EIGENVALUES OF THE VERTEX SET HECKE ALGEBRA OF AN AFFINE BUILDING

A. M. MANTERO AND A. ZAPPA

Abstract. The aim of this paper is to describe the eigenvalues of the vertex set Hecke algebra of an affine building. We prove, by a direct approach, the invariance (with respect to the Weyl group) of any eigenvalue associated to a character. Moreover we construct the Satake isomorphism of the Hecke algebra and we prove, by this isomorphism, that every eigenvalue arises from a character.

1. Introduction

The aim of this paper is to discuss the eigenvalues of the vertex set Hecke algebra $\mathcal{H}(\Delta)$ of any affine building Δ, using only its geometric properties. We avoid making use of the structure of any group acting on Δ.

To every multiplicative function χ on the fundamental apartment $\mathbb{A}$ of the building we associate an eigenvalue Λ_χ that can be expressed in terms of the Poisson kernel relative to the character χ. We prove the invariance of the eigenvalue Λ_χ with respect to the action of the finite Weyl group $\mathbf{W}$ on the characters. Moreover we prove that every eigenvalue arises from a character. Following the method used by Macdonald in his paper [8], the basic tool we use to obtain this characterization is the definition of the Satake isomorphism between the algebra $\mathcal{H}(\Delta)$ and the Hecke algebra of all $\mathbf{W}$-invariant operators on the fundamental apartment $\mathbb{A}$.

Our approach strongly depends on the definition of an α-boundary Ω_α, for every simple root α. Indeed we associate to every point of Ω a tree, called tree at infinity, and we define the α-boundary Ω_α as the collection of all such isomorphic trees. Thus we can show that the maximal boundary splits as the product of Ω_α and the boundary ∂T of the tree at infinity, and so any probability measure on Ω decomposes as the product of a probability measure on Ω_α and the standard measure on ∂T.

Our goal is to present a proof of the results which puts the geometry of the building front and center. Since we intend to address a non-specialized audience, we make use of a language that reduces to a minimum the algebraic knowledge required about affine buildings. This makes the paper as self-contained as possible. Hence we give, without claim of originality except possibly in the presentation, the main results about buildings and their maximal boundary Ω.

In a forthcoming paper we will use our results here to construct the Macdonald formula for the spherical functions on the building.

For buildings of type $\widetilde{A}_2, \widetilde{B}_2$ and $\widetilde{G}_2$ the eigenvalues of the algebra $\mathcal{H}(\Delta)$ are described in detail in [10], [11] and [12] respectively.

We point out that an exhaustive presentation of the features of an affine building and its maximal boundary can be found in the paper [13] of J. Parkinson. Moreover the same author obtains in [14] the results about the eigenvalues of the algebra $\mathcal{H}(\Delta)$, by expressing all algebra homomorphisms $h : \mathcal{H}(\Delta) \to \mathbb{C}$ in terms of the Macdonald spherical functions.

2. Affine buildings

In this section we collect the fundamental definitions and properties concerning buildings and we fix notation we shall use in the following. Our presentation is based on [3], [15] and [16] and we refer the reader to these books for more details about the argument. We also point out the paper [13] for a similar presentation about buildings.

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2.1. **Labelled chamber complexes.** A *simplicial complex* (with vertex set $\mathcal{V}$) is a collection Δ of finite subsets of $\mathcal{V}$ (called *simplices*) such that every singleton $\{v\}$ is a simplex and every subset of a simplex A is a simplex (called a *face* of A).The cardinality r of A is called the *rank* of A, and $r-1$ is called the *dimension* of A. Moreover a simplicial complex is said to be a *chamber complex* if all maximal simplices have the same dimension d and any two can be connected by a *gallery*, that is by a sequence of maximal simplices in which any two consecutive ones are adjacent, that is have a common codimension 1 face. The maximal simplices will then be called *chambers* and the rank $d+1$ (respectively the dimension d) of any chamber is called the *rank* (respectively the *dimension*) of Δ. The chamber complex is said to be *thin* (respectively *thick*) if every codimension 1 simplex is a face of exactly two chambers (respectively at least three chambers).

A *labelling* of the chamber complex Δ by a set I is a function τ which assigns to each vertex an element of I (the *type* of the vertex), in such a way that the vertices of every chamber are mapped bijectively onto I. The number of labels or types used is required to be the rank of Δ (that is the number of vertices of any chamber), and joinable vertices are required to have different types. When a chamber complex Δ is endowed by a labelling τ, we say that Δ is a *labelled chamber complex.* For every $A \in \Delta$, we will call $\tau(A)$ the type of A, that is the subset of I consisting of the types of the vertices of A; moreover we call $I \setminus \tau(A)$ the co-type of A.

A *chamber system* over a set I is a set $\mathcal{C}$, such that each $i \in I$ determines a partition of $\mathcal{C}$, two elements in the same class of this partition being called *i-adjacent.* The elements of $\mathcal{C}$ are called chambers and we write $c \sim_i d$ to mean that the chambers c and d are i-adjacent. Then a labelled chamber complex is a chamber system over the set I of the types and two chambers are i-adjacent if they share a face of co-type i.

2.2. **Coxeter systems.** Let W be a group (possibly infinite) and S be a set of generators of W of order 2. Then W is called a *Coxeter group* and the pair (W, S) is called a *Coxeter system*, if W admits the presentation

$$\langle S\,;(st)^{m(s,t)} = 1\rangle,$$

where $m(s,t)$ is the order of st and there is one relation for each pair s, t, with $m(s,t) \le \infty$. We shall assume that S is finite, and denote by N the cardinality of S; then, if I is an arbitrary index set with $|I| = N$, we can write $S = (s_i)_{i\in I}$ and

$$W = \langle (s_i)_{i\in I}\,;(s_i s_j)^{m_{ij}} = 1\rangle,$$

where $m(s_i s_j) = m_{ij}$. When $w \in W$ is written as $w = s_{i_1} s_{i_2} \cdots s_{i_k}$, with $i_j \in I$ and k minimal, we say that the expression is reduced and we call *length* $|w|$ of w the number k. The matrix $M = (m_{ij})_{i,j\in I}$, with entries $m_{ij} \in \mathbb{Z} \cup \{\infty\}$, is called the *Coxeter matrix* of W. We shall represent M by its diagram D : the nodes of D are the elements of I (or of S) and between two nodes there is a bond if $m_{ij} \ge 3$, with the label m_{ij} over the bond if $m_{ij} \ge 4$. We call D the *Coxeter diagram* or the *Coxeter graph* of W. We often say that W has type M, if M is its Coxeter matrix.

2.3. **Coxeter complexes.** Let (W, S) be a Coxeter system, with $S = (s_i)_{i\in I}$ finite. We define a *special coset* to be a coset $w\langle S'\rangle$, with $w \in W$ and $S' \subset S$, and we define $\Sigma = \Sigma(W, S)$ to be the set of special cosets, partially ordered by the opposite of the inclusion relation: $B \le A$ in Σ if and only if $B \supseteq A$ as subsets of W, in which case we say that B is a *face* of A. The set Σ is a simplicial complex; moreover it is a thin, labellable chamber complex of rank $N = \text{card } S$ and the W-action on Σ is type-preserving. We remark that the chambers of Σ are the elements of W and, for each $i \in I$, $w \sim_i w'$ means that $w' = ws_i$ or $w' = w$. Following Tits, we shall call Σ the *Coxeter complex* associated to (W, S), or the *Coxeter complex of type* M, if M is the Coxeter matrix of W.

2.4. **Buildings.** Let (W, S) be a Coxeter system, and let $M = (m_{ij})_{i,j\in I}$ its Coxeter matrix. A *building of type* M (see Tits [16]) is a simplicial complex Δ, which can be expressed as the union of subcomplexes $\mathcal{A}$ (called *apartments*) satisfying the following axioms:

- (B_0) each apartment $\mathcal{A}$ is isomorphic to the Coxeter complex $\Sigma(W, S)$ of type M of W;
- (B_1) for any two simplices $A, B \in \Delta$, there is an apartment $\mathcal{A}(A, B)$ containing both of them;
- (B_2) if $\mathcal{A}$ and $\mathcal{A}'$ are two apartments containing A and B, there is an isomorphism $\mathcal{A} \to \mathcal{A}'$ fixing A and B point-wise.

Hence each apartment of Δ is a thin, labelled chamber complex over I of rank $N = |I|$. It can be proved that a building of type M is a chamber system over the set I with the properties:

(i) for each chamber $c \in \Delta$ and $i \in I$, there is a chamber $d \neq c$ in Δ such that $d \sim_i c$;

(ii) there exists a W-distance function

$$\delta \;:\; \Delta \times \Delta \to W$$

such that, if $f = i_1 \cdots i_k$ is a reduced word in the free monoid on I and $w_f = s_{i_1} \cdots s_{i_k} \in W$, then

$$\delta(c,d) = w_f$$

when c and d can be joined by a gallery of type f. We write $d = c \cdot \delta(c,d)$.

Actually it can be proved that each chamber system over a set I satisfying these properties is in fact a building.

To ensure that the labelling of Δ and $\Sigma(W,S)$ are compatible, we assume that the isomorphisms in (B_0) and (B_2) are *type-preserving*; this implies that the isomorphism in (B_2) is unique. We write $\mathcal{C}(\Delta)$ for the chamber set of Δ. We call *rank of* Δ the cardinality N of the index set I.

We always assume that Δ is irreducible, that is the associated Coxeter group W is irreducible (that is its Coxeter graph is connected). Moreover we confine ourselves to consider *thick* buildings.

2.5. **Regularity and parameter system.** Let Δ be a (irreducible) building of type M, with associated Coxeter group W over index set I, with $|I| = N$. We say that Δ is *locally finite* if

$$|\{d \in \mathcal{C}(\Delta),\ c \sim_i d\}| < \infty, \quad \forall i \in I,\ \forall c \in \mathcal{C}(\Delta).$$

Moreover we say that Δ is *regular* if this number does not depend on the chamber c. We shall assume that Δ is locally finite and regular. Since, for every $i \in I$, the set

$$\mathcal{C}_i(c) = \{d \in \mathcal{C}(\Delta),\ c \sim_i d\}$$

has a cardinality which does not depend on the choice of the chamber c, we put

$$q_i = |\mathcal{C}_i(c)|, \quad \forall c \in \mathcal{C}(\Delta).$$

The set $\{q_i\}_{i\in I}$ is called the *parameter system* of the building. We notice that the parameter system has the following properties (see for instance [13] for the proof):

(i) $q_i = q_j$, whenever $m_{i,j} < \infty$ is odd;
(ii) if $s_j = w s_i w^{-1}$, for some $w \in W$, then $q_i = q_j$.

The property (ii) implies (see [2]) that, for $w \in W$, the monomial $q_{i_1} \cdots q_{i_k}$ is independent of the particular reduced decomposition $w = s_{i_1} \cdots s_{i_k}$ of w. So we define, for every $w \in W$,

$$q_w = q_{i_1} \cdots q_{i_k}$$

if $s_{i_1} \cdots s_{i_k}$ is any reduced expression for w. If we set, for every $c \in \mathcal{C}(\Delta)$ and every $w \in W$,

$$\mathcal{C}_w(c) = \{d \in \mathcal{C}(\Delta),\ \delta(c,d) = w\},$$

it can be proved that

$$|\mathcal{C}_w(c)| = q_w = q_{i_1} \cdots q_{i_k},$$

whenever $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for w. Hence the cardinality of the set $\mathcal{C}_w(c)$ does not depend on the choice of the chamber c. Obviously, $q_w = q_{w^{-1}}$.

If U is any finite subset of W, we define

$$U(q) = \sum_{w \in U} q_w$$

and we call it the *Poincaré polynomial* of U.

2.6. **Affine buildings.** According to [2], W is called an *affine reflection group* if W is a group of affine isometries of a Euclidean space $\mathbb{V}$ (of dimension $n \geq 1$) generated by reflections s_H, where H ranges over a set locally finite $\mathcal{H}$ of affine hyperplanes of $\mathbb{V}$, which is W-invariant. We also assume W infinite. It is known that an affine reflection group is in fact a Coxeter group, because it has a finite set S of $n+1$ generators and admits the presentation

$$\langle S\,;(st)^{m(s,t)} = 1\rangle.$$

A building Δ (of type M) is said *affine* if the associated Coxeter group W is an affine reflection group. It is well known that each affine reflection group can be seen as the affine Weyl group of a root system. So we can define an affine building as a building associated to the affine Weyl group of a root system.

For the purpose of fixing notation, we shall give a brief discussion of root systems and its affine Weyl group, and we shall describe the geometric realization of the Coxeter complex associated to this group. We refer to [2] for an exhaustive reference to this subject.

2.7. **Root systems.** Let $\mathbb{V}$ be a vector space over $\mathbb{R}$, of dimension $n \geq 1$, with the inner product $\langle \cdot, \cdot \rangle$. For every $v \in \mathbb{V} \setminus \{0\}$ we define

$$v^\vee = \frac{2v}{\langle v, v \rangle}.$$

Let R be an irreducible, but not necessarily reduced, *root system* in $\mathbb{V}$ (see [2]). The elements of R are called *roots* and the rank of R is n.

Let $B = \{\alpha_i,\ i \in I_0\}$ be a basis of R, where $I_0 = \{1, \cdots, n\}$. Thus B is a subset of R, such that

(i) B is a vector space basis of $\mathbb{V}$;
(ii) each root in R can be written as a linear combination

$$\sum_{i \in I_0} k_i \alpha_i,$$

with integer coefficients k_i which are either all nonnegative or all nonpositive.

The roots in B are called *simple*. We say that $\alpha \in R$ is *positive* (respectively *negative*) if $k_i \geq 0, \forall i \in I_0$ (respectively $k_i \leq 0, \forall i \in I_0$). We denote by R^+ (respectively R^-) the set of all positive (respectively negative) roots. Thus $R^- = -R^+$ and $R = R^+ \cup R^-$ (as disjoint union). Define the *height* (with respect to B) of $\alpha = \sum_{i \in I_0} k_i \alpha_i$ by

$$ht(\alpha) = \sum_{i \in I_0} k_i.$$

There exists a unique root $\alpha_0 \in R$ whose height is maximal, and if we wright $\alpha_0 = \sum_{i \in I_0} m_i \alpha_i$, then $m_i \geq k_i$ for every root $\alpha = \sum_{i \in I_0} k_i \alpha_i$; in particular $m_i > 0$, $\forall i \in I_0$ (see [2]).

The set $R^\vee = \{\alpha^\vee,\ \alpha \in R\}$ is an irreducible root system, which is reduced if and only if R is. We call $R^\vee$ the *dual* (or *inverse*) of R and we call co-roots its elements.

For each $\alpha \in R$, we denote by H_α the linear hyperplane of $\mathbb{V}$ defined by $\langle v, \alpha \rangle = 0$ and we denote by $\mathcal{H}_0$ the family of all linear hyperplanes H_α. For every $\alpha \in R$, let s_α be the reflection with reflecting hyperplane H_α; we denote by $\mathbf{W}$ the subgroup of $GL(\mathbb{V})$ generated by $\{s_\alpha,\ \alpha \in R\}$. $\mathbf{W}$ permutes the set R and is a finite group, called the *Weyl group* of R. Note that $\mathbf{W}(R) = \mathbf{W}(R^\vee)$.

The hyperplanes in $\mathcal{H}_0$ split up $\mathbb{V}$ into finitely many regions; the connected components of $\mathbb{V} \setminus \cup_\alpha H_\alpha$ are (open) sectors based at 0, called the (open) *Weyl chambers* of $\mathbb{V}$ (with respect to R). The so called *fundamental Weyl chamber* or *fundamental sector* based at 0 (with respect to the basis B) is the Weyl chamber

$$\mathbb{Q}_0 = \{v \in \mathbb{V} \ :\ \langle v, \alpha_i \rangle > 0,\ i \in I_0\}.$$

It is known that

(i) $\mathbf{W}$ is generated by $S_0 = \{s_i = s_{\alpha_i},\ i \in I_0\}$ and hence $(\mathbf{W}, S_0)$ is a finite Coxeter system;
(ii) $\mathbf{W}$ acts simply transitively on Weyl chambers;
(iii) $\overline{\mathbb{Q}_0}$ is a fundamental domain for the action of $\mathbf{W}$ on $\mathbb{V}$.

Moreover, for every $\mathbf{w} \in \mathbf{W}$, we have $|\mathbf{w}| = n(\mathbf{w})$, if $n(\mathbf{w})$ is the number of positive roots α for which $\mathbf{w}(\alpha) < 0$. We recall that at most two root lengths occur in R, if R is reduced, and all roots of a given length are conjugate under $\mathbf{W}$. When there are in R two distinct root lengths, we speak of *long* and *short* roots. In this case, the highest root α_0 is long.

The root system (or the associated Weyl group) can be characterized by the Dynkin diagram, which is the usual Coxeter graph D_0 of the group $\mathbf{W}$, where we add an arrow pointing to the shorter of the two roots. We refer to [2] for the classification of (irreducible) root systems. We notice that, for every $n \geq 1$, there is exactly one irreducible non-reduced root system (up to isomorphism) of rank n, denoted by BC_n. If we take $\mathbb{V} = \mathbb{R}^n$, with the usual inner product, the root system BC_n is the following:

$$R = \{\pm e_k,\ \pm 2e_k,\ 1 \leq k \leq n\} \cup \{\pm e_i \pm e_j,\ 1 \leq i < j \leq n\}.$$

Hence we can choose $B = \{\alpha_i,\ 1 \leq i \leq n\}$, if $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1$ and $\alpha_n = e_n$. Moreover

$$R^+ = \{e_k,\ 2e_k,\ 1 \leq k \leq n\} \cup \{e_i \pm e_j,\ 1 \leq i < j \leq n\}$$

and $\alpha_0 = 2e_1$. In this case $R^\vee = R$ and $\mathbf{W}(BC_n) = \mathbf{W}(C_n) = \mathbf{W}(B_n)$.
It will be useful to decompose $R = R_1 \cup R_2 \cup R_0$, as disjoint union, by defining

$$\begin{aligned} R_1 &= \{\alpha \in R \ :\ \ \alpha/2 \in R, 2\alpha \notin R\} \\ R_2 &= \{\alpha \in R \ :\ \ \alpha/2 \notin R, 2\alpha \in R\} \\ R_0 &= \{\alpha \in R \ :\ \ \alpha/2, 2\alpha \notin R\}. \end{aligned}$$

Then $\alpha_0 \in R_1,\ \alpha_n \in R_2$, and $\alpha_i \in R_0,\ \forall i = 1, \cdots, n-1$, and $\mathbf{W}$ stabilizes each R_j.

The $\mathbb{Z}$-span $L(R)$ of the root system R is called the *root lattice* of $\mathbb{V}$ and $L(R^\vee)$ is called the *co-root lattice* of $\mathbb{V}$ associated to R. Notice that $L(BC_n) = L(C_n) = L(B_n^\vee)$. We simply denote by L the *co-root lattice* of $\mathbb{V}$ associated to R. Moreover we set

$$L^+ = \{ \sum_{\alpha\in R^+} n_\alpha \alpha,\ n_\alpha \in \mathbb{N}\}.$$

2.8. **Affine Weyl group of a root system.** Let R be an irreducible root system, not necessarily reduced. For every $\alpha \in R$ and $k \in \mathbb{Z}$, define an affine hyperplane

$$H_\alpha^k = \{v \in \mathbb{V}\ :\ \langle v, \alpha\rangle = k\}.$$

We remark that $H_\alpha^k = H_{-\alpha}^{-k}$ and $H_\alpha^0 = H_\alpha$; moreover H_α^k can be obtained by translating H_α^0 by $\frac{k}{2}\alpha^\vee$.

When R is reduced we define $\mathcal{H} = \cup_{\alpha\in R^+}\mathcal{H}(\alpha)$, where, for every $\alpha \in R^+$,

$$\mathcal{H}(\alpha) = \{H_\alpha^k,\ \text{for all } k \in \mathbb{Z}\}.$$

When R is not reduced, we note that, for every $\alpha \in R_2,\ H_\alpha^k = H_{2\alpha}^{2k}$; then we define

$$\begin{aligned}
\mathcal{H}_1 &= \{H_\alpha^k\ :\quad \alpha \in R_1,\quad k \in 2\mathbb{Z}+1\}\\
\mathcal{H}_2 &= \{H_\alpha^k\ :\quad \alpha \in R_2,\quad k \in \mathbb{Z}\}\\
\mathcal{H}_0 &= \{H_\alpha^k\ :\quad \alpha \in R_0,\quad k \in \mathbb{Z}\},
\end{aligned}$$

and $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_0$, as disjoint union. Since $\mathcal{H}_1 \cup \mathcal{H}_2 = \{H_\alpha^k,\ \alpha \in R_1,\ k \in \mathbb{Z}\}$, we can write

$$\mathcal{H} = \cup_{\alpha\in R_1\cup R_0}\mathcal{H}(\alpha),$$

by setting, for every $\alpha \in R_0$ or $\alpha \in R_1,\ \mathcal{H}(\alpha) = \{H_\alpha^k,\ \text{for all } k \in \mathbb{Z}\}$, as in the reduced case. Actually, $R_1 \cup R_0$ is the root system of type C_n and the hyperplanes described before are these associated with this reduced root system.

Given an affine hyperplane $H_\alpha^k \in \mathcal{H}$, the affine reflection with respect to H_α^k is the map s_α^k defined by

$$s_\alpha^k(v) = v - (\langle v, \alpha\rangle - k)\alpha^\vee, \quad \forall v \in \mathbb{V}.$$

The reflection s_α^k fixes H_α^k and sends 0 to $k\alpha^\vee$; in particular $s_\alpha^0 = s_\alpha,\ \forall\alpha \in R$. We denote by $\mathcal{S}$ the set of all affine reflections defined above. We define the *affine Weyl group* W of R to be the subgroup of Aff($\mathbb{V}$) generated by all affine reflections $s_\alpha^k,\ \alpha \in R,\ k \in \mathbb{Z}$. (Here Aff($\mathbb{V}$) is the set of maps $v \mapsto Tv + \lambda$, for all $T \in GL(\mathbb{V})$ and $\lambda \in \mathbb{V}$).

Let $s_0 = s_{\alpha_0}^1$ and $I = I_0 \cup \{0\}$; then it can be proved that W is a Coxeter group over I, generated by the set $S = \{s_i,\ i \in I\}$. Writing t_λ for the translation $v \mapsto v + \lambda$, we can consider $\mathbb{V}$ as a subgroup of Aff($\mathbb{V}$), by identifying λ and t_λ. In this sense we have Aff($\mathbb{V}$) $= GL(\mathbb{V}) \ltimes \mathbb{V}$. In the same sense, if we consider the affine Weyl group W, the co-root lattice L can be seen as a subgroup of W, since $t_\lambda,\ \lambda \in L$, are the only translations of $\mathbb{V}$ belonging to W, and we have

$$W = \mathbf{W} \ltimes L.$$

We point out that $W(BC_n) = W(C_n)$, whereas $W(BC_n) \supset W(B_n)$. Hence we can write each $w \in W$ in a unique way as $w = \mathbf{w}t_\lambda$, for some $\mathbf{w} \in \mathbf{W}$ and $\lambda \in L$; moreover, if $w_1 = \mathbf{w}_1 t_{\lambda_1}$ and $w_2 = \mathbf{w}_2 t_{\lambda_2}$, then $w_2^{-1}w_1 \in L$ if and only if $\mathbf{w}_1 = \mathbf{w}_2$. This implies that there is a bijection between the quotient W/L and $\mathbf{W}$, in the sense that each coset wL determines a unique $\mathbf{w} \in \mathbf{W}$. So we denote by $\mathbf{w}$ the coset whose representative in W is w, and we shall write $w \in \mathbf{w}$ to intend that $w = \mathbf{w}t_\lambda$, for some $\lambda \in L$.

It is not difficult to construct, for each irreducible root system R, the Coxeter graph D of the affine Weyl group W; one just needs to work out the order of $s_i s_0$, for each $i \in I_0$, to see what new bonds and labels occur when the new node is adjoined to the Coxeter graph D_0 of $\mathbf{W}$,that is of R. We refer to [6] for the classification of all affine Weyl groups.

2.9. **Co-weight lattice.** Following standard notation, we call *weight lattice* of $\mathbb{V}$ associated to the root system R the $\mathbb{Z}$-span $\widehat{L}(R)$ of the vectors $\{\lambda_i^\vee,\ i \in I_0\}$, defined by $\langle\lambda_i^\vee, \alpha_j^\vee\rangle = \delta_{ij}$ and we call $\widehat{L}(R^\vee)$ the *co-weight lattice* of $\mathbb{V}$ associated to the root system R. We simply set $\widehat{L} = \widehat{L}(R^\vee)$. Then $\widehat{L}$ is the $\mathbb{Z}$-span of the vectors $\{\lambda_i,\ i \in I_0\}$, defined by

$$\langle\lambda_i, \alpha_j\rangle = \delta_{ij}, \quad \forall\ i, j \in I_0.$$

It is easy to see that, when R is reduced, $\widehat{L}$ contains L as a subgroup of finite index $\mathbf{f}$, the so called *index of connection*, with $1 \le \mathbf{f} \le n+1$. Instead, when R is non reduced, that is when R has type BC_n, then $\widehat{L}(BC_n) = L(BC_n)$; thus, in this case

$$L(C_n) = L(BC_n) = \widehat{L}(BC_n) \not\subseteq \widehat{L}(C_n).$$

A co-weight λ is said *dominant* (respectively *strongly dominant*) if $\langle \lambda, \alpha_i \rangle \geq 0$ (respectively $\langle \lambda, \alpha_i \rangle > 0$) for every $i \in I_0$. We denote by $\widehat{L}^+$ (resp. $\widehat{L}^{++}$) the set of all dominant (respectively strongly dominant) co-weights. Thus $\lambda \in \widehat{L}^+$ if and only if $\lambda \in \overline{\mathbb{Q}}_0$ and $\lambda \in \widehat{L}^{++}$ if and only if $\lambda \in \mathbb{Q}_0$. Remark that L^+ does not lie on $\widehat{L}^+$ and $L^+ \cap \widehat{L}^+$ consists of all dominant coweights of type 0.

2.10. **Geometric realization of an affine Coxeter complex.** Let W be the affine Weyl group of a root system R; let $\mathcal{H}$ be the collection of the affine hyperplanes associated to W. The open connected components of $\mathbb{V} \backslash \cup_{\alpha,k} H_\alpha^k$ are called *chambers*. Since R is irreducible, each chamber is an open (geometric) simplex of rank $n+1$ and dimension n. The extreme points of the closure of any chamber are called *vertices* and the 1 codimension faces of any chamber are called *panels*.

We write $\mathbb{A} = \mathbb{A}(R)$ for the vector space $\mathbb{V}$ equipped with chambers, vertices, panels as defined above. Hence $\mathbb{A}$ is a geometric simplicial complex of rank $n+1$ and dimension n, realized as a tessellation of the vector space $\mathbb{V}$ in which all chambers are isomorphic.

It is convenient to single out one chamber, called *fundamental chamber* of $\mathbb{A}$, in the following way:

$$C_0 = \{v \in \mathbb{V} \,:\, 0 < \langle v, \alpha \rangle < 1,\ \forall \alpha \in R^+\} = \{v \in \mathbb{V} \,:\, \langle v, \alpha_i \rangle > 0,\ \forall i \in I_0, \quad \langle v, \alpha_0 \rangle < 1\}.$$

Define *walls* of C_0 the hyperplanes $H_{\alpha_i},\ i \in I_0$ and $H^1_{\alpha_0}$; then the group W is generated by the set of the reflections with respect to the walls of the fundamental chamber C_0.

We denote by $\mathcal{C}(\mathbb{A})$ the set of chambers and by $\mathcal{V}(\mathbb{A})$ the set of vertices of $\mathbb{A}$. It can be proved that W acts simply transitively on $\mathcal{C}(\mathbb{A})$ and $\overline{C_0}$ is a fundamental domain for the action of W on $\mathbb{V}$. Moreover $\mathbf{W}$ acts simply transitively on the set $\mathcal{C}(0)$ of all chambers C, such that $0 \in \overline{C}$. Hence, we have well-defined walls for each chamber $C \in \mathcal{C}(\mathbb{A})$: the walls of C are the images of the walls of C_0 under w, if $C = wC_0$. If we declare $wC_0 \sim_i wC_0$ and $wC_0 \sim_i ws_iC_0$, for each $w \in W$ and each $i \in I$, then the map

$$w \mapsto wC_0$$

is an isomorphism of the labelled chamber complex of W onto $\mathcal{C}(\mathbb{A})$. For every $w \in W$, we set $C_w = wC_0$. The vertices of C_0 are $X_0^0, X_1^0, \ldots, X_n^0$, where $X_0^0 = 0$ and $X_i^0 = \lambda_i/m_i,\ i \in I_0$.

We declare $\tau(0) = 0$ and $\tau(\lambda_i/m_i) = i$, for $i \in I_0$; more generally we declare that a vertex X of $\mathbb{A}$ has type $i,\ i \in I$, if $X = w(X_i^0)$ for some $w \in W$. This define a unique labelling

$$\tau : \mathcal{V}(\mathbb{A}) \to I,$$

and the action of W on $\mathbb{A}$ is type-preserving. We say that a panel of C_0 has *co-type* i, for any $i \in I$, if i is the type of the vertex of C_0 not lying on the panel; this extends to a unique labelling on the panels of $\mathbb{A}$.

Hence, if we consider the Coxeter complex $\Sigma(W,S)$ associated to the affine Weyl group W, there is a unique isomorphism type-preserving of $\Sigma(W,S)$ onto $\mathbb{A}$; thus $\mathbb{A}$ may be regarded as the geometric realization of Σ; up to this isomorphism, the co-root lattice L consists of all type 0 vertices of $\mathbb{A}$ and W acts on L. We point out that, for every $w \in W$, the chamber C_w can be joined to C_0 by a gallery $\gamma(C_0, C_w)$ of type $f = i_1 \cdots i_k$, if $w = s_{i_1} \cdots s_{i_k}$; so, recalling the definition of the W-distance function given in Section 2.4, we have $w = \delta(C_0, C_w)$. This suggests to denote by $C_0 \cdot w$, the chamber C_w.

According to [2], a vertex X is a *special vertex* of $\mathbb{A}$ if, for every $\alpha \in R^+$, there exists $k \in \mathbb{Z}$ such that $X \in H_\alpha^k$. In particular the vertex 0 is special and hence every vertex of type 0 is special, but in general not all vertices of $\mathbb{A}$ are special. We shall denote by $\mathcal{V}_{sp}(\mathbb{A})$ the set of all special vertices of $\mathbb{A}$. We point out that, when R is reduced, $\mathcal{V}_{sp}(\mathbb{A}) = \widehat{L}$. More precisely , if R has type A_n, all $n+1$ types are special; furthermore, if R has type D_n, E_6 and G_2, occur respectively four, three and only one special type; in all other cases the special types are two. In particular, if R has type B_n or C_n, the special vertices have type 0 or n. We refer the reader to [6] for more details.

Remark 2.10.1. *When R has type C_n and $\alpha = \alpha_n$, then all vertices of type 0 lie on hyperplanes H_α^{2k}, for $k \in \mathbb{Z}$, whereas all vertices of type n lie on hyperplanes H_α^{2k+1}, for $k \in \mathbb{Z}$. Actually, the reflection s_{α_0} fixes each hyperplane H_α^h and the panel of co-type n, containing 0, of the hyperplane $H^0_{\alpha_0}$ and, for every j, the reflection with respect to $H^j_{\alpha_0}$ fixes its panel and each hyperplane H_α^h. The same is true for every long root. If R has type B_n the previous property holds for each simple root $\alpha = \alpha_i,\ i = 1, \cdots, n-1$, and then for every long root.*

When R is non reduced, the Coxeter complex $\Sigma(W,S)$ associated to the root system of type BC_n has the same geometric realization as the one associated to the root system of type C_n. Then the special types are type 0 and type n, and they are arranged according to Remark 2.10.1. Since $\widehat{L}(BC_n) = L(BC_n)$, the lattice $\widehat{L}(BC_n)$ is a proper subset of $\mathcal{V}_{sp}(\mathbb{A})$ and it consists of all type 0 vertices, lying on the hyperplanes H_i^{2k}, for $k \in \mathbb{Z}$ and $i = 0, n$.

In general we denote by $\widehat{\mathcal{V}}(\mathbb{A})$ the set of all special vertices of $\mathbb{A}$ belonging to $\widehat{L}$; so $\widehat{\mathcal{V}}(\mathbb{A})$ inherits the group structure of $\widehat{L}$. If we define $\widehat{I} := \{\tau(\lambda) \ : \ \lambda \in \widehat{L}\}$, then $\widehat{\mathcal{V}}(\mathbb{A})$ is the set of all special vertices of $\mathbb{A}$ whose type belongs to $\widehat{I}$. We remark that $\widehat{I} = \{i \in I \ : \ m_i = 1\}$. See [13] for a proof of this property.

For every $\lambda \in \widehat{L}^+$, we define

$$\mathbf{W}_\lambda = \{\mathbf{w} \in \mathbf{W} \ : \ \mathbf{w}\lambda = \lambda\}.$$

If X_λ is the special vertex of $\mathbb{A}$ associated with λ and C_λ is the chamber containing X_λ in the minimal gallery connecting C_0 to X_λ, that is the chamber of $\mathbb{Q}_0$ containing X_λ and nearest to C_0, then the set $\mathbf{W}_\lambda$ is the stabilizer of X_λ in $\mathbf{W}$. Moreover we denote by w_λ the unique element of W such that $C_\lambda = w_\lambda(C_0)$.

Finally, for each $i \in \widehat{I}$, we denote by $\mathbf{W}_i$ the stabilizer in W of the vertex X_i^0 of type i lying on the fundamental chamber C_0, that is the Weyl group associated with $I_i = I \setminus \{i\}$. Obviously $\mathbf{W}_0 = \mathbf{W}$.

2.11. **Extended affine Weyl group of R.** Let us consider in $\mathrm{Aff}(\mathbb{V})$ the translation group corresponding to $\widehat{L}$; since this group is also normalized by $\mathbf{W}$, we can form the semi-direct product

$$\widehat{W} = \mathbf{W} \ltimes \widehat{L},$$

called the *extended affine Weyl group* of R. We notice that $\widehat{W}/W$ is isomorphic to $\widehat{L}/L$; hence $\widehat{W}$ contains W as a normal subgroup of finite index $\mathbf{f}$. In particular when R is non reduced, then $\widehat{W}(BC_n) = W(BC_n)$, as in this case $\widehat{L}(BC_n) = L(BC_n)$; moreover $\widehat{W}(BC_n)$ is not isomorphic to $\widehat{W}(C_n)$, since $\widehat{W}(C_n)$ is larger than $W(C_n)$. Notice that $\widehat{W}$ permutes the hyperplanes in $\mathcal{H}$ and acts transitively, but not simply transitively, on $\mathcal{C}(\mathbb{A})$.

Given any two special vertices X, Y of $\mathbb{A}$, there exists a unique $\widehat{w} \in \widehat{W}$ such that $\widehat{w}(X) = 0$ and $\widehat{w}(Y)$ belongs to $\overline{\mathbb{Q}}_0$. We call *shape* of Y with respect to X the element $\lambda = \widehat{w}(Y)$ of $\widehat{L}^+$ and we denoted it by $\sigma(X,Y)$. For every $\lambda \in \widehat{L}^+$, we set

$$\mathcal{V}_\lambda(X) = \{Y \in \mathcal{V}(\mathbb{A}) \ : \ \sigma(X,Y) = \lambda\}.$$

As for W/L, there is a bijection between the quotient $\widehat{W}/\widehat{L}$ and $\mathbf{W}$, in the sense that each coset $\widehat{w}\widehat{L}$ determines a unique $\mathbf{w} \in \mathbf{W}$; so we denote by $\mathbf{w}$ the coset whose representative in $\mathbf{W}$ is $\mathbf{w}$. Hence we shall write $\widehat{w} \in \mathbf{w}$ to mean that $\widehat{w} = \mathbf{w}t_\lambda$, for some $\lambda \in \widehat{L}$.

For every $\widehat{w} \in \widehat{W}$, let define

$$\mathcal{L}(\widehat{w}) = |\{H \in \mathcal{H} \ : \ H \text{ separates } C_0 \text{ and } \widehat{w}(C_0)\}|.$$

If $w \in W$, then $\mathcal{L}(w) = |w|$. The subgroup $G = \{g \in \widehat{W} : \mathcal{L}(g) = 0\}$ is the stabilizer of C_0 in $\widehat{W}$ and

$$\widehat{W} \cong G \ltimes W.$$

Hence $G \cong \widehat{L}/L$ and is a finite abelian group. If R is reduced, it can be proved that $G = \{g_i, \ i \in \widehat{I}\}$, where $g_0 = 1$ and, for every $i \in I_0$, $g_i = t_{\lambda_i}\mathbf{w}^0_{\lambda_i}\mathbf{w}_0$, if $\mathbf{w}_0$ and $\mathbf{w}^0_{\lambda_i}$ denote the longest elements of $\mathbf{W}$ and $\mathbf{W}_{\lambda_i} = \{\mathbf{w} \in \mathbf{W} \ : \ \mathbf{w}\lambda_i = \lambda_i\}$ respectively. A proof of this property can be found in [13]. Obviously, if R is non reduced, then G is trivial.

We extend to $\widehat{W}$ the definition of q_w given in Section 2.5, for every $w \in W$, by setting

$$q_{\widehat{w}} = q_w \text{ if } \widehat{w} = wg,$$

where $w \in W$ and $g \in G$. In particular, for each $\lambda \in \widehat{L}$, $q_{t_\lambda} = q_{u_\lambda}$ if $t_\lambda = u_\lambda g$.

2.12. **Automorphisms of $\mathbb{A}$ and D.** As usual, an automorphism of $\mathbb{A}$ is a bijection φ on $\mathbb{V}$ mapping chambers to chambers, with the property that $\varphi(C)$ and $\varphi(C')$ are adjacent if and only if C and C' are adjacent. If D denotes the Coxeter graph of W, then an automorphism of D is a permutation σ of I, such that $m_{\sigma(i),\sigma(j)} = m_{i,j}$, $\forall i,j \in I$. We denote by $Aut(\mathbb{A})$ and $Aut(D)$ the automorphism group of $\mathbb{A}$ and D respectively. It can be proved (see for instance [13]) that, for every $\varphi \in Aut(\mathbb{A})$, there exists $\sigma \in Aut(D)$, such that , for every $X \in \mathcal{V}(\mathbb{A})$,

$$\tau \circ \varphi(X) = \sigma \circ \tau(X),$$

and $\varphi(C) \sim_{\sigma(i)} \varphi(C')$, if $C \sim_i C'$.

Obviously W, $\mathbf{W}$ and $\widehat{W}$ can be seen as subgroups of $Aut(\mathbb{A})$ such that $\mathbf{W} \le W \le \widehat{W} \le Aut(\mathbb{A})$ (in some cases $\widehat{W}$ is a proper subgroup). Consider in particular the finite abelian group G and, for every $i \in \widehat{I}$, denote by σ_i the automorphism of D such that $\tau \circ g_i = \sigma_i \circ \tau$; then $\sigma_i(0) = i$, for every $i \in \widehat{I}$. Hence we call *type-rotating* every σ_i, $i \in \widehat{I}$, and denote

$$Aut_{tr}(D) = \{\sigma_i, \ i \in \widehat{I}\}.$$

In particular $\sigma_0 = 1$. We note that $Aut(D) = Aut(D_0) \ltimes Aut_{tr}(D)$, if D_0 is the Coxeter graph of $\mathbf{W}$, and $Aut_{tr}(D)$ acts simply transitively on $\widehat{I}$. Since each $w \in W$ is type-preserving, it corresponds to the element $\sigma_0 = 1$ of $Aut_{tr}(D)$; actually W is the subgroup of all type-preserving automorphisms of $\mathbb{A}$. Keeping in mind the formula $\widehat{W} \cong G \ltimes W$, we call *type-rotating* automorphism of $\mathbb{A}$ any element of $\widehat{W}$.

The group $Aut_{tr}(D)$ acts on W as following: for every $\sigma \in Aut_{tr}(D)$ and $w = s_{i_1} \cdots s_{i_k} \in W$, then

$$\sigma(w) = s_{\sigma(i_1)} \cdots s_{\sigma(i_k)}.$$

In particular, for every $i \in \widehat{I}$, we have $\mathbf{W}_i = \sigma_i(\mathbf{W})$.

Consider now the map

$$\iota(\mu) = -\mathbf{w}_0(\mu), \quad \forall \mu \in \mathbb{A}.$$

Since the map $\mu \mapsto -\mu$ is an automorphism of $\mathbb{A}$, then $\iota \in Aut(\mathbb{A})$; moreover $\iota^2 = 1$ and $\iota(\mathbb{Q}_0) = \mathbb{Q}_0$. Therefore either ι is the identity or it permutes the walls of the sector $\mathbb{Q}_0$. Since the identity is the unique element of $\mathbf{W}$ which fixes the sector $\mathbb{Q}_0$, by virtue of the simple transitivity of $\mathbf{W}$ on the sectors based at 0, it follows that ι belongs to $\mathbf{W}$ only when is the identity. This happens when the map $\mu \mapsto -\mu$ belongs to $\mathbf{W}$, that is when $\mathbf{w}_0 = -1$. Hence, if we consider the automorphism $\sigma_\star$ of D induced by ι, then in general $\sigma_\star$ is not an element of $Aut_{tr}(D)$, but $\sigma_\star \in Aut_{tr}(D)$ if and only if $\sigma_\star = 1$. Moreover, when $\sigma_\star \neq 1$, then it belongs to $Aut(D_0)$. On the other hand, $Aut(D_0)$ is non trivial only for a root system of type A_l $(l \geq 2)$, D_l $(l \geq 4)$ and E_6. Hence, apart these three cases, ι is always the identity, or equivalently, the map $\mu \mapsto -\mu$ belongs to $\mathbf{W}$.

Simple computations allow to state if ι is trivial or not for a Dynkin diagram D_0 of type A_l $(l \geq 2)$, D_l $(l \geq 4)$ and E_6. The results are listed in the following proposition.

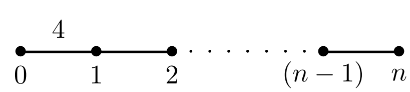
Proposition 2.12.1. *Let R be an irreducible root system.*

(i) If R has type A_l $(l \geq 2)$, then ι induces the unique automorphism non trivial of the diagram D_0;
(ii) if R has type D_l $(l \geq 4)$, then ι is the identity for n even and it induces the unique automorphism non trivial of the diagram D_0 for n odd;
(iii) if R has type E_6, then ι induces the unique automorphism non trivial of the diagram D_0.

For every $\mu \in \mathcal{V}_{sp}(\mathbb{A})$, we denote $\mu^\star = \iota(\mu)$; then $\mu^\star \in \overline{\mathbb{Q}}_0$ for each $\mu \in \overline{\mathbb{Q}}_0$.

2.13. **Affine buildings of type $\widetilde{X}_n$.** Let Δ be an affine building; we assume Δ is irreducible, locally finite, regular and we denote by $\{q_i\}_{i \in I}$ its parameter system. By definition, there is a Coxeter group W canonically associated to Δ and W is an affine reflection group, which can be interpreted as the affine Weyl group of a (irreducible) root system R. Hence there is a root system R canonically associated to each (irreducible, locally finite, regular) affine building. The choice of R is in most cases "straightforward", since in general different root systems have different affine Weyl group.

The only exceptions to this rule are the root systems of type C_n and BC_n, which have the same affine Weyl group. So, when the group W associated to the building is the affine Weyl group of the root systems of type C_n and BC_n, we have to choose the root system. We assume to operate this choice according to the parameter system of the building. Actually, we choose R to ensure that in each case the group $Aut_{tr}(D)$ preserves the parameter system of the building, that is in order to have, for each $\sigma \in Aut_{tr}(D)$, $q_{\sigma(i)} = q_i$, for all $i \in I$. Actually, in the case $R = C_n$ or BC_n, the Coxeter graph of W is



Hence $q_1 = q_2 = \cdots = q_{n-1}$, but in general $q_0 \neq q_1 \neq q_n$. On the other hand, if $R = C_n$, then $Aut_{tr}(D) = \{1, \sigma\}$, while, if $R = BC_n$, then $Aut_{tr}(D) = \{1\}$. Thus, if $R = C_n$, the condition $q_{\sigma(0)} = q_0$ implies $q_n = q_0$, while, if $R = BC_n$, q_0 and q_n can have different values.

Keeping in mind the above choice and the classification of root systems, we shall say that

(1) Δ is an affine building of type $\widetilde{X}_n$, if R has type X_n, in the following cases:

$$X_n = A_n \ (n \geq 2), \quad B_n \ (n \geq 3), \quad D_n \ (n \geq 4), \quad E_n \ (n = 6, 7, 8), \quad F_4, \quad G_2;$$

(2) Δ is an affine building of type
   (i) $\widetilde{A}_1$, associated to a root system of type A_1, if $q_0 = q_1$ (homogeneous tree);
   (ii) $\widetilde{BC}_1$, associated to a root system of type BC_1, if $q_0 \neq q_1$ (semi-homogeneous tree);

(3) Δ is an affine building of type
   (i) $\widetilde{C}_n$, $n \geq 2$, associated to a root system of type C_n, if $q_0 = q_n$;
   (ii) $\widetilde{BC}_n$, $n \geq 2$, associated to a root system of type BC_n, if $q_0 \neq q_n$.

We refer to Appendix of [13] for the classification of all irreducible, locally finite, regular affine buildings, in terms of diagram and parameter system.

In each case $Aut_{tr}(D)$ preserves the parameter system of the building. Actually, if we define

$$Aut_q(D) = \{\sigma \in Aut(D) \ : \ q_{\sigma(i)} = q_i, \ i \in I\},$$

then in each case $Aut_{tr}(D) \cup \{\sigma_\star\} \subset Aut_q(D)$.

2.14. **Subgroups of G.** We are interested to determine the subsets of the set $\widehat{I}$ of special types corresponding to sublattices of $\widehat{L}$. In order to solve this problem we have to determine all the subgroups of the finite group $G = \widehat{L}/L$ of order **f**. We only consider buildings of type $\widetilde{A}_n, \widetilde{D}_n$ and $\widetilde{E}_6$, as only in these cases **f** is greater than 2 and hence there is the possibility to have proper subgroups of $\widehat{L}/L$. Since the order of a proper subgroup of a finite group must be a divisor of the order of the group, then in the cases $\widetilde{E}_6$ and $\widetilde{A}_n, \ n = 2k+1$, we have no one proper subgroup of $\widehat{L}/L$. So the only cases to consider are the case $\widetilde{A}_n$, if n is an even number, and the case $\widetilde{D}_n$. The following results can be proved by direct computations.

Proposition 2.14.1. *Let Δ be a building of type $\widetilde{D}_n$; then*

(i) if n is even, G has three subgroups of order two: $G_{0,1} = \langle g_0, g_1\rangle$, $G_{0,n-1} = \langle g_0, g_{n-1}\rangle$ and $G_{0,n} = \langle g_0, g_n\rangle$, corresponding to types $\{0,1\}$, $\{0, n-1\}$ and $\{0,n\}$ respectively;

(ii) if n is odd, then $G_{0,1} = \langle g_0, g_1\rangle$ is the unique subgroup of order two of G corresponding to the types $\{0,1\}$.

Proposition 2.14.2. *Let Δ be a building of type $\widetilde{A}_n$; assume $n = lm$, for some $l, m \in \mathbb{Z}, 1 < l, m < n$. Then $\{g_0, g_l, g_{2l}, \cdots, g_{(m-1)l}\}$ generate the unique subgroup of order m in G.*

Proposition 2.14.1 implies that, for a building of type $\widetilde{D}_n$, the vertices of $\mathbb{A}$ of types 0 and 1 form an sublattice of $\widehat{L}$, for every n; moreover, when n is even, also the vertices of types $\{0, n-1\}$ and the vertices of type $\{0,n\}$ form a sublattice of $\widehat{L}$. Besides the types $\{n-1, n\}$ do not correspond to a subgroup of order two in $\widehat{L}/L$, but to its complement; this means that the vertices of $\mathbb{A}$ of types $n-1$ and n form an affine lattice which does not contain the origin 0. The same is true, when n is even, for the types $\{1, n-1\}$ and $\{1, n\}$.

As a consequence of Proposition 2.14.2, the vertices of $\mathbb{A}$ of types $\{0, l, 2l, \ldots, (m-1)l\}$ form a sublattice of $\widehat{L}$, whereas the types $\{j, j+l, j+2l, \ldots, j+(m-1)l\}$, for $0 < j < l$, do not correspond to any subgroup of order m in $\widehat{L}/L$, but to a lateral of this subgroup. This means that the vertices of $\mathbb{A}$ of types $\{j, j+l, j+2l, \ldots, j+(m-1)l\}$, for $0 < j < l$, form an affine lattice which does not contain the origin 0.

2.15. **Geometric realization of an affine building.** Let Δ be any affine building of type $\widetilde{X}_n$. The affine Coxeter complex $\mathbb{A}$ associated to W is called the *fundamental apartment* of the building. By definition, each apartment $\mathcal{A}$ of Δ is isomorphic to $\mathbb{A}$ and hence it can be regarded as a Euclidean space, tessellated by a family of affine hyperplanes isomorphic to the family $\mathcal{H}$. Moreover every such isomorphism is type-preserving or type-rotating. If $\psi : \mathcal{A} \to \mathbb{A}$ is any type-preserving isomorphism, then, for each $\widehat{w} \in \widehat{W}$, $\psi' = \widehat{w}\psi$ is a type-rotating isomorphism and for every vertex x of type i, the type of $\psi'(x)$ is $\sigma_j(i)$, if $\widehat{w} = wg_j$. Moreover each type-rotating isomorphism $\psi' : \mathcal{A} \to \mathbb{A}$ is obtained in this way.

For any apartment $\mathcal{A}$, we denote by $\mathcal{H}(\mathcal{A})$ the family of all hyperplanes h of $\mathcal{A}$. If $\psi \ : \ \mathcal{A} \to \mathbb{A}$ is any type-rotating isomorphism, we set $h = h^k_\alpha$, if $\psi(h) = H^k_\alpha$. Obviously k and α depend on ψ.

We denote by $\mathcal{V}(\Delta)$ the set of all vertices of the building and, for each $i \in I$, we denote by $\mathcal{V}_i(\Delta)$ the set of all type i vertices in Δ.

There is a natural way to extend to Δ the definition of special vertices given in $\mathbb{A}$; we call special each vertex x of Δ such that its image on $\mathbb{A}$ (under any isomorphism type-preserving between any apartment containing x and the fundamental apartment) is a special vertex of $\mathbb{A}$. We point out that all types are special for a building of type $\tilde{A}_n$; furthermore for a building of type $\tilde{D}_n$, $\tilde{E}_6$ and $\tilde{G}_2$ occur respectively four, three and only one special type; in all other cases the special types are two. We denote by $\mathcal{V}_{sp}(\Delta)$ the set of all special vertices of Δ.

Finally, we denote by $\widehat{\mathcal{V}}(\Delta)$ the set of all vertices of type $i \in \widehat{I}$, that is the set of all vertices x such that its image on $\mathbb{A}$ (under any isomorphism type-preserving between any apartment containing x and the fundamental apartment) belongs to $\widehat{L}$. It is obvious that $\widehat{\mathcal{V}}(\Delta) = \mathcal{V}_{sp}(\Delta)$, if Δ is reduced, while $\widehat{\mathcal{V}}(\Delta) = \mathcal{V}_0(\Delta)$, if Δ is not reduced. We always refer vertices of $\widehat{\mathcal{V}}(\Delta)$.

We recall that, for every pair of chambers $c, d \in \mathcal{C}(\Delta)$, there exists a minimal gallery $\gamma(c,d)$ from c to d, lying on any apartment containing both chambers; the type of $\gamma(c,d)$ is $f = i_1 \cdots i_k$ if $\delta(c,d) = w_f$. If $\{q_i\}_{i \in I}$ is the parameter system of the building, for every $c \in \mathcal{C}(\Delta)$ and $w \in W$, we have $|\mathcal{C}_w(c)| = q_w$, if $\mathcal{C}_w(c) = \{d \in \mathcal{C}(\Delta) \ : \ \delta(c,d) = w\}$.

Analogously, given a vertex $x \in \widehat{\mathcal{V}}(\Delta)$, and a chamber d, there exists a minimal gallery $\gamma(x,d)$ from x to d, lying on any apartment containing x and d; if c is the chambers of $\gamma(x,d)$ containing x, then the type of this gallery is $f = i_1 \cdots i_k$, if $\delta(c,d) = w_f$, and we set $\delta(x,d) = \delta(c,d)$. Hence we define, for every $x \in \widehat{\mathcal{V}}(\Delta)$ and $w \in W$,

$$\mathcal{C}_w(x) = \{d \in \mathcal{C}(\Delta) \ : \ \delta(x,d) = w\}.$$

If, for every $x \in \widehat{\mathcal{V}}(\Delta)$, we denote by $\mathcal{C}(x)$ the set of all chambers containing x, then $\mathcal{C}_w(x) = \cup_{c \in \mathcal{C}(x)} \mathcal{C}_w(c)$, as a disjoint union. We notice that, for every x of type $i \in \widehat{I}$, then, fixed any chamber c containing x,

$$\mathcal{C}(x) = \{c' \in \mathcal{C}(\Delta) \ : \ \delta(c,c') = w, \ \forall w \in \mathbf{W}_i\},$$

if $\mathbf{W}_i = \sigma_i(\mathbf{W})$ is the stabilizer of the type i vertex of C_0. Hence the cardinality of the set $\mathcal{C}(x)$ is the Poicaré polynomial $\mathbf{W}_i(q)$ of $\mathbf{W}_i$. On the other hand, $\mathbf{W}_i(q) = \mathbf{W}_{\sigma_i(0)}(q) = \mathbf{W}(q)$; so, in each case,

$$|\mathcal{C}(x)| = \mathbf{W}(q).$$

Therefore, for every $x \in \mathcal{V}_{sp}(\Delta)$ and $w \in W$, the cardinality of the set $\mathcal{C}_w(x)$ does not depend on x and

$$|\mathcal{C}_w(x)| = \mathbf{W}(q) \ q_w.$$

For any pair of facets $\mathcal{F}_1, \mathcal{F}_2$ of the building, there exists an apartment $\mathcal{A}(\mathcal{F}_1, \mathcal{F}_2)$ containing them. We call *convex hull* of $\{\mathcal{F}_1, \mathcal{F}_2\}$ the minimal convex region $[\mathcal{F}_1, \mathcal{F}_2]$ delimited by hyperplanes of $\mathcal{A}(\mathcal{F}_1, \mathcal{F}_2)$ containing $\{\mathcal{F}_1, \mathcal{F}_2\}$.

Given two special vertices x, y, there exists a minimal gallery $\gamma(x,y)$ from x to y, lying on any apartment $\mathcal{A}(x,y)$ containing x and y. If c and d are the chambers of $\gamma(x,y)$ containing x and y respectively, and $\delta(c,d) = w_f$, then the type of this gallery is $f = i_1 \cdots i_k$. Moreover, if we denote by φ any type-preserving isomorphism from $\mathcal{A}(x,y)$ onto $\mathbb{A}$, we define the *shape* of y with respect to x as

$$\sigma(x,y) = \sigma(X,Y), \quad \text{if} \quad X = \varphi(x), \ Y = \varphi(y).$$

Hence, by definition of $\sigma(X,Y)$, the shape $\sigma(x,y)$ is an element of $\widehat{L}^+$ and, if $\sigma(x,y) = \lambda$, there exists a type-rotating isomorphism $\psi : \mathcal{A}(x,y) \to \mathbb{A}$, such that $\psi(x) = 0$ and $\psi(y) = \lambda$.

For every vertex $x \in \widehat{\mathcal{V}}(\Delta)$ and every $\lambda \in \widehat{L}^+$, we define

$$\mathcal{V}_\lambda(x) = \{y \in \widehat{\mathcal{V}}(\Delta) \ : \ \sigma(x,y) = \lambda\}.$$

It is easy to prove that, for every $x \in \widehat{\mathcal{V}}(\Delta)$, we have $\widehat{\mathcal{V}}(\Delta) = \cup_{\lambda \in \widehat{L}^+} \mathcal{V}_\lambda(x)$ as a disjoint union.

The following proposition provides a formula for the cardinality of the set $\mathcal{V}_\lambda(x)$.

Proposition 2.15.1. *Let $x \in \widehat{\mathcal{V}}(\Delta)$ and $\lambda \in \widehat{L}^+$. If $\tau(x) = i, \ \tau(X_\lambda) = l$ and $j = \sigma_i(l)$, then*

$$|\mathcal{V}_\lambda(x)| \ = \ \frac{1}{\mathbf{W}(q)} \sum_{w \in \mathbf{W} w_\lambda \mathbf{W}_j} q_w \ = \ \frac{\mathbf{W}(q)}{\mathbf{W}_\lambda(q)} \ q_{w_\lambda}.$$

In particular $|\mathcal{V}_\lambda(x)| = \mathbf{W}(q) \ q_{w_\lambda}$, if $\lambda \in L^{++}$.

Proof. For every chamber c of Δ and for every $i \in I$, we denote by $v_i(c)$ the vertex of type i of c. Then

$$\mathcal{V}_\lambda(x) = \{y = v_j(d), \ d \in \mathcal{C}(\Delta) \ : \ \delta(x,d) = \sigma_i(w_\lambda)\}.$$

If we define

$$\mathcal{C}_\lambda(x) = \{d \in \mathcal{C}(\Delta) \ : \ v_j(d) \in \mathcal{V}_\lambda(x)\},$$

then it is immediate to note that, for each $y \in \mathcal{V}_\lambda(x)$, there are $\mathbf{W}(q)$ chambers in $\mathcal{C}_\lambda(x)$ containing y; hence $|\mathcal{C}_\lambda(x)| = \mathbf{W}(q)|\mathcal{V}_\lambda(x)|$. On the other hand, if c denotes any chamber in the set $\mathcal{C}(x)$, it can be proved that, as disjoint union,

$$\mathcal{C}_\lambda(x) = \bigcup_{w \in \mathbf{W}_i \sigma_i(w_\lambda) \mathbf{W}_j} \mathcal{C}_w(c).$$

This implies that $|\mathcal{C}_\lambda(x)| = \sum_{w \in \mathbf{W}_i \sigma_i(w_\lambda) \mathbf{W}_j} |\mathcal{C}_w(c)|$. Since $\mathbf{W}_i \sigma_i(w_\lambda) \mathbf{W}_j = \sigma_i(\mathbf{W} w_\lambda \mathbf{W}_j)$ and $q_{\sigma_i(w)} \ = \ q_w$, it follows that

$$|\mathcal{C}_\lambda(x)| = \sum_{w \in \mathbf{W} w_\lambda \mathbf{W}_j} q_w.$$

So the first formula is proved.

Furthermore we notice that, if f_λ is the type of the gallery $\gamma(C_0, C_\lambda)$, then , for each $c \in \mathcal{C}(x)$, the gallery $\gamma(c,y)$ has type $\sigma_i(f_\lambda)$. Since, for each $c \in \mathcal{C}(x)$, the number of galleries $\gamma(c,y)$ is $q_{w_\lambda}/\mathbf{W}_\lambda(q)$ and $|\mathcal{C}(x)| = \mathbf{W}(q)$, also the last formula is proved. $\square$

Proposition 2.15.1 shows that $|\mathcal{V}_\lambda(x)|$ does not depend on x; so we can set, for every vertex $x \in \widehat{\mathcal{V}}(\Delta)$,

$$N_\lambda = |\mathcal{V}_\lambda(x)|.$$

We notice that, if we set $\lambda^\star = \iota(\lambda)$, then $y \in \mathcal{V}_\lambda(x)$ if and only if $x \in \mathcal{V}_{\lambda^\star}(y)$. Hence $N_\lambda = N_{\lambda^\star}$.

We provide an alternative formula for N_λ, in terms of q_{t_λ}.

Proposition 2.15.2. *Let $\lambda \in \widehat{L}^+$; then*

$$N_\lambda = \frac{\mathbf{W}(q^{-1})}{\mathbf{W}_\lambda(q^{-1})} q_{t_\lambda}.$$

In particular, if $\lambda \in L^{++}$, we have

$$N_\lambda = \mathbf{W}(q^{-1}) q_{t_\lambda}.$$

Proof. For any $x \in \widehat{\mathcal{V}}(\Delta)$ and $y \in \mathcal{V}_\lambda(x)$, we denote by c_x and c_y the chambers containing x and y respectively in any minimal gallery connecting x to y. Then, defining

$$\mathcal{C}_{t_\lambda}(x,y) = \{d \in \mathcal{C}(\Delta) \ : \ y \in d, \ \delta(x,d) = t_\lambda\},$$

it is easy to check that

$$\mathcal{C}_{t_\lambda}(x,y) = \{d \in \mathcal{C}(\Delta) \ : \ \delta(c_y,d) = w_j^0 w_{j,\lambda}^0\},$$

if w_j^0 and $w_{j,\lambda}^0$ are the longest elements of $\mathbf{W}_j$ and $\mathbf{W}_{j,\lambda} = \{w \in \mathbf{W}_j, \ : \ w\lambda = \lambda\}$ respectively. Therefore,

$$|\mathcal{C}_{t_\lambda}(x,y)| = q_{w_j^0 w_{j,\lambda}^0} = q_{w_j^0} q^{-1}_{w_{j,\lambda}^0} = q_{\mathbf{w}_0} q^{-1}_{\mathbf{w}_\lambda^0}$$

and

$$q_{t_\lambda} = q_{w_\lambda} q_{\mathbf{w}_0} q^{-1}_{\mathbf{w}_\lambda^0}.$$

Hence

$$N_\lambda = \frac{\mathbf{W}(q)}{\mathbf{W}_\lambda(q)} q^{-1}_{\mathbf{w}_0} q_{\mathbf{w}_\lambda^0} \ q_{t_\lambda}.$$

Since $\mathbf{W}(q) = q_{\mathbf{w}_0} \mathbf{W}(q^{-1})$ and $\mathbf{W}_\lambda(q) = q_{\mathbf{w}_\lambda^0} \mathbf{W}_\lambda(q^{-1})$, we conclude that

$$N_\lambda = \frac{\mathbf{W}(q^{-1})}{\mathbf{W}_\lambda(q^{-1})} q_{t_\lambda}.$$

In particular, if $\lambda \in L^{++}$, we have

$$N_\lambda = \mathbf{W}(q^{-1}) q_{t_\lambda}.$$

□

2.16. **Parameter system of R.** Let Δ be a building of type $\widetilde{X}_n$ and let $\{q_i\}_{i \in I}$ the parameter system of Δ. As we said in section 2.13, $q_{\sigma(i)} = q_i$, for every $i \in I$ and every $\sigma \in Aut_{tr}(D)$. Moreover we notice that $q_i = q_j$, if there exists an hyperplane h on any apartment of the building which contains two panels π_i and π_j of co-type i and j respectively. Hence for every hyperplane h of the building we may define $q_h = q_i$ if there is a panel of co-type i lying on h. We notice that if h and h' are two hyperplanes of the building, lying on $\mathcal{A}$ and $\mathcal{A}'$ respectively, and there exists a type-rotating isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$, such that $h' = \psi(h)$, then $q_{h'} = q_h$; actually, if π_i is a panel lying on h, then h' contains a panel of co-type $\sigma(i)$, for some $\sigma \in Aut_{tr}(D)$.

Consider any apartment $\mathcal{A}$ of Δ and the set $\mathcal{H}(\mathcal{A})$ of all the hyperplanes of $\mathcal{A}$. Let $\psi : \mathcal{A} \to \mathbb{A}$ any type-rotating isomorphism. According to notation of Section 2.15, we set $h = h_\alpha^k$ if $\psi(h) = H_\alpha^k$, for any positive root α and any $k \in \mathbb{Z}$. In this case we define

$$q_{\alpha,k} = q_h.$$

This definition is independent of the particular choice of $\mathcal{A}$ and ψ. Actually, if $\psi' : \mathcal{A}' \to \mathbb{A}$ is another type-rotating isomorphism and $\psi(h) = \psi'(h') = H_\alpha^k$, then $q_{h'} = q_h$, since ${\psi'}^{-1}\psi$ is a type-rotating automorphism mapping h onto h'.

If R is reduced, it is easy to check that $q_{\alpha,k} = q_{\alpha',k'}$, if $H_{\alpha'}^{k'} = \widehat{w}(H_\alpha^k)$, for some $\widehat{w} \in \widehat{W}$; actually $q_{h'} = q_h$, if $\psi(h) = H_\alpha^k$ and $\psi(h') = H_{\alpha'}^{k'}$, for any $\psi : \mathcal{A} \to \mathbb{A}$. In particular $q_{\alpha,0} = q_{\alpha',0}$, if $\alpha' = \mathbf{w}(\alpha)$, for some $\mathbf{w} \in \mathbf{W}$ and, for every $\alpha \in R^+$, $q_{\alpha,k} = q_{\alpha,0}$, for every $k \in \mathbb{Z}$. Moreover $q_{\alpha_i,0} = q_i$, $i = 1, \cdots, n$, and $q_{\alpha_0,1} = q_0$. These properties suggest to define, for every $\alpha \in R^+$,

$$q_\alpha = q_{\alpha,k}, \quad \forall k \in \mathbb{Z}.$$

Then $q_{\alpha_i} = q_i$, $\forall i \in I$, and for every $\alpha \in R^+$, $q_\alpha = q_{\alpha_i}$, if $\alpha = \mathbf{w}\alpha_i$, for some $\mathbf{w} \in \mathbf{W}$. Hence $q_\alpha = q_{\alpha_i}$, if $|\alpha| = |\alpha_i|$. It turns out that, if all roots have the same length (as for R of type A_n), then $q_i = q$, for every $i \in I$ and $q_\alpha = q$, for every $\alpha \in R$. Moreover, if R contains long and short roots, then $q_i = q$, if α_i is long, and $q_i = p$, if α is short; so $q_\alpha = q$, for all long α, and $q_\beta = p$, for all short β.

Consider now the case of a non reduced root system of type BC_n. Since $\widehat{L} = L$ and $\widehat{W} = W$, then every isomorphism of an apartment $\mathcal{A}$ onto $\mathbb{A}$ is type-preserving and $q_{\alpha,k} = q_{\alpha',k'}$, if $H^{k'}_{\alpha'} = w(H^k_\alpha)$, for some $w \in W$. Hence it is easy to prove that, for all $k \in \mathbb{Z}$,

$$\begin{aligned}
&q_{\alpha,2k+1} = q_{\alpha,1} = q_{\alpha_0,1}, \quad \forall \alpha \in R_1,\\
&q_{\alpha,k} = q_{\alpha,0} = q_{\alpha_n,0}, \quad \forall \alpha \in R_2,\\
&q_{\alpha,k} = q_{\alpha,0} = q_{\alpha_i,0}, \quad i = 1,\cdots,n-1, \text{ if } \alpha \in R_0 \text{ and } \alpha = \mathbf{w}\alpha_i, \quad \text{for some } \mathbf{w} \in \mathbf{W}.
\end{aligned}$$

Moreover

$$q_{\alpha_0,1} = q_1, \quad q_{\alpha_i,0} = q_0, \quad \text{for every} \quad i = 1,\cdots,n-1 \quad \text{and} \quad q_{\alpha_n,0} = q_n.$$

So, if we define

$$q_\alpha = \begin{cases} q_{\alpha,2k+1}, & \forall \alpha \in R_1, \quad \forall k \in \mathbb{Z},\\ q_{\alpha,k}, & \forall \alpha \in R_2 \cup R_0, \quad \forall k \in \mathbb{Z}, \end{cases}$$

we have

$$q_\alpha = \begin{cases} q_1, & \forall \alpha \in R_1,\\ q_0, & \forall \alpha \in R_0,\\ q_n, & \forall \alpha \in R_2. \end{cases}$$

For ease of notation, we set $q_1 = p,\ q_0 = q,\ q_n = r$. In each case it is convenient to extend the definition of q_α, by setting $q_\alpha = 1$, if $\alpha \notin R$. Thus, $q_\alpha = p,\ q_{\alpha/2} = r$, if $\alpha \in R_1$, $q_\alpha = q,\ q_{\alpha/2} = 1$, if $\alpha \in R_0$, and $q_\alpha = r,\ q_{\alpha/2} = 1$, if $\alpha \in R_2$.

It will be useful to give the following alternative characterization of q_{t_λ}, for every $\lambda \in \widehat{L}^+$.

Proposition 2.16.1. *For every $\lambda \in \widehat{L}^+$, then*

$$q_{t_\lambda} = \prod_{\alpha \in R^+} q_\alpha^{\langle\lambda,\alpha\rangle}\ q_{2\alpha}^{-\langle\lambda,\alpha\rangle}.$$

Proof. In order to prove this formula, we recall that q_{u_λ} denotes the number of chambers c' connected to any chamber c by a gallery of type u_λ. Moreover $q_{t_\lambda} = q_{u_\lambda} = q_{i_1}\cdots q_{i_r}$, if $t_\lambda = u_\lambda g_l$ and $u_\lambda = s_{i_1}\cdots s_{i_r}$.

Fix in the building Δ two chambers c, c' such that $\delta(c,c') = u_\lambda$; denote by $\mathcal{A}$ any apartment containing c, c' (and hence the gallery $\gamma(c,c')$ of type u_λ), and consider the isomorphism $\psi : \mathcal{A} \to \mathbb{A}$ such that $\psi(c) = C_0$. Through this isomorphism, the chamber c' maps to the chamber $u_\lambda(C_0)$, lying on $\mathbb{Q}_0$. For every $i_1,\cdots,i_r$, the panel π_{i_j} of the gallery belongs to a hyperplane h of $\mathcal{A}$ such that $\psi(h) = H^j_\alpha$, for some $\alpha \in R^+$ and $j \in \mathbb{Z}$; therefore it follows that

$$q_{t_\lambda} = \prod_{\alpha \in R^+} q_\alpha^{k_\alpha},$$

if, for each $\alpha \in R^+$, k_α denotes the number of hyperplanes in $\mathcal{H}(\alpha)$ separating C_0 and $u_\lambda(C_0)$. Since $v_l(u_\lambda(C_0)) = \lambda$, we notice that $k_\alpha = \langle\lambda,\alpha\rangle$, when $\alpha/2 \notin R$, and $k_\alpha = \langle\lambda,\alpha/2\rangle$, otherwise; so we get the required formula. □

Corollary 2.16.2. *Let $\lambda \in \widehat{L}^+$; then*

$$N_\lambda = \frac{\mathbf{W}(q^{-1})}{\mathbf{W}_\lambda(q^{-1})} \prod_{\alpha \in R^+} q_\alpha^{\langle\lambda,\alpha\rangle}\ q_{2\alpha}^{-\langle\lambda,\alpha\rangle}.$$

In particular, if $\lambda \in \widehat{L}^{++}$, we have

$$N_\lambda = \mathbf{W}(q^{-1}) \prod_{\alpha \in R^+} q_\alpha^{\langle\lambda,\alpha\rangle}\ q_{2\alpha}^{-\langle\lambda,\alpha\rangle}.$$

2.17. **The algebra $\mathcal{H}(\mathcal{C})$.** We denote by $\mathcal{L}(\mathcal{C})$ the space of all finitely supported functions on $\mathcal{C} = \mathcal{C}(\Delta)$. Each function $f \in \mathcal{L}(\mathcal{C})$ can be written uniquely as $f = \sum_c f(c)\mathbb{I}_c$, where, for each chamber $c \in \mathcal{C}(\Delta)$,

$$\mathbb{I}_c(c') = \begin{cases} 1, & c' = c\\ 0, & c' \neq c. \end{cases}$$

For each $w \in W$, we define

$$T_w \mathbb{I}_c = \sum_{\delta(c',c) = w} \mathbb{I}_{c'}.$$

The operator T_w may be extended by linearity to the space $\mathcal{L}(\mathcal{C})$, by setting $T_w f = \sum_c f(c)\ T_w \mathbb{I}_c$, if $f = \sum_c\ f(c)\mathbb{I}_c$. It is easy to prove that, for every c,

$$T_w f(c) = \sum_{\delta(c,c')=w} f(c').$$

Actually

$$T_w f(c) = \langle T_w f, \mathbb{I}_c\rangle = \sum_{c'} f(c') \sum_{\delta(c'',c')=w} \langle \mathbb{I}_{c''}, \mathbb{I}_c\rangle = \sum_{\delta(c,c')=w} f(c'),$$

since we can choose $c'' = c$ in the sum only in the case $\delta(c,c') = w$ and $\langle \mathbb{I}_{c''}, \mathbb{I}_c\rangle = 0$ for $c'' \neq c$.

We denote by $\mathcal{H}(\mathcal{C})$ the linear span of $\{T_w,\ w \in W\}$. We shall prove that in fact $\mathcal{H}(\mathcal{C})$ is an algebra.

Lemma 2.17.1. *Let S be the finite set of generators of W; for every $s \in S$,*

$$T_s^2 = q_s I + (q_s - 1)T_s,$$

if $q_s = q_\alpha$, when $s = s_\alpha$.

Proof. Fix $s \in S$; then, for every chamber c,

$$T_s^2 \mathbb{I}_c = \sum_{\delta(c',c)=s} T_s \mathbb{I}_{c'} = \sum_{\delta(c',c)=s} \sum_{\delta(c'',c')=s} \mathbb{I}_{c''} = \sum_{\delta(c',c)=s} \left(\mathbb{I}_c + \sum_{\delta(c'',c')=s, c''\neq c} \mathbb{I}_{c''}, \right).$$

Since q_s is the number of chambers c' such that $\delta(c,c') = \delta(c',c) = s$, we conclude that

$$T_s^2 = q_s \mathbb{I}_c + (q_s - 1) \sum_{\delta(c',c)=s} \mathbb{I}_{c'} = q_s I + (q_s - 1)T_s.$$

□

Proposition 2.17.2. *For every $w \in W$, and $s \in S$, then*

$$T_w T_s = \begin{cases} T_{ws}, & \text{if} \quad |ws| = |w| + 1, \\ q_s T_{ws} + (q_s - 1)T_w, & \text{if} \quad |ws| = |w| - 1. \end{cases}$$

Proof. For each function $f \in \mathcal{L}(\mathcal{C})$, and each chamber c, we have by definition

$$(T_w T_s) f(c) = \sum_{\delta(c,c')=w} \sum_{\delta(c',c'')=s} f(c'') \quad \text{and} \quad T_{ws} f(c) = \sum_{\delta(c,\tilde{c})=ws} f(\tilde{c}).$$

If $|ws| = |w| + 1$, then, for every $\tilde{c}$, there exists c' such that $\delta(c,c') = w$ and $\delta(c',\tilde{c}) = s$; hence $\mathcal{C}_{ws}(c) = \{\tilde{c} \ :\ \delta(c,\tilde{c}) = ws\} = \cup_{\delta(c,c')=w}\{c'' \ :\ \delta(c',c'') = s\}$. Therefore $(T_w T_s) f(c) = T_{ws} f(c)$.

Assume now $|ws| = |w| - 1$ and define $w_1 = ws$. In this case $w = w_1 s$, with $|w_1 s| = |w_1| + 1$. Therefore $T_w = T_{w_1 s} = T_{w_1} T_s$ and, by Lemma 2.17.1,

$$T_w T_s = T_{w_1} T_s^2 = q_s T_{w_1} + (q_s - 1) T_{w_1} T_s = q_s T_{w_1} + (q_s - 1) T_{w_1 s} = q_s T_{ws} + (q_s - 1) T_w.$$

□

Theorem 2.17.3. *Let $w_1, w_2 \in W$; for every $w \in W$ there exists $N_w(w_1,w_2)$, such that*

$$T_{w_1} T_{w_2} = \sum_{w\in W} N_w(w_1,w_2)\ T_w.$$

Moreover the set $\{w \in W \ :\ N_w(w_1,w_2) \neq 0\}$ is finite, for all $w_1, w_2 \in W$.

Proof. We use induction on $|w_2|$. If $|w_2| = 1$, then $w_2 = s$, for some $s \in S$, and the identity follows from Proposition 2.17.2. If $|w_2| = n$, for $n > 1$, we write $w_2 = w's$, for some s and w' such that $|w'| = n - 1$. Hence $T_{w_1} T_{w_2} = T_{w_1} T_{w'} T_s$. If we assume that the identity is true for each $k < n$, then

$$T_{w_1} T_{w_2} = (T_{w_1} T_{w'}) T_s = \left(\sum_{w\in W} N_w(w_1,w')\ T_w \right)\ T_s = \sum_{w\in W} N_w(w_1,w')\ (T_w\ T_s).$$

Therefore the identity follows from Proposition 2.17.2. □

Corollary 2.17.4. *Let $w_1, w_2 \in W$; if $|w_1 w_2| = |w_1| + |w_2|$, then $T_{w_1} T_{w_2} = T_{w_1 w_2}$.*

Proof. If $|w_2| = 1$, the identity follows from Proposition 2.17.2. If $|w_2| = n$, for $n > 1$, and $w_2 = w's$, for some s and w' such that $|w'| = n-1$, then $|w_1w'| = |w_1| + |w'|$, and $|w_1w_2| = |w_1w'| + |s|$. Thus, if we assume the identity true for each $k < n$, we have, by Proposition 3.1.2,

$$T_{w_1}T_{w_2} = T_{w_1}T_{w'}T_s = T_{w_1w'}T_s = T_{w_1w's} = T_{w_1w_2}.$$

□

Theorem 2.17.3 shows that $\mathcal{H}(\mathcal{C})$ is an associative algebra, generated by $\{T_s,\ s \in S\}$. We refer to the numbers $N_w(w_1, w')$ as the *structure constants* of the algebra $\mathcal{H}(\mathcal{C})$. We notice that $\mathcal{H}(\mathcal{C})$ is (up to an isomorphism) the Hecke algebra $\mathcal{H}(q_s, q_s - 1)$associated to W and S (see [6], Chapter 7).

It will be useful to exhibit some particular operators of the algebra $\mathcal{H}(\mathcal{C})$. For every $i \in \widehat{I}$ and for any chamber c, we set

$$T_i\mathbb{I}_c = \sum_{v_i(c')=v_i(c)} \mathbb{I}_{c'},$$

if, as usual, $v_i(c)$ denotes the vertex of type i lying in c. We extend T_i to the space $\mathcal{L}(\mathcal{C})$ by linearity.

Proposition 2.17.5. *For every $i \in \widehat{I}$, the operator T_i belongs to the algebra $\mathcal{H}(\mathcal{C})$. Moreover $T_i^\star = T_i$.*

Proof. We observe that $T_i \in \mathcal{H}(\mathcal{C})$, for every $i \in \widehat{I}$, because $T_i = \sum_{w\in W_i} T_w$; actually

$$\{c' : v_i(c') = v_i(c)\} = \cup_{w\in W_i}\{c' : \delta(c, c') = w\}.$$

To prove that T_i is selfadjoint, we consider, for all c_1, c_2,

$$\langle T_i\mathbb{I}_{c_1}, \mathbb{I}_{c_2}\rangle = \sum_{v_i(c')=v_i(c_1)} \langle \mathbb{I}_{c'}, \mathbb{I}_{c_2}\rangle \quad \text{and} \quad \langle \mathbb{I}_{c_1}, T_i\mathbb{I}_{c_2}\rangle = \sum_{v_i(c'')=v_i(c_2)} \langle \mathbb{I}_{c_1}, \mathbb{I}_{c''}\rangle.$$

We notice that $\langle \mathbb{I}_{c'}, \mathbb{I}_{c_2}\rangle \neq 0$ only for $c' = c_2$ and we can choose $c' = c_2$ in the set $\{c' \ :\ v_i(c') = v_i(c_1)\}$ only if $v_i(c_1) = v_i(c_2)$. Analogously, $\langle \mathbb{I}_{c_1}, \mathbb{I}_{c''}\rangle \neq 0$ only for $c'' = c_1$ and we can choose $c'' = c_1$ in the set $\{c'' \ :\ v_i(c'') = v_i(c_2)\}$ only if $v_i(c_1) = v_i(c_2)$. Therefore we conclude that

$$\langle T_i\mathbb{I}_{c_1}, \mathbb{I}_{c_2}\rangle = \langle \mathbb{I}_{c_1}, T_i\mathbb{I}_{c_2}\rangle = \begin{cases} 1, & \text{if} \quad v_i(c_1) = v_i(c_2), \\ 0, & \text{if} \quad v_i(c_1) \neq v_i(c_2). \end{cases}$$

□

2.18. **Chamber and vertex regularity of the building.** For every triple $w_0, w_1, w_2 \in W$ and every pair of chambers c_1, c_2, such that $\delta(c_1, c_2) = w_0$, consider the set

$$\{c' \in \mathcal{C}(\Delta) \ :\ \delta(c_1, c') = w_1,\ \delta(c_2, c') = w_2\}.$$

We say that the building Δ is *chamber regular* if the cardinality of this set does not depend on the choice of the chambers , but only depends on w_0, w_1, w_2.

Proposition 2.18.1. *The building Δ is chamber regular.*

Proof. Fix a triple $w_0, w_1, w_2 \in W$ and a pair of chambers c_1, c_2, such that $\delta(c_1, c_2) = w_0$. Consider the operator $T_{w_1}T_{w_2^{-1}}$. For any chamber c,

$$(T_{w_1}T_{w_2^{-1}})\mathbb{I}_c = \sum_{\delta(c',c)=w_2^{-1}} \ \sum_{\delta(c'',c')=w_1} \mathbb{I}_{c''} = \sum_{\delta(c,c')=w_2} \ \sum_{\delta(c'',c')=w_1} \mathbb{I}_{c''}.$$

Let $c_1, c_2 \in \mathcal{C}(\Delta)$ and assume that $\delta(c_1, c_2) = w_0$. Then

$$\langle (T_{w_1}T_{w_2^{-1}})\mathbb{I}_{c_2}, \mathbb{I}_{c_1}\rangle = \sum_{\delta(c_2,c')=w_2} \ \sum_{\delta(c'',c')=w_1} \langle \mathbb{I}_{c''}, \mathbb{I}_{c_1}\rangle = |\{c' \ :\ \delta(c_1, c') = w_1,\ \delta(c_2, c') = w_2\}|,$$

since $\langle \mathbb{I}_{c''}, \mathbb{I}_{c_1}\rangle = 1$, if $c'' = c_1$ and $\langle \mathbb{I}_{c''}, \mathbb{I}_{c_1}\rangle = 0$ otherwise. On the other hand, as we have proved in Section 2.17, there exist constants $N_w(w_1, w_2^{-1}),\ w \in W$, such that

$$T_{w_1}T_{w_2^{-1}} = \sum_{w\in W} N_w(w_1, w_2^{-1})\ T_w.$$

Therefore

$$\begin{aligned} \langle (T_{w_1}T_{w_2^{-1}})\mathbb{I}_{c_2}, \mathbb{I}_{c_1}\rangle &= \sum_{w\in W} N_w(w_1, w_2^{-1})\langle T_w\mathbb{I}_{c_2}, \mathbb{I}_{c_1}\rangle \\ &= \sum_{w\in W} N_w(w_1, w_2^{-1}) \sum_{\delta(d,c_2)=w} \langle \mathbb{I}_d, \mathbb{I}_{c_1}\rangle = N_{w_0}(w_1, w_2^{-1}), \end{aligned}$$

since $\langle \mathbb{I}_d, \mathbb{I}_{c_1}\rangle \neq 0$ only if $d = c_1$ and this equality is possible only in the case $w = w_0$, as we assumed $\delta(c_1, c_2) = w_0$. So we conclude that

$$|\{c' \ : \ \delta(c_1, c') = w_1,\ \delta(c_2, c') = w_2\}| = N_{w_0}(w_1, w_2^{-1}).$$

This prove the required statement. □

Using the operators T_i, defined in Section 2.17, we extend the previous result to every set

$$\{c' \in \mathcal{C}(\Delta) \ : \ \delta(c_1, c') = w_1,\ \delta(c_2, c') = w_2\}.$$

Proposition 2.18.2. *Let $w_0, w_1, w_2 \in W$. If $x \in \mathcal{V}_{sp}(\Delta)$ and $c \in \mathcal{C}(\Delta)$ satisfy $\delta(x, c) = w_0$, then*

$$|\{c' \in \mathcal{C}(\Delta) \ : \ \delta(x, c') = w_1,\ \delta(c, c') = w_2\}|$$

does not depend on x and c, but only on w_0, w_1, w_2.

Proof. Let x be a special vertex and let c be a chamber; assume $\delta(x, c) = w_0$. This means that $\delta(c_x, c) = w_0$, if c_x denotes the chamber containing x in a minimal gallery $\gamma(x, c)$. If $\tau(x) = i$, we have

$$\begin{aligned}\langle (T_{w_1} T_{w_2^{-1}})\mathbb{I}_c, T_i \mathbb{I}_{c_x}\rangle &= \sum_{c'_x : x \in c'_x} \langle (T_{w_1} T_{w_2^{-1}})\mathbb{I}_c, \mathbb{I}_{c'_x}\rangle = \sum_{c'_x : x \in c'_x} |\{c' \ : \ \delta(c'_x, c') = w_1,\ \delta(c, c') = w_2\}| \\ &= |\{c' \ : \ \delta(x, c') = w_1,\ \delta(c, c') = w_2\}|.\end{aligned}$$

On the other hand T_i is a selfadjoint operator of the algebra generated by $\{T_w, w \in W\}$; hence

$$\langle (T_{w_1} T_{w_2^{-1}})\mathbb{I}_c, T_i \mathbb{I}_{c_x}\rangle = \langle (T_i T_{w_1} T_{w_2^{-1}})\mathbb{I}_c, \mathbb{I}_{c_x}\rangle$$

and there exist constants $n^i_w(w_1, w_2^{-1})$ such that $T_i T_{w_1} T_{w_2^{-1}} = \sum_{w \in W} n^i_w(w_1, w_2^{-1}) T_w$. Therefore, by the same argument used in Proposition 2.18.1,

$$\langle (T_{w_1} T_{w_2^{-1}})\mathbb{I}_c, T_i \mathbb{I}_{c_x}\rangle = \sum_{w \in W} n^i_w(w_1, w_2^{-1}) \langle T_w \mathbb{I}_c, \mathbb{I}_{c_x}\rangle = n^i_{w_0}(w_1, w_2^{-1}).$$

This proves the required statement, as

$$|\{c' \ : \ \delta(x, c') = w_1,\ \delta(c, c') = w_2\}| = n^i_{w_0}(w_1, w_2^{-1}).$$

□

Corollary 2.18.3. *Let $\lambda \in \widehat{L}^+$ and $w_1, w_2 \in W$. If $x, y \in \widehat{\mathcal{V}}(\Delta)$, and $\sigma(x, y) = \lambda$, then*

$$|\{c' \in \mathcal{C}(\Delta) \ : \ \delta(x, c') = w_1,\ \delta(y, c') = w_2\}|$$

does not depend on x and y, but only on λ, w_1, w_2.

For every triple $\lambda, \mu, \nu \in \widehat{L}$ and every pair $x, y \in \widehat{\mathcal{V}}(\Delta)$, such that $\sigma(x, y) = \lambda$, consider the set

$$\{z \in \widehat{\mathcal{V}}(\Delta) \ : \ \sigma(x, z) = \mu,\ \sigma(y, z) = \nu\}.$$

We say that the building Δ is *vertex regular* if the cardinality of this set does not depend on the choice of the vertices , but only depends on λ, μ, ν.

Proposition 2.18.4. *The building is vertex regular. Moreover*

$$|\{z \in \widehat{\mathcal{V}}(\Delta) \ : \ \sigma(x, z) = \mu,\ \sigma(y, z) = \nu\}| = |\{z \in \widehat{\mathcal{V}}(\Delta) \ : \ \sigma(x, z) = \nu^\star,\ \sigma(y, z) = \mu^\star\}|.$$

Proof. Let $\lambda \in \widehat{L}^+$ and $\sigma(x, y) = \lambda$. Consider in W the elements $\sigma_i(w_\mu), \sigma_j(w_\nu)$, if $i = \tau(x), j = \tau(y)$. By Corollary 2.18.3, the cardinality of the set

$$A = \{c' \in \mathcal{C}(\Delta) \ : \ \delta(x, c') = \sigma_i(w_\mu),\ \delta(y, c') = \sigma_j(w_\nu)\}$$

does not depend on x and y. On the other hand $\sigma(x, z) = \mu,\ \sigma(y, z) = \nu$ if and only if $z = v_l(c')$, for some $c' \in A$, and some $l \in \widehat{I}$. This proves that the set $\{z \in \widehat{\mathcal{V}}(\Delta) \ : \ \sigma(x, z) = \mu,\ \sigma(y, z) = \nu\}$ has a cardinality independent of x and y. Moreover we notice that, if $\sigma(x, y) = \lambda$, then $\sigma(y, x) = \lambda^\star$; hence

$$|\{z \in \widehat{\mathcal{V}}(\Delta) \ : \ \sigma(x, z) = \mu,\ \sigma(y, z) = \nu\}| = |\{z' \in \widehat{\mathcal{V}}(\Delta) \ : \ \sigma(y, z') = \mu^\star,\ \sigma(x, z') = \nu^\star\}|.$$

This completes the proof. □

We set

$$N(\lambda, \mu, \nu) = |\{z \in \widehat{\mathcal{V}}(\Delta) \ : \ \sigma(x, z) = \mu,\ \sigma(y, z) = \nu\}| = N(\lambda, \nu^\star, \mu^\star), \quad \text{if} \quad \sigma(x, y) = \lambda. \tag{2.18.1}$$

2.19. **Partial ordering on $\mathbb{A}$.** We define a partial order on $\widehat{L}$, by setting

$$\mu \preceq \lambda, \quad \text{if} \quad \lambda - \mu \in L^+.$$

Since $\widehat{\mathcal{V}}(\mathbb{A})$ may be identified with the co-weight lattice $\widehat{L}$, the partial ordering defined on $\widehat{L}$ applies to $\widehat{\mathcal{V}}(\mathbb{A})$. For every $\lambda \in \widehat{L}^+$, we define

$$\Pi_\lambda = \{\mathbf{w}\mu \ : \ \mu \in \widehat{L}^+, \ \mu \preceq \lambda, \ \mathbf{w} \in \mathbf{W}\}.$$

This set is *saturated*: for every $\eta \in \Pi_\lambda$ and every $\alpha \in R$, then $\eta - j\alpha^\vee \in \Pi_\lambda$, for every $0 \le j \le \langle \eta, \alpha \rangle$. Hence it is stable under $\mathbf{W}$. Moreover λ is the highest co-weight of Π_λ. It is easy to prove that $\Pi_\lambda + \Pi_\mu \subset \Pi_{\lambda+\mu}$, for every $\lambda, \mu \in \widehat{L}^+$. We recall that W is endowed with the Bruhat ordering, defined as follows (see [7]). We declare $w_1 < w_2$ if there exists a sequence $w_1 = u_0 \to u_1, \cdots, u_{k-1} \to u_k = w_2$, where $u_j \to u_{j+1}$ means that $u_{j+1} = u_j s$, for some $s \in S$, and $|u_j| < |u_{j+1}|$. This defines a partial order on W that can be extended to $\widehat{W}$, by setting $\widehat{w}_1 \le \widehat{w}_2$, if $\widehat{w}_1 = w_1 g_1$ and $\widehat{w}_2 = w_2 g_2$ with $w_1 < w_2$. We remark that $w_1 \le w_2$ if and only if w_1 can be obtained as a sub-expression $s_{i_{k_1}} \cdots s_{i_{k_m}}$ of any reduced expression $s_{i_1} \cdots s_{i_r}$ for w_2. We notice that, for every $\lambda \in \widehat{L}^+$, if $\widehat{w}(0) \in \Pi_\lambda$, then $\widehat{w}'(0) \in \Pi_\lambda$, for each $\widehat{w}' \le \widehat{w}$.

We define also a partial ordering on $\mathcal{C}(\mathbb{A})$, in the following way. Given two chambers C_1, C_2 consider all the hyperplanes H_α^k separating C_1 and C_2. We declare $C_1 \prec C_2$, if C_2 belongs to the positive half-space determined by each of these hyperplanes. It is clear that the resulting relation $C_1 \preceq C_2$ is a partial ordering of $\mathcal{C}(\mathbb{A})$. We notice that, by definition of $\mathbb{Q}_0$, we have $C_0 \prec C$ if and only if $C \subset \mathbb{Q}_0$. Moreover, if C is any chamber and $s = s_\alpha^k$ is the affine reflection with respect to the hyperplane containing a panel of C, then $C \prec s(C)$ or $s(C) \prec C$, since C and $s(C)$ are adjacent. Since $\mathcal{C}(\mathbb{A})$ may be identified with W, the previous definition induces a partial ordering on W. We point out that this ordering is different from the Bruhat order. Nevertheless, if $w_1(C_0)$ and $w_2(C_0)$ belong to $\mathbb{Q}_0$, then $w_1(C_0) \prec w_2(C_0)$ if and only if $w_1 < w_2$. Moreover, on $\mathbf{W}$, we have

$$\mathbf{w}_1(C_0) \prec \mathbf{w}_2(C_0) \quad \text{if and only if} \quad \mathbf{w}_1 > \mathbf{w}_2.$$

Proposition 2.19.1. *Let C be a chamber of $\mathbb{A}$; let $s = s_\alpha^k$ be the affine reflection with respect to the hyperplane H_α^k containing a panel of C and $\mathbf{s} = s_\alpha^0$. Assume that $C \prec s(C)$. Let $w \in W$; if $w = \mathbf{w}t_\lambda$ for some $\mathbf{w} \in \mathbf{W}$ and $\lambda \in L$, then*

(i) if $w(C) \prec ws(C)$, then $\mathbf{w} < \mathbf{ws}$;
(ii) if $ws(C) \prec w(C)$, then $\mathbf{ws} < \mathbf{w}$.

Proof. Since α is positive and $C \prec s(C)$, then C and $s(C)$ belong respectively to the negative and the positive half-space determined by the affine hyperplane H_α^k, that is, for every vertex v lying in C,

$$\langle v, \alpha \rangle \le k, \qquad \langle s(v), \alpha \rangle \ge k.$$

The adjacent chambers $w(C)$ and $ws(C)$ share a panel which belongs to the hyperplane $w(H_\alpha^k) = H_{\mathbf{w}(\alpha)}^{k'}$; moreover, for every $v \in C$,

$$\langle w(v), \mathbf{w}(\alpha) \rangle \le k' \quad \text{and} \quad \langle ws(v), \mathbf{w}(\alpha) \rangle \ge k'.$$

Actually, if we set $k' = k + \langle \lambda, \alpha \rangle$, then

$$\langle w(v), \mathbf{w}(\alpha) \rangle = \langle \mathbf{w}t_\lambda(v), \mathbf{w}(\alpha) \rangle = \langle t_\lambda(v), \alpha \rangle = \langle v, \alpha \rangle + \langle \lambda, \alpha \rangle \le k'$$
$$\langle ws(v), \mathbf{w}(\alpha) \rangle = \langle \mathbf{w}t_\lambda s(v), \mathbf{w}(\alpha) \rangle = \langle t_\lambda s(v), \alpha \rangle = \langle s(v), \alpha \rangle + \langle \lambda, \alpha \rangle \ge k'.$$

This implies that $\mathbf{w}(\alpha)$ is positive in the case (i) and negative in the case (ii).

If $\mathbf{w}(\alpha) > 0$, then, for every $v \in \mathbb{Q}_0$, we have

$$\langle \mathbf{w}^{-1}v, \alpha \rangle = \langle v, \mathbf{w}(\alpha) \rangle > 0, \qquad \langle (\mathbf{ws})^{-1}v, \alpha \rangle = \langle v, \mathbf{ws}(\alpha) \rangle = -\langle v, \mathbf{w}(\alpha) \rangle < 0,$$

since $\langle v, \mathbf{s}(\alpha) \rangle = -\langle v, \alpha \rangle$. Therefore $\mathbb{Q}_0$ and $\mathbf{w}^{-1}(\mathbb{Q}_0)$ belong to the same half-space determined by H_α, while H_α separates $(\mathbf{ws})^{-1}(\mathbb{Q}_0)$ and $\mathbb{Q}_0$. So the number of hyperplanes separating $\mathbb{Q}_0$ and $(\mathbf{ws})^{-1}(\mathbb{Q}_0)$ is bigger than the number of hyperplanes separating $\mathbb{Q}_0$ and $(\mathbf{w})^{-1}(\mathbb{Q}_0)$, and we conclude that $\mathbf{w} < \mathbf{ws}$.

On the contrary, if $\mathbf{w}(\alpha) < 0$, then, for every $v \in \mathbb{Q}_0$, we have

$$\langle \mathbf{w}^{-1}v, \alpha \rangle < 0, \qquad \langle (\mathbf{ws})^{-1}v, \alpha \rangle > 0,$$

and therefore we conclude that $\mathbf{w} > \mathbf{ws}$. $\square$

2.20. **Retraction ρ_x.** Let x be any special vertex of Δ (say $\tau(x) = i$). For every $c \in \mathcal{C}(\Delta)$, we denote by $proj_x(c)$ the chamber containing x in any minimal gallery $\gamma(x, c)$. In particular we write $proj_0(c)$ when x is the fundamental vertex e. We note that $proj_x(c)$ does not depend on the minimal gallery we consider.

In the fundamental apartment $\mathbb{A}$, let $\mathbb{Q}_0^- = \mathbf{w}_0(\mathbb{Q}_0)$ and C_0^- the base chamber of $\mathbb{Q}_0^-$.

Definition 2.20.1. *For every $c \in \mathcal{C}(\Delta)$, the retraction of c with respect to x is defined as*

$$\rho_x(c) = C_0^- \cdot \delta_i(proj_x(c), c),$$

if, for every pair c, d of chambers, we set $\delta_i(c, d) = w_{\sigma_i^{-1}(f)}$ when $\delta(c, d) = w_f$. In particular, if $\tau(x) = 0$,

$$\rho_x(c) = C_0^- \cdot \delta(proj_x(c), c).$$

Obviously, $\rho_x(c)$ belongs to $\mathbb{Q}_0^-$, for every c. We extend the previous definition to all special vertices. For every $y \in \mathcal{V}_{sp}(\Delta)$, say $\tau(y) = j$, we set

$$\rho_x(y) = v_l(\rho_x(c)),$$

if c is any chamber containing y, and $l = \sigma_i^{-1}(j)$. Actually this definition does not depend on the choice of the chamber containing the vertices y, since $v_l(c_1) = v_l(c_2)$ implies $v_l(\rho_x(c_1)) = v_l(\rho_x(c_2))$. In particular, we denote by ρ_0 the retraction with respect to the fundamental vertex e. It will be useful to remark that, if $\lambda \in \widehat{L}^+$, and $t_\lambda = u_\lambda g_l$, then, for every c such that $\delta(proj_0(c), c) = u_\lambda$, we have $\rho_0(c) = \mathbf{w}_0 u_\lambda(C_0)$. Therefore, if $\sigma(e, x) = \lambda$, then $\rho_0(x) = \mathbf{w}_0\lambda$.

2.21. **Extended chambers.** We recall that the action of $\widehat{W}$ on the set $\mathcal{C}(\mathbb{A})$ is transitive but not simply transitive; actually, if $\widehat{w}_i = wg_i$, then $\widehat{w}_i(C_0) = w(C_0)$, for every $w \in W$ and for every $i \in \widehat{I}$. Nevertheless, the action of the elements $\widehat{w}_i$ on the special vertices $v_j(C_0)$ of C_0 depends on i, because

$$\widehat{w}_i(v_j(C_0)) = v_{\sigma_i(j)}(w(C_0)).$$

This suggest to enlarge the set $\mathcal{C}(\mathbb{A})$ in the following way. We call extended chamber of $\mathbb{A}$ a pair $\widehat{C} = (C, \sigma)$, for every $C \in \mathcal{C}(\mathbb{A})$ and for every $\sigma \in Aut_{tr}(D)$; we denote by $\widehat{\mathcal{C}}(\mathbb{A})$ the set of all extended chambers. A straightforward consequence of this definition is that $\widehat{W}$ acts simply transitively on $\widehat{\mathcal{C}}(\mathbb{A})$: for every couple of extended chambers $\widehat{C}_1 = (C_1, \sigma_{i_1})$ and $\widehat{C}_2 = (C_2, \sigma_{i_2})$, there exists a unique element $\widehat{w} \in \widehat{W}$ such that $\widehat{C}_2 = \widehat{w}(\widehat{C}_1)$. Actually, if $C_2 = w(C_1)$, $g = g_{i_2} g_{i_1}^{-1}$ and σ is the automorphism of D corresponding to g, then $\widehat{w} = wg = g\sigma(w)$. In particular, for every $\widehat{C} = (C, \sigma_i)$, then $\widehat{w} = wg_i = g_i\sigma_i(w)$ is the unique element of $\widehat{W}$ such that $\widehat{w}(C_0) = \widehat{C}$, if $C = w(C_0)$.

In the same way we enlarge the set $\mathcal{C}(\Delta)$ and we define

$$\widehat{\mathcal{C}}(\Delta) = \{\widehat{c} = (c, \sigma_i),\ c \in \mathcal{C}(\Delta),\ i \in \widehat{I}\}.$$

We notice that for every $c \in \mathcal{C}(\Delta)$ and $i \in \widehat{I}$, there exists a unique $\widehat{c}$ such that $v_i(c) = v_0(\widehat{c})$; actually, this element is $\widehat{c} = (c, \sigma_i)$. The W-distance on $\mathcal{C}(\Delta)$ can be extended to a $\widehat{W}$-distance on $\widehat{\mathcal{C}}(\Delta)$ in the following way: for every couple of extended chambers $\widehat{c}_1 = (c_1, \sigma_{i_1})$ and $\widehat{c}_2 = (c_2, \sigma_{i_2})$, we set

$$\widehat{\delta}(\widehat{c}_1, \widehat{c}_2) = \delta(c_1, c_2) g_{i_2} g_{i_1}^{-1}.$$

For every $\lambda \in \widehat{L}^+$, with $\tau(\lambda) = l$, consider the translation $t_\lambda = u_\lambda g_l$; then $t_\lambda(C_0) = (u_\lambda(C_0), g_l)$ and $v_0(t_\lambda(C_0)) = v_l(u_\lambda(C_0))$.

3. Maximal boundary

3.1. **Sectors of $\mathbb{A}$.** Let R be a root system and let $\mathbb{A} = \mathbb{A}(R)$. In Section 2.7 we defined a sector of $\mathbb{A}$, based at 0, as any connected component of $\mathbb{V} \setminus \cup_\alpha H_\alpha$; in particular $\mathbb{Q}_0 = \{v \in \mathbb{V}\ :\ \langle v, \alpha\rangle > 0,\ i \in I_0\}$ is the fundamental sector based at 0. For every chamber C containing 0, we denote by $Q_0(C)$ the sector based at 0, of base chamber C; in particular, C_0 is the base chamber of $\mathbb{Q}_0$. We notice that $Q_0(C) = \mathbf{w}\mathbb{Q}_0$, for some $\mathbf{w} \in \mathbf{W}$.

More generally, for each special vertex X of $\mathbb{A}$, in particular for every $X \in \widehat{\mathcal{V}}(\mathbb{A})$, we call *sector* of $\mathbb{A}$, based at X, any connected component of $\mathbb{V} \setminus \cup_{H_\alpha^k \in \mathcal{H}_X} H_\alpha^k$, if $\mathcal{H}_X$ denotes the collection of all hyperplanes of $\mathcal{H}$ sharing X. For every chamber C containing X, we denote by $Q_X(C)$ the sector based at X, of base chamber C. We remark that, for every $X \in \widehat{\mathcal{V}}(\mathbb{A})$, and every C containing X, there exists a unique $\widehat{w} \in \widehat{W}$, such that $Q_X(C) = \widehat{w}\ \mathbb{Q}_0$.

3.2. **Maximal boundary.** We extend to any irreducible regular affine building Δ the definition of sector given on its fundamental apartment $\mathbb{A} = \mathbb{A}(R)$, declaring that, for any $x \in \mathcal{V}_{sp}(\Delta)$, a *sector* of Δ, with base vertex x, is a subcomplex Q_x of any apartment $\mathcal{A}$ of the building, such that $\psi_{tp}(Q_x) = Q_X$, if X is any special vertex such that $\tau(X) = \tau(x)$, and $\psi_{tp} : \mathcal{A} \to \mathbb{A}$ is a type-preserving isomorphism mapping x to X. We note that, given any apartment $\mathcal{A}$ of the building, for every sector $Q_x \subset \mathcal{A}$, there exists a unique type-rotating isomorphism $\psi_{tr} : \mathcal{A} \to \mathbb{A}$ mapping Q_x to $\mathbb{Q}_0$.

We say that a sector Q_y is a *subsector* of a sector Q_x if $Q_y \subset Q_x$. Two sectors Q_x and Q_y are said to be *equivalent* if they share a subsector Q_z. Each equivalence class of sectors is called a *boundary point* of the building and it is denoted by ω; the set of all equivalence classes of sectors is called the *maximal boundary* of the building and it is denoted by Ω. As an immediate consequence of definition, for every special vertex x and $\omega \in \Omega$, there is one and only one sector in the class ω, based at x, denoted by $Q_x(\omega)$.

For every special vertex $x \in \mathcal{V}_{sp}(\Delta)$ and every $\omega \in \Omega$, there exists an apartment $\mathcal{A}(x, \omega)$ containing x and ω (in fact containing $Q_x(\omega)$). Analogously, for every chamber c and every $\omega \in \Omega$, there exists an apartment $\mathcal{A}(c, \omega)$ containing c and ω, that is c and a sector in the class ω. On this apartment we denote by $Q_c(\omega)$ the intersection of all sectors in the class ω containing c.

For every $x \in \mathcal{V}_{sp}(\Delta)$ and every chamber $c \in \mathcal{C}(\Delta)$, we define on the maximal boundary Ω the set

$$\Omega(x, c) = \{\omega \in \Omega \ : \ Q_x(\omega) \supset c\}.$$

Analogously, for every pair of special vertices x, y, we can define the set $\Omega(x, y)$ of Ω given by

$$\Omega(x, y) = \{\omega \in \Omega \ : \ y \in Q_x(\omega)\}.$$

We note that , for every x,

$$\Omega(x, c'),\ \Omega(x, z) \supset \Omega(x, c), \quad \text{for every} \quad c',\ z \quad \text{in the convex hull of} \quad \{x, c\},$$
$$\Omega(x, c'),\ \Omega(x, z) \supset \Omega(x, y), \quad \text{for every} \quad c',\ z \quad \text{in the convex hull of} \quad \{x, y\}.$$

From now on we shall limit to consider sectors based at a vertex of $\widehat{\mathcal{V}}(\Delta)$.

3.3. **Retraction ρ^x_ω.** Let $\omega \in \Omega$ and $x \in \widehat{\mathcal{V}}(\Delta)$; for every apartment $\mathcal{A} = \mathcal{A}(x, \omega)$ containing ω and x, there exists a unique type-rotating isomorphism $\psi_{tr} : \mathcal{A} \to \mathbb{A}$, such that $\psi_{tr}(Q_x(\omega)) = \mathbb{Q}_0$. On the other hand, if $\mathcal{A}'$ contains a subsector $Q_y(\omega)$ of $Q_x(\omega)$, but not x, then there exists a type-preserving isomorphism $\phi : \mathcal{A}' \to \mathcal{A}(x, \omega)$ fixing $Q_y(\omega)$; hence it is well defined the type-rotating isomorphism $\psi'_{tr} = \psi_{tr}\ \phi : \mathcal{A}' \to \mathbb{A}$. Since every facet $\mathcal{F}$ of the building lies on an apartment $\mathcal{A}'$ containing a subsector $Q_y(\omega)$ of $Q_x(\omega)$ (possibly $Q_x(\omega)$), then, according to previous notation, $\mathcal{F}$ maps uniquely on the facet $\mathbf{F} = \psi'_{tr}(\mathcal{F})$ of $\mathbb{A}$.

Definition 3.3.1. *We call retraction of* Δ *on* $\mathbb{A}$, *with respect to* ω *and of center* x, *the map*

$$\rho^x_\omega : \Delta \to \mathbb{A},$$

such that , for every apartment $\mathcal{A}'$ *and for every facet* $\mathcal{F} \in \mathcal{A}'$, $\rho^x_\omega(\mathcal{F}) = \mathbf{F} = \psi'_{tr}(\mathcal{F})$.

In particular we remark that $\rho^x_\omega(x) = 0$, and, if we denote by c^x_ω the base chamber of $Q_x(\omega)$, then $\rho^x_\omega(c^x_\omega) = C_0$. Moreover, for every chamber $c \in Q_x(\omega)$, and for every special vertex $y \in Q_x(\omega)$, then

$$\rho^x_\omega(c) = C_0 \cdot \delta(c^x_\omega, c), \quad \text{and} \quad \rho^x_\omega(y) = X_\mu,$$

if X_μ is the special vertex associated with $\mu = \sigma(x, y)$. For ease of notation, we simply set $\rho^x_\omega(z) = \mu$, to mean that $\rho^x_\omega(y) = X_\mu$. In the case $x = e$, we set $\rho_\omega = \rho^e_\omega$.

Proposition 3.3.2. *Let* $x \in \widehat{\mathcal{V}}(\Delta)$, $c \in \mathcal{C}(\Delta)$ *and* $\omega \in \Omega$. *If* $d \subset Q_x(\omega) \cap Q_c(\omega)$, *then* $\delta(x, d)\ \delta(d, c)$ *is independent of* d. *Moreover*

$$\rho^x_\omega(c) = C_0 \cdot \delta(x, d)\ \delta(d, c).$$

Proof. Fix $d \in Q_x(\omega) \cap Q_c(\omega)$; for every $d' \in Q_d(\omega)$, we have

$$\delta(x, d') = \delta(c^x_\omega, d') = \delta(c^x_\omega, d)\ \delta(d, d') \ \text{ and } \ \delta(c, d') = \delta(c, d)\ \delta(d, d'),$$

if c^x_ω is the base chamber of the sector $Q_x(\omega)$. Hence $\delta(c^x_\omega, d')\ \delta(c, d')^{-1} = \delta(c^x_\omega, d)\ \delta(c, d)^{-1}$. Given d_1 and d_2 in $Q_x(\omega) \cap Q_c(\omega)$, and chosen $d' \in Q_{d_1}(\omega) \cap Q_{d_2}(\omega)$, we conclude that

$$\delta(c^x_\omega, d_1)\ \delta(c, d_1)^{-1} = \delta(c^x_\omega, d')\ \delta(c, d')^{-1} = \delta(c^x_\omega, d_2)\ \delta(c, d_2)^{-1}.$$

By definition of ρ^x_ω, we have

$$\rho^x_\omega(d) = \rho^x_\omega(c^x_\omega) \cdot \delta(c^x_\omega, d) = C_0 \cdot \delta(c^x_\omega, d) \quad \text{and} \quad \rho^x_\omega(d) = \rho^x_\omega(c) \cdot \delta(c, d).$$

Actually, since $d \subset Q_x(\omega) \cap Q_c(\omega)$, the retraction of a gallery $\gamma(c^x_\omega, d)$ is a gallery $\Gamma(\rho^x_\omega(c^x_\omega), \rho^x_\omega(d))$ of the same type as $\gamma(c^x_\omega, d)$ and the retraction of a gallery $\gamma(c, d)$ is a gallery $\Gamma(\rho^x_\omega(c), \rho^x_\omega(d))$ of the same type as $\gamma(c, d)$. Therefore

$$\rho^x_\omega(c) = \rho^x_\omega(d) \cdot \delta(c, d)^{-1} = \rho^x_\omega(d) \cdot \delta(d, c) = C_0 \cdot \delta(c^x_\omega, d)\ \delta(d, c).$$

□

An analogous of Proposition 3.3.2 holds for the retraction ρ^x_ω of special vertices of the building.

Proposition 3.3.3. *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$. For every $z \in Q_x(\omega) \cap Q_y(\omega)$, $\sigma(x, z) - \sigma(y, z)$ is independent of z. Moreover*

$$\rho^x_\omega(y) = \sigma(x, z) - \sigma(y, z).$$

Proof. Fix $z \in Q_x(\omega) \cap Q_y(\omega)$ and assume that $\sigma(x, z) = \mu$ and $\sigma(y, z) = \nu$; for every $z' \in Q_z(\omega)$, we have $\sigma(x, z') = \mu + \lambda'$, $\sigma(y, z') = \nu + \lambda'$, if $\sigma(z, z') = \lambda'$; hence $\sigma(x, z') - \sigma(y, z') = \mu - \nu$. Given z_1 and z_2 in $Q_x(\omega) \cap Q_y(\omega)$, and chosen $z' \in Q_{z_1}(\omega) \cap Q_{z_2}(\omega)$, we conclude that

$$\sigma(x, z_1) - \sigma(y, z_1) = \sigma(x, z') - \sigma(y, z') = \sigma(x, z_2) - \sigma(y, z_2).$$

This proves that $\sigma(x, z) - \sigma(y, z)$ does not depend on the choice of z in $Q_x(\omega) \cap Q_y(\omega)$.

In order to prove that $\rho^x_\omega(y) = \sigma(x, z) - \sigma(y, z)$, for every $z \in Q_x(\omega) \cap Q_y(\omega)$, we fix any apartment $\mathcal{A}(x, \omega)$ containing $Q_x(\omega)$. If $y \in \mathcal{A}(x, \omega)$, and $z \in Q_x(\omega) \cap Q_y(\omega)$, then $\rho^x_\omega(x) = 0$, $\rho^x_\omega(z) = \mu$; moreover, if we set $\rho^x_\omega(y) = \eta$, then $\tau_{-\eta}(Q_\eta) = \mathbb{Q}_0$, and in particular $\mu - \eta = \tau_{-\eta}(\rho^x_\omega(z)) = \nu$. If, instead, $y \notin \mathcal{A}(x, \omega)$, there is $y' \in \mathcal{A}(x, \omega)$, such that $\rho^x_\omega(y) = \rho^x_\omega(y')$ and we have $\sigma(y, z) = \sigma(y', z) = \mu - \nu$; hence, as before, $\mu - \eta = \tau_{-\eta}(\rho^x_\omega(z)) = \nu$. □

Corollary 3.3.4. *For all x, y, z in $\widehat{\mathcal{V}}(\Delta)$ and for each $\omega \in \Omega$,*

$$\rho^y_\omega(z) = \rho^x_\omega(z) - \rho^x_\omega(y).$$

Proof. If $z' \in Q_x(\omega) \cap Q_y(\omega) \cap Q_z(\omega)$, then

$$\rho^x_\omega(y) = \sigma(x, z') - \sigma(y, z'), \quad \rho^x_\omega(z) = \sigma(x, z') - \sigma(z, z'), \quad \rho^y_\omega(z) = \sigma(y, z') - \sigma(z, z')$$

and hence

$$\rho^x_\omega(z) - \rho^x_\omega(y) = \sigma(y, z') - \sigma(z, z') = \rho^y_\omega(z).$$

□

We notice that if $z = x$, then $\rho^y_\omega(x) = -\rho^x_\omega(y)$. In particular, for all x, y special and for each $\omega \in \Omega$,

$$\rho^x_\omega(y) = \rho_\omega(y) - \rho_\omega(x).$$

We point out that in fact this formula is independent of the choice of the fundamental vertex e.

We shall prove that, for every $\lambda \in \widehat{L}^+$, it is possible to choose μ large enough with respect to λ, such that Proposition 3.3.3 holds for every $y \in V_\lambda(x)$ and every $\omega \in \Omega$. For every chamber c we denote by $\mathcal{L}(x, c)$ the length of the element $w = \delta(x, c)$, that is the number of hyperplanes separating x and c. On the fundamental apartment $\mathbb{A}$ we define, for every $v \in \mathbb{Q}_0$,

$$\partial(v, \partial\mathbb{Q}_0) = min\{\langle v, \alpha_i\rangle,\ i \in I_0\}.$$

We extend this definition to all special vertices of $Q_x(\omega)$, for any x and ω, in the following way: for each special vertex $y \in Q_x(\omega)$,

$$\partial(y, \partial Q_x(\omega)) = \partial(\rho^x_\omega(y), \partial\mathbb{Q}_0).$$

We define, for $k \in \mathbb{N}$,

$$Q^k_x(\omega) = \{y \in Q_x(\omega)\ :\ \partial(y, \partial Q_x(\omega)) \geq k\}.$$

Lemma 3.3.5. *Let $x \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$; let $k > 0$. Then*

$$Q^k_x(\omega) \subset Q_c(\omega), \tag{3.3.1}$$

for every $c \in \mathcal{C}(\Delta)$ such that $\mathcal{L}(x, c) \leq k$.

Proof. We use induction with respect to k. If $k = 0$, then $x \in c$, and hence $Q_x(\omega) \subset Q_c(\omega)$. Since $\{y \in Q_x(\omega)\ :\ \partial(y, \partial Q_x(\omega)) \geq 0\} = Q_x(\omega)$, we have the required formula. Assume now that (3.3.1) holds for every c such that $\mathcal{L}(x, c) \leq k$; let c_1 such that $\mathcal{L}(x, c_1) = k + 1$. If $\gamma(x, c_1)$ is a gallery joining x to c_1, we denote by d_1 the chamber of this gallery adjacent to c_1; then $\mathcal{L}(x, d_1) = k$ and then

$$\{y \in Q_x(\omega)\ :\ \partial(y, \partial Q_x(\omega)) \geq k\} \subset Q_{d_1}(\omega).$$

Hence, if $Q_{c_1} \supset Q_{d_1}$, the result follows immediately. Otherwise, we have $Q_{c_1} \subset Q_{d_1}$ and for every $y \in (Q_{d_1} \setminus Q_{c_1}) \cap Q_x(\omega)$, we have $\langle \rho^x_\omega(y), \alpha \rangle = k$, for some $\alpha \in R^+$, and $\langle \rho^x_\omega(y), \alpha' \rangle = k \geq k$, for $\alpha' \neq \alpha$. On the other hand,

$$\{y \in Q_x(\omega) : \partial(y, \partial Q_x(\omega)) \geq k+1\} = \{y \in Q_x(\omega) : \partial(y, \partial Q_x(\omega)) \geq k\} \setminus \{y \in Q_x(\omega) : \partial(y, \partial Q_x(\omega)) = k\}$$

and $\{y \in Q_x(\omega) : \partial(y, \partial Q_x(\omega)) = k\}$ is the set of all $y \in Q_x(\omega)$ such that $\langle \rho^x_\omega(y), \alpha \rangle = k$, for some $\alpha \in R^+$, and $\langle \rho^x_\omega(y), \alpha' \rangle = k' \geq k$, for $\alpha' \neq \alpha$. Thus (3.3.1) is true also in this case. □

Let $x \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$; for every $w \in W$, we denote by $Q_w(\omega)$ the intersection of all sectors in the class ω containing the chamber d_w such that $\delta(c_x(\omega), d_w) = w$.

Proposition 3.3.6. *Let $w_1 \in W$; there exists $w_0 \in W$ such that, for every x and c such that $\delta(x, c) = w_1$, and for every $\omega \in \Omega$,*

$$Q_{w_0}(\omega) \subset Q_x(\omega) \cap Q_c(\omega).$$

Moreover, for every chamber d of $Q_{w_0}(\omega)$,

$$\rho^x_\omega(c) = C_0 \cdot \delta(c_x(\omega), d)\delta(d, c).$$

Proof. Let $k > 0$ and $Q_k = \{v \in \mathbb{Q}_0 : \langle v, \alpha_i \rangle \geq k, \ \forall i \in I_0\}$. Choose a chamber $D \subset Q_k$ and let w_k be the element of W such that $D = C_0 \cdot w_k$. For every ω, consider the chamber d_{w_k} such that $\delta(c_x(\omega), d_{w_k}) = w_k$ and the sector $Q_{w_k}(\omega)$. If k is bigger than the length of w_1, that is $\mathcal{L}(x, c) \leq k$, then Lemma 3.3.5 implies that, for every ω, the sector $Q_{w_k}(\omega)$ lies on $Q_x(\omega) \cap Q_c(\omega)$. Therefore $w_0 = w_k$ is the required element of W. Moreover, Proposition 3.3.2 implies that, for every chamber d of $Q_{w_0}(\omega)$,

$$\rho^x_\omega(c) = C_0 \cdot \delta(c_x(\omega), d)\delta(d, c).$$

□

Fix x and ω; for every $\lambda \in \widehat{L}^+$, we denote by z_λ the unique vertex of $Q_x(\omega)$ such that $\sigma(x, z_\lambda) = \lambda$ and by $Q_\lambda(\omega)$ the subsector of $Q_x(\omega)$ of base vertex z_λ. Moreover we denote by k_λ the number of hyperplanes separating 0 and λ.

Proposition 3.3.7. *Let $\lambda \in \widehat{L}^+$; there exists