$.

(b) In dimension 2 an ij-cell (for $0 \leqslant i < j$) amounts to: two $\mathbf{C}_i$-morphisms r, s (its j-faces), two $\mathbf{C}_j$-morphisms u, v (its i-faces) and a sequence $\varphi = (\varphi_0, ..., \varphi_j)$ of 2-cells in $\mathbf{C}$

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "v"] \\ C \arrow[r, "s"'] & D \end{tikzcd} \qquad \varphi_h \colon v_h r_h \to s_h u_h \quad (1 \leqslant h \leqslant j), \qquad \begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d, "j"] & {} \\ {} \end{tikzcd} \tag{53}$$

where the sequences (r_h) and (s_h) are constant for $h \geqslant i$.

(c) In dimension 3 an ijk-cell (for $0 \leqslant i < j < k$) is a cubical diagram

$$\begin{tikzcd} \bullet \arrow[rr, "r"] \arrow[dr, "u"] \arrow[dd, "x"'] & \arrow[d, phantom, "\varphi"] & \bullet \arrow[dr, "v"] & \\ \arrow[r, phantom, "\omega"] & \bullet \arrow[rr, "s"] \arrow[dd, "y"] & \arrow[d, phantom, "\rho"] & \bullet \arrow[dd, "y'"] \\ \bullet \arrow[dr, "u'"'] & & {} & \\ & \bullet \arrow[rr, "s'"'] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[rr, "r"] \arrow[dd, "x"'] & \arrow[d, phantom, "\pi"] & \bullet \arrow[dr, "v"] \arrow[dd, "x'"'] & \\ & {} & \arrow[r, phantom, "\zeta"] & \bullet \arrow[dd, "y'"] \\ \bullet \arrow[rr, "r'"] \arrow[dr, "u'"'] & \arrow[d, phantom, "\psi"] & \bullet \arrow[dr, "v'"'] & \\ & \bullet \arrow[rr, "s'"'] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d, "k"'] \arrow[dr, "j"] & {} \\ {} & {} \end{tikzcd} \tag{54}$$

Its six faces are determined by their boundary-arrows and by bundles of 2-cells of $\mathbf{C}$

- ij-cells: $\quad \varphi_h \colon v_h r_h \to s_h u_h, \quad \psi_h \colon v'_h r'_h \to s'_h u'_h \qquad (1 \leqslant h \leqslant j)$,
- ik-cells: $\quad \pi_h \colon x'_h r_h \to r'_h x_h, \quad \rho_h \colon y'_h s_h \to s'_h . y_h \qquad (1 \leqslant h \leqslant k)$,
- jk-cells: $\quad \omega_h \colon y_h u_h \to u'_h x_h, \quad \zeta_h \colon y'_h v_h \to v'_h x'_h \qquad (1 \leqslant h \leqslant k)$.

They must satisfy the following commutativity relations, in $\mathbf{C}$

$$y'_h \varphi_h \otimes \rho_h u_h \otimes s'_h \omega_h = \zeta_h r_h \otimes v'_h \pi_h \otimes \psi_h x_h \colon y'_h v_h r_h \to s'_h u'_h x_h \quad (1 \leqslant h \leqslant k).$$

(d) Every cell of dimension $n > 3$ is an n-cube whose 2-dimensional faces are U-quintets, under the condition that each 3-dimensional face in direction ijk be an ijk-cell, as defined above.

5. The triple category of pseudo algebras

For a 2-monad T on a 2-category $\mathbf{C}$ we build a triple category of pseudo T-algebras, of quintet type. Then we show how algebras and normal pseudo algebras of graphs of categories are related to strict and weak double categories.

Algebras and pseudo algebras for a 2-monad are studied in [Bu, BlKP, Fi].

5.1. Reviewing pseudo algebras. We have a 2-monad $T = (T, h, m)$ on the 2-category $\mathbf{C}$. A *pseudo algebra* in $\mathbf{C}$ is a quadruple (A, a, ω, κ) consisting of an object A of $\mathbf{C}$, a map $a \colon TA \to A$ (the *structure*) and two vertically invertible cells (the *comparisons*)

$$\begin{aligned} &\omega \colon 1_A \to a.hA && \text{(the } \textit{normaliser}\text{)}, \\ &\kappa \colon a.Ta \to a.mA && \text{(the } \textit{extended associator}\text{)}, \end{aligned} \tag{55}$$

$$
\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "hA"'] \arrow[d, phantom, shift left=4ex, "\swarrow\omega"] & A \arrow[d, "1"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "Ta"] \arrow[d, "mA"'] \arrow[d, phantom, shift left=4ex, "\swarrow\kappa"] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
$$

These data are to satisfy three conditions of coherence:

$$
\begin{aligned}
a.T\omega \otimes \kappa.ThA = 1_a = \omega a \otimes \kappa.hTA, \\
\kappa.T^2a \otimes \kappa.mTA = a.T\kappa \otimes \kappa.TmA,
\end{aligned}
\tag{56}
$$

$$
\begin{tikzcd}
a \arrow[r, "\omega a"] \arrow[d, "a.T\omega"'] \arrow[ddrr, "1"] & a.hA.a \arrow[r, equal] & a.Ta.hTA \arrow[d, "\kappa.hTA"] \\
a.Ta.ThA \arrow[d, "\kappa.ThA"'] & & a.mA.hTA \arrow[d, equal] \\
a.mA.ThA \arrow[rr, equal] & & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a.Ta.T^2a \arrow[r, "a.T\kappa"] \arrow[d, "\kappa.T^2a"'] & a.Ta.TmA \arrow[d, "\kappa.TmA"] \\
a.mA.T^2a \arrow[d, equal] & a.mA.TmA \arrow[d, equal] \\
a.Ta.mTA \arrow[r, "\kappa.mTA"'] & a.mA.mTA
\end{tikzcd}
$$

A (strict) *morphism of pseudo algebras* $f\colon (A, a, \omega, \kappa) \to (B, b, \omega, \kappa)$ is a morphism $f\colon A \to B$ of $\mathbf{C}$ which preserves the structure (note that the comparisons of pseudo algebras are always denoted by the same letters)

$$
b.Tf = f.a, \qquad \omega f = f\omega, \qquad \kappa.T^2f = f\kappa. \tag{57}
$$

We write as $\mathbf{Psa}_2(T)$ the category of pseudo algebras and their strict morphisms.

5.2. Normal pseudo algebras. We say that a pseudo algebra (A, a, ω, κ) is *normal* if the normaliser ω is the identity, and therefore $a.hA = 1_A$.

Normal pseudo algebras are important, and should not be called 'unitary', as the following example shows. We consider the 2-monad $T\colon \mathbf{Cat} \to \mathbf{Cat}$, where $T(A)$ is the free strict monoidal category over the category A; its objects are the finite families $(X_1, ..., X_n)$ of objects of A.

A strict monoidal structure $a\colon TA \to A$ over a (small) category A

$$
a(X_1, ..., X_n) = \textstyle\bigotimes_i X_i, \tag{58}
$$

gives all finite tensor products in A; the identity object $I = a(\underline{e})$ comes from the empty family $\underline{e}$.

A *normal* pseudo algebra (A, a, κ) amounts to an unbiased monoidal category, *with a trivial unary tensor* $\bigotimes(X) = X$ (of single objects of A); the two unitors and the binary associator all come from the extended associator κ (that operates on finite tensor products)

$$
\begin{aligned}
&\textstyle\bigotimes(\bigotimes(\underline{e}), \bigotimes(X)) \to \bigotimes(X), \qquad \bigotimes(\bigotimes(X), \bigotimes(\underline{e})) \to \bigotimes(X), \\
&\textstyle\bigotimes(\bigotimes(X), \bigotimes(Y, Z)) \longrightarrow \bigotimes(X, Y, Z) \longleftarrow \bigotimes(\bigotimes(X, Y), \bigotimes(Z))
\end{aligned}
\tag{59}
$$

In the general case each object X has an associated object $\bigotimes(X)$, isomorphic to X, which - when the procedure is idempotent - can be viewed as a 'normal form'.

For instance, let A be the monoidal category of finite totally ordered sets, with $X \otimes Y$ the ordinal sum (extending the sum $X + Y$ by letting $x < y$ for all $x \in X$, $y \in Y$). Redefining $\bigotimes'(X_1, ..., X_n)$ as the ordinal isomorphic to $\bigotimes(X_1, ..., X_n)$ we have a pseudo algebra (A, a, ω, κ) which is not normal, but has a trivial associator $\kappa = \mathrm{id}$.

5.3. Lax, colax and pseudo morphisms. Let us come back to a 2-monad (T, h, m) on the 2-category $\mathbf{C}$ and its pseudo algebras.

(a) A *lax morphism of pseudo algebras* $\mathbf{f} = (f, \varphi)\colon (A, a, \omega, \kappa) \to (B, b, \omega, \kappa)$ is a morphism $f\colon A \to B$ of $\mathbf{C}$ with a comparison cell φ such that:

$$\begin{gathered} \varphi\colon b.Tf \to f.a, \qquad\qquad \omega f \otimes \varphi.hA = f\omega, \\ \kappa.T^2 f \otimes \varphi.mA = b.T\varphi \otimes \varphi.Ta \otimes f\kappa, \end{gathered} \tag{60}$$

$$\begin{tikzcd} f \arrow[r, "f\omega"] \arrow[d, "\omega f"'] & fa.hA & b.Tb.T^2 f \arrow[r, "b.T\varphi"] \arrow[d, "\kappa.T^2 f"'] & b.Tf.Ta \arrow[r, "\varphi.Ta"] & fa.TA \arrow[d, "f\kappa"] \\ b.hB.f \arrow[r, equal] & b.Tf.hA \arrow[u, "\varphi.hA"'] & b.mB.T^2 f \arrow[r, equal] & b.Tf.mA \arrow[r, "\varphi.mA"'] & fa.mA \end{tikzcd}$$

These morphisms compose: given $(g, \gamma)\colon (B, b, \omega, \kappa) \to (C, c, \omega, \kappa)$, we let

$$(g, \gamma).(f, \varphi) = (gf, \gamma.Tf \otimes g\varphi), \qquad \gamma.Tf \otimes g\varphi\colon c.T(gf) \to gb.Tf \to gfa. \tag{61}$$

We have thus a category $\mathrm{Lx}\mathbf{Psa}_2(T)$, with identities $\mathrm{id}(A, a, \omega, \kappa) = (\mathrm{id}A, 1_a)$. A *pseudo morphism* is a lax morphism (f, φ) where the cell φ is vertically invertible.

(b) A *colax morphism of pseudo algebras* $\mathbf{r} = (r, \rho)\colon (A, a, \omega, \kappa) \to (B, b, \omega, \kappa)$ is a morphism $r\colon A \to B$ of $\mathbf{C}$ with a comparison cell ρ such that:

$$\begin{gathered} \rho\colon r.a \to b.Tr, \qquad\qquad r\omega \otimes \rho.hA = \omega r, \\ r\kappa \otimes \rho.mA = \rho.Ta \otimes b.T\rho \otimes \kappa.T^2 r, \end{gathered} \tag{62}$$

$$\begin{tikzcd} r \arrow[r, "r\omega"] \arrow[d, "\omega r"'] & ra.hA \arrow[d, "\rho.hA"] & ra.Ta \arrow[r, "\rho.Ta"] \arrow[d, "r\kappa"'] & b.Tr.Ta \arrow[r, "b.T\rho"] & b.Tb.T^2 r \arrow[d, "\kappa.T^2 r"] \\ b.hB.r \arrow[r, equal] & b.Tr.hA & ra.mA \arrow[r, "\rho.mA"'] & b.Tr.mA \arrow[r, equal] & b.mB.T^2 r \end{tikzcd}$$

Given a second colax morphism $(s, \sigma)\colon (B, b, \omega, \kappa) \to (C, c, \omega, \kappa)$, we let

$$(s, \sigma).(r, \rho) = (sr, \sigma.Tr \otimes s\rho), \qquad \sigma.Tr \otimes s\rho\colon sra \to sb.Tr \to c.T(sr). \tag{63}$$

This gives a category $\mathrm{Cx}\mathbf{Psa}_2(T)$, with identities as above.

5.4. A DOUBLE CATEGORY OF PSEUDO ALGEBRAS. We form now a double category $\mathbb{P}\mathrm{sa}_2(T)$ *of pseudo algebras* of T, with lax morphisms in horizontal and colax morphisms in vertical. The construction is similar to that of $\mathbb{D}\mathrm{bl}$ in [GP2], and we shall see that it extends it.

Its objects are the pseudo T-algebras $\mathbb{A} = (A, a, \omega, \kappa)$, $\mathbb{B} = (B, a, \omega, \kappa)$,...; its horizontal arrows are the *lax* morphisms $\mathbf{f} = (f, \varphi)$, $\mathbf{g} = (g, \gamma)$...; its vertical arrows are the *colax* morphisms $\mathbf{r} = (r, \rho)$, $\mathbf{s} = (s, \sigma)$... A cell $\boldsymbol{\pi}\colon (\mathbf{r}\ {}^{\mathbf{f}}_{\mathbf{g}}\ \mathbf{s})$ consists of four morphisms as above together with a cell $\pi\colon sf \to gr\colon A \to D$ in $\mathbf{C}$ (where sf and gr are *just* morphisms of $\mathbf{C}$, not of algebras)

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "\mathbf{f}"] \arrow[d, "\mathbf{r}"'] \arrow[dr, phantom, "\boldsymbol{\pi}"] & \mathbb{B} \arrow[d, "\mathbf{s}"] \\
\mathbb{C} \arrow[r, "\mathbf{g}"'] & \mathbb{D}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "r"'] \arrow[dr, phantom, "\pi"] & B \arrow[d, "s"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\qquad (64)
$$

$$
\begin{array}{lllll}
f\colon A \to B, & \varphi\colon b.Tf \to f.a, & g\colon C \to D, & \gamma\colon d.Tg \to g.c & \text{(lax morphisms)},\\
r\colon A \to C, & \rho\colon r.a \to c.Tr, & s\colon B \to D, & \sigma\colon s.b \to d.Ts & \text{(colax morphisms)},\\
 & \pi\colon sf \to gr\colon A \to D & & & \text{(a 2-cell of } \mathbf{C}).
\end{array}
$$

(Let us note that $\boldsymbol{\pi}$ consists of its boundary and a double cell $\pi\colon (r\ {}^{f}_{g}\ s)$ of the double category $\mathbb{Q}(\mathbf{C})$ of quintets over the 2-category $\mathbf{C}$, as displayed in the right diagram above.)

These data must satisfy the following coherence condition, in the 2-category $\mathbf{C}$

(coh) $\quad s\varphi \otimes \pi a \otimes g\rho = \sigma.Tf \otimes d.T\pi \otimes \gamma.Tr,$

$$
\begin{tikzcd}
 & TB \arrow[rr, "b"] \arrow[d, phantom, "\varphi"] & & B \arrow[dr, "s"] & \\
TA \arrow[ur, "Tf"] \arrow[rr, "a"] \arrow[dr, "Tr"'] & \, & A \arrow[ur, "f"] \arrow[dr, "r"'] \arrow[rr, phantom, "\pi"] & & D \\
 & TC \arrow[rr, "c"'] \arrow[u, phantom, "\rho"] & & C \arrow[ur, "g"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & TB \arrow[rr, "b"] \arrow[dr, "Ts"] & & B \arrow[dr, "s"] \arrow[d, phantom, "\sigma"] & \\
TA \arrow[ur, "Tf"] \arrow[dr, "Tr"'] \arrow[rr, phantom, "T\pi"] & & TD \arrow[rr, "d"] & \, & D \\
 & TC \arrow[rr, "c"'] \arrow[ur, "Tg"'] & & C \arrow[ur, "g"'] \arrow[u, phantom, "\gamma"] &
\end{tikzcd}
\qquad (65)
$$

The horizontal and vertical composition of double cells are both defined using the vertical composition of the 2-category $\mathbf{C}$. Namely, for a consistent matrix of double cells

$$
\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathbf{f}"] \arrow[d, "\mathbf{r}"'] \arrow[dr, phantom, "\boldsymbol{\pi}"] & \mathsf{B} \arrow[r, "\mathbf{f}'"] \arrow[d, "\mathbf{s}"] \arrow[dr, phantom, "\boldsymbol{\vartheta}"] & \mathsf{B}' \arrow[d, "\mathbf{t}"] \\
\mathsf{C} \arrow[r, "\mathbf{g}"] \arrow[d, "\mathbf{r}'"'] \arrow[dr, phantom, "\boldsymbol{\zeta}"] & \mathsf{D} \arrow[r, "\mathbf{g}'"] \arrow[d, "\mathbf{s}'"] \arrow[dr, phantom, "\boldsymbol{\omega}"] & \mathsf{D}' \arrow[d, "\mathbf{t}'"] \\
\mathsf{E} \arrow[r, "\mathbf{h}"'] & \mathsf{F} \arrow[r, "\mathbf{h}'"'] & \mathsf{F}'
\end{tikzcd}
\qquad (66)
$$

we let:

$$(\pi \,|\, \vartheta) = \vartheta f \otimes g'\pi, \qquad\qquad (\frac{\pi}{\zeta}) = s'\pi \otimes \zeta r. \tag{67}$$

We prove below that these cells are indeed coherent. Then, plainly, these composition laws are strictly associative and unitary. Moreover, they satisfy the middle-four interchange law because this holds in the double category $\mathbb{Q}(\mathbf{C})$ of quintets over the 2-category $\mathbf{C}$.

We have thus a forgetful double functor, which is cellwise faithful

$$U\colon \mathbb{P}\mathrm{sa}_2(T) \to \mathbb{Q}(\mathbf{C}), \qquad \mathbb{A} \mapsto A, \qquad \mathbf{f} \mapsto f, \qquad \mathbf{r} \mapsto r, \qquad \boldsymbol{\pi} \mapsto \pi. \tag{68}$$

Finally we verify the axiom (coh) for $(\pi \,|\, \vartheta)$, which means that

$$t(\varphi'.Tf \otimes f'\varphi) \otimes (\vartheta f \otimes g'\pi)a \otimes g'g\rho = \tau.T(f'f) \otimes d'.T(\vartheta f \otimes g'\pi) \otimes (\gamma'.Tg \otimes g'\gamma).Tr,$$

where $a\colon TA \to A$ and $d'\colon TD' \to D'$ are the structures of the pseudo algeras A and D'.

The proof is similar to that of Theorem 2.8. Writing cells as arrows between morphisms, our property amounts to the commutativity of the outer diagram below

$$\begin{tikzcd}
tf'fa \arrow[r, "\vartheta fa"] & g'sfa \arrow[r, "g'\pi a"] & g'gra \arrow[d, "g'g\rho"] \\
tf'b.Tf \arrow[u, "tf'\varphi"] \arrow[r, "\vartheta b.Tf"] & g'sb.Tf \arrow[u, "g's\varphi"'] \arrow[d, "g'\sigma.Tf"] & g'gc.Tr \\
tb'.T(f'f) \arrow[u, "t\varphi' Tf"] \arrow[d, "\tau.T(f'f)"'] & g'd.T(sf) \arrow[r, "g'd.T\pi"] & g'd.T(gr) \arrow[u, "g'\gamma.Tr"'] \\
d'.T(tf'f) \arrow[r, "d'.T(\vartheta f)"'] & d'.T(g'sf) \arrow[u, "\gamma' T(sf)"'] \arrow[r, "d'.T(g'\pi)"'] & d'.T(g'gr) \arrow[u, "\gamma'.T(gr)"']
\end{tikzcd} \tag{69}$$

Here the two hexagons commute by coherence of the double cells $\boldsymbol{\pi}$ and $\boldsymbol{\vartheta}$, and the two rectangles by interchange of 2-cells in $\mathbf{C}$.

5.5. Triple categories of pseudo algebras. The double category $\mathbb{P}\mathrm{sa}_2(T)$ can be extended to a triple category $\mathsf{P} = \mathsf{Ps}\mathbb{P}\mathrm{sa}_2(T)$ of pseudo algebras, adding the pseudo morphisms as transversal arrows.

More precisely, the sets P_0, P_1, P_2 of arrows of P consist of the pseudo, lax and colax morphisms of pseudo algebras, respectively. In dimension 2, the new 01- and 02-cells are obvious, since P_0 can be viewed as a subset of P_1 and P_2. Finally a 3-dimensional 012-cell is a 'commutative cube' determined by its six faces, as in (7).

$\mathsf{Ps}\mathbb{P}\mathrm{sa}_2(T)$ is a triple category of quintet type, with respect to the obvious forgetful functor with values in the triple category $\mathrm{trc}_3(\mathsf{Q}(\mathbf{C}))$. The latter has the same objects as $\mathbf{C}$, the same arrows in all three directions, quintets for all three kinds of double cells and commutative cubes for three dimensional cells.

Our construction is foreshadowed in Shulman's [Sh], Remark (4.8). As he points out (private communication): a 2-monad on $\mathbf{C}$ extends canonically to a triple monad on the triple category $\mathrm{trc}_3(\mathsf{Q}(\mathbf{C}))$, and its Eilenberg-Moore object (in the 2-category of strict triple categories, strict functors and strict transformations) consists of strict algebras, with strict, lax, and colax morphisms respectively, and higher cells as above.

In fact, we are more interested in the triple subcategory $\mathsf{Ps}(\mathrm{N}\mathbb{P}\mathrm{sa}_2(T))$ *of normal pseudo algebras*, which - for a convenient 2-monad T, will be proved to be equivalent to the triple category $\mathsf{Ps}(\mathbb{D}\mathrm{bl})$, obtained by extending the double category $\mathbb{D}\mathrm{bl}$ as above.

(Taking strict morphisms and double functors in the transversal direction, one would not get an equivalence, as will be remarked at the end of the proof of Theorem 5.8).

5.6. Graphs of categories. To examine double categories in the present framework, we let $\mathbf{C} = \mathrm{Gph}\mathbf{Cat}$ be the 2-category of graphs $A = (A_i, \partial^\alpha)$ in $\mathbf{Cat}$

$$\partial^\alpha \colon A_1 \rightrightarrows A_0, \tag{70}$$

where a 2-cell $\varphi\colon F \to G\colon A \to B$ is a pair of natural transformations of ordinary functors, consistent with the faces

$$\varphi_i\colon F_i \to G_i\colon A_i \to B_i, \qquad \partial^\alpha\varphi_1 = \varphi_0\partial^\alpha \qquad\qquad (i = 0, 1;\ \alpha = \pm). \tag{71}$$

A double category $\mathbb{D}$ has an underlying graph $U(\mathbb{D}) = (\mathrm{Hor}_i(\mathbb{D}), \partial^\alpha)$, formed by the category $\mathrm{Hor}_0(\mathbb{D})$ of objects and horizontal arrows, the category $\mathrm{Hor}_1(\mathbb{D})$ of vertical arrows and double cells (both with horizontal composition), linked by two ordinary functors, the vertical faces ∂^α.

This defines a forgetful 2-functor

$$U\colon \mathbf{Dbl} \to \mathrm{Gph}\mathbf{Cat} = \mathbf{C}, \tag{72}$$

on the 2-category of (small) double categories, double functors and horizontal transformations.

5.7. Theorem. [Strict double categories as algebras] *The 2-functor U defined above is 2-monadic: it gives a comparison 2-isomorphism $K\colon \mathbf{Dbl} \to \mathbf{Alg}(T)$ with the 2-category of T-algebras for the associated 2-monad $T = UD$.*

Proof. A graph of categories $A = (A_i, \partial^\alpha)$ generates a free double category $D(A)$, described as follows.

(a) $\mathrm{Hor}_0(DA)$ is the category A_0.

(b) $\mathrm{Ver}_0(DA)$ is the free category generated by the graph of sets $\mathrm{Ob}A = (\mathrm{Ob}A_i, \partial^\alpha)$; its arrows give the vertical arrows $(u_1, ..., u_n)$ of DA, including the vertical unit $e(x)$ on an object x of A_0 (the *empty path* at x).

(c) $\mathrm{Ver}_1(DA)$ is the free category generated by the graph of sets $\mathrm{Mor}A = (\mathrm{Mor}A_i, \partial^\alpha)$; its arrows give the double cells $(a_1, ..., a_n)$ of DA, including the vertical unit $e(f)$ on a morphism f of A_0

$$
\begin{tikzcd}
\bullet \arrow[r, "f_0"] \arrow[d, "u_1"'] \arrow[dr, phantom, "a_1"] & \bullet \arrow[d, "v_1"] \\
\bullet \arrow[r] \arrow[d, dotted, no head] & \bullet \arrow[d, dotted, no head] \\
\bullet \arrow[r] \arrow[d, "u_n"'] \arrow[dr, phantom, "a_n"] & \bullet \arrow[d, "v_n"] \\
\bullet \arrow[r, "f_n"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "e(x)"'] \arrow[dr, phantom, "e(f)"] & y \arrow[d, "e(y)"] \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}
\tag{73}
$$

(d) The horizontal composition of these double cells is a concatenation of compositions in A_1, which we write as $(a_i \,|\, b_i)$

$$((a_1, ..., a_n) \,|\, (b_1, ..., b_n)) = ((a_1 \,|\, b_1), ..., (a_n \,|\, b_n)).$$

The obvious embedding $hA\colon A \to UD(A)$ is the 2-universal arrow from A to U. This gives the left 2-adjoint $D\colon \mathbf{C} \to \mathbf{Dbl}$ and the associated 2-monad (T, h, m) on $\mathbf{C}$, with $T = UD$.

The comparison K is plainly an isomorphism of 2-categories. ■

5.8. Theorem. [Weak double categories as normal pseudo algebras] *The triple categories* $\mathsf{Ps}(\mathrm{N}\mathbb{P}\mathrm{sa}_2(T))$ *and* $\mathsf{Ps}\mathbb{D}\mathrm{bl}$ *are linked by an adjoint equivalence of triple categories ([GP10], Section 5.4)*

$$
\begin{gathered}
V\colon \mathsf{Ps}(\mathrm{N}\mathbb{P}\mathrm{sa}_2(T)) \rightleftarrows \mathsf{Ps}\mathbb{D}\mathrm{bl} : J, \\
VJ = 1, \qquad \varepsilon\colon JV \cong 1, \qquad V\varepsilon = 1, \qquad \varepsilon J = 1.
\end{gathered}
\tag{74}
$$

Proof. (a) A normal pseudo algebra $\mathbb{A} = (A, c, \kappa)$ for T is a graph of categories $A = (A_i, \partial^\alpha)$ with an assigned vertical composition of finite paths of vertical arrows and cells

$$
\begin{aligned}
u_1 \otimes ... \otimes u_n &= c(u_1, ..., u_n), & a_1 \otimes ... \otimes a_n &= c(a_1, ..., a_n), \\
e_x &= c(e(x)), & e_f &= c(e(f)),
\end{aligned}
\tag{75}
$$

and an (invertible) extended associator $\kappa\colon c.Dc \to c.mA$.

As a first consequence of normality, the unary vertical composition is trivial: $c(u) = u$ and $c(a) = a$, for all items of the category A_1. Second, $\mathbb{A}$ is trivial in degree 0, in the sense that $(TA)_0 = \mathrm{Hor}_0(DA) = A_0$, while the functor c_0 and the natural transformation κ_0

$$c_0\colon (TA)_0 \to A_0, \qquad \kappa_0\colon c_0.Dc_0 \to c_0.(mA)_0\colon (T^2A)_0 \to A_0, \tag{76}$$

are identities: this follows easily from the coherence conditions (56), where ωA, $(hA)_0$ and $(mA)_0$ are identities.

(b) $\mathbb{A}$ can be viewed as an 'unbiased' weak double category, where all finite vertical compositions are assigned. The fact that κ_0 is trivial says that the comparison cells of the unbiased associator κ_1 are special, i.e. their horizontal arrows are identities - as required in the axioms of weak double categories, for the binary associator and the unitors.

The normal pseudo algebra $\mathbb{A}$ has un underlying weak double category $V(\mathbb{A})$, obtained by extracting the binary and zeroary vertical operations and their comparisons. We get a canonical triple functor V, that sends:

- a pseudo, lax or colax morphism of normal pseudo algebras to the corresponding pseudo, lax or colax double functor of weak double categories, reducing the unbiased comparisons of finite vertical composition to the 'biased ones', of binary and zeroary composition,

- a 2-dimensional cell of type 01 (resp. 02, 12)

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\varphi"] & \mathbb{A}' \arrow[d] \\ \mathbb{B} \arrow[r] & \mathbb{B}' \end{tikzcd} \tag{77}$$

to the corresponding quintet (resp. quintet, generalised quintet) of 'functors': the latter are reduced to their biased comparisons, but the components of φ (on the objects and vertical arrows of $\mathbb{A}$) stay the same,

- a 'commutative cube' of 2-dimensional cells to the 'commutative cube' of the modified cells.

(c) The other way round, we construct a triple functor J such that $VJ = 1$, by choosing *a* 'bracketing' of n-ary compositions. Namely, a weak double category $\mathbb{D}$ can be extended to a normal pseudo algebra $J(\mathbb{D})$ by defining the n-ary vertical composition (of vertical arrows and cells) as

$$x_1 \otimes ... \otimes x_n = (...((x_1 \otimes x_2) \otimes x_3)... \otimes x_n), \tag{78}$$

and extending the comparisons. A strict, or lax, or colax double functor becomes a strict, or lax, or colax morphism, by extending the comparisons (in the last two cases). For a 2-dimensional cell we just note that its coherence with the unbiased comparisons implies coherence with the biased ones.

(d) Finally, a normal pseudo algebra $\mathbb{A} = (A, c, \kappa)$ produces an object $JV(\mathbb{A}) = (A, c', \kappa')$ with modified unbiased vertical operations and modified unbiased comparisons. The identity $\mathrm{id}(A)$ of the underlying graph extends to an invertible *pseudo morphism*

$$\varepsilon\mathbb{A} \colon JV(\mathbb{A}) \to_0 \mathbb{A}$$

satisfying the triangular conditions, as in (74). For a lax morphism $F \colon \mathbb{A} \to \mathbb{B}$ we get an obvious 01-cell $\varepsilon F \colon JV(F) \to_0 F$, inhabited by an identity; similarly for colax morphisms and generalised quintets.

This point fails if we restrict to the triple categories $\mathsf{S}(\mathrm{N}\mathbb{P}\mathrm{sa}_2(T))$ and $\mathsf{S}\mathbb{D}\mathrm{bl}$ (with strict items in the transversal direction), because the component $\varepsilon\mathbb{A}$ is not a strict morphism. ■

References

[ABS] F.A.A. Al-Agl - R. Brown - R. Steiner, Multiple categories: the equivalence of a globular and a cubical approach, Adv. Math. 170 (2002), 71-118.

[BE] A. Bastiani - C. Ehresmann, Multiple functors I. Limits relative to double categories, Cahiers Top. Géom. Diff. 15 (1974), 215-292.

[BlKP] R. Blackwell - G.M. Kelly - J. Power, Two-dimensional monad theory, J. Pure Appl. Algebra 59 (1989), 1-41.

[Bo] P. Booth, Sequences of adjoint functors, Arch. Math. 23 (1972), 489-493.

[BrM] R. Brown - G. Mosa, Double categories, 2-categories, thin structures and connections, Theory Appl. Categ. 5 (1999), 163-175.

[Bu] A. Burroni, Structures pseudo-algébriques. I, Cah. Topol. Géom. Differ. 16 (1975), 343-393.

[DP1] R. Dawson - R. Paré, General associativity and general composition for double categories, Cahiers Top. Géom. Diff. Catég. 34 (1993), 57-79.

[DP2] R. Dawson - R. Paré, What is a free double category like?, J. Pure Appl. Algebra 168 (2002), 19-34.

[DPP1] R. Dawson - R. Paré - D.A. Pronk, Free extensions of double categories, Cah. Topol. Géom. Différ. Catég. 45 (2004), 35-80.

[DPP2] R. Dawson - R. Paré - D.A. Pronk, The span construction, Theory Appl. Categ. 24 (2010), 302-377.

[EE1] A. Ehresmann - C. Ehresmann, Multiple functors II. The monoidal closed category of multiple categories, Cahiers Top. Géom. Diff. 19 (1978), 295-333.

[EE2] A. Ehresmann - C. Ehresmann, Multiple functors III. The Cartesian closed category $\mathrm{Cat_n}$, Cahiers Top. Géom. Diff. 19 (1978), 387-443.

[EE3] A. Ehresmann - C. Ehresmann, Multiple functors IV. Monoidal closed structures on $\mathrm{Cat_n}$, Cahiers Top. Géom. Diff. 20 (1979), 59-104.

[Eh] C. Ehresmann, Catégories structurées, Ann. Sci. Ecole Norm. Sup. 80 (1963), 349-425.

[Fi] T.M. Fiore, Pseudo algebras and pseudo double categories, J. Homotopy Relat. Struct. 2 (2007), 119-170.

[FGK1] T.M. Fiore - N. Gambino - J. Kock, Monads in double categories, J. Pure Appl. Algebra 215 (2011), 1174-1197.

[FGK2] T.M. Fiore - N. Gambino - J. Kock, Double adjunctions and free monads, Cah. Topol. Géom. Différ. Catég. 53 (2012), 242-306.

[Ga] R. Garner, Double clubs, Cah. Topol. Géom. Différ. Catég. 47 (2006), 261-317.

[G1] M. Grandis, Higher cospans and weak cubical categories (Cospans in Algebraic Topology, I), Theory Appl. Categ. 18 (2007), No. 12, 321-347.

[G2] M. Grandis, The role of symmetries in cubical sets and cubical categories (On weak cubical categories, I), Cah. Topol. Géom. Différ. Catég. 50 (2009), 102-143.

[G3] M. Grandis, Adjoints for symmetric cubical categories (On weak cubical categories, III), Cah. Topol. Géom. Différ. Catég. 54 (2013), 91-126.

[GP1] M. Grandis - R. Paré, Limits in double categories, Cah. Topol. Géom. Différ. Catég. 40 (1999), 162-220.

[GP2] M. Grandis - R. Paré, Adjoint for double categories, Cah. Topol. Géom. Différ. Catég. 45 (2004), 193-240.

[GP3] M. Grandis - R. Paré, Kan extensions in double categories (On weak double categories, Part III), Theory Appl. Categ. 20 (2008), No. 8, 152-185.

[GP4] M. Grandis - R. Paré, Lax Kan extensions for double categories (On weak double categories, Part IV), Cah. Topol. Géom. Différ. Catég. 48 (2007), 163-199.

[GP5] M. Grandis - R. Paré, From cubical to globular higher categories, in: Liber Amicorum en l'honneur de Mme A.C. Ehresmann, Diagrammes 67-68 suppl. (2012), 117-148. Preprint available at: http://www.dima.unige.it/~grandis/CGlb.pdf

[GP6] M. Grandis - R. Paré, Intercategories, Theory Appl. Categ. 30 (2015), No. 38, 1215-1255.
http://www.tac.mta.ca/tac/volumes/30/38/30-38.pdf

[GP7] M. Grandis - R. Paré, Intercategories: a framework for three dimensional category theory, J. Pure Appl. Algebra 221 (2017), 999-1054. Online publication (30 August 2016):
http://www.sciencedirect.com/science/article/pii/S0022404916301256

[GP8] M. Grandis - R. Paré, An introduction to multiple categories (On weak and lax multiple categories, I), Cah. Topol. Gom. Diffr. Catg. 57 (2016), 103-159.
Preprint available at: http://www.dima.unige.it/~grandis/Mlc1.pdf

[GP9] M. Grandis - R. Paré, Limits in multiple categories (On weak and lax multiple categories, II), Cah. Topol. Géom. Différ. Catég., to appear.
Preprint available at: http://www.dima.unige.it/~grandis/Mlc2.pdf

[GP10] M. Grandis - R. Paré, Adjoints for multiple categories (On weak and lax multiple categories, III), Cah. Topol. Géom. Différ. Catég., to appear.
Preprint available at: http://www.dima.unige.it/~grandis/Mlc3.pdf

[GP11] M. Grandis - R. Paré, Span and cospan representations of weak double categories, Categ. Gen. Algebr. Struct. Appl. 6 (2017), 85-105.
Online publication: (18 Nov 2016): http://www.cgasa.ir/volume_3074.html

[Ko1] S.R. Koudenburg, On pointwise Kan extensions in double categories, Theory Appl. Categ. 29 (2014), 781-818.

[Ko2] S.R. Koudenburg, Algebraic Kan extensions in double categories, Theory Appl. Categ. 30 (2015), 86-146.

[Ni] S. Niefield, Span, cospan, and other double categories, Theory Appl. Categ. 26 (2012), 729-742.

[P1] R. Paré, Yoneda theory for double categories, Theory Appl. Categ. 25 (2011), 436-489.

[P2] R. Paré, Composition of modules for lax functors, Theory Appl. Categ. 27 (2012), 393-444.

[Sh] M. Shulman, Comparing composites of left and right derived functors New York J. Math. 17 (2011), 75-125.

Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146-Genova, Italy

Department of Mathematics and Statistics
Dalhousie University
Halifax NS
Canada B3H 4R2

Email: grandis@dima.unige.it, R.Pare@Dal.Ca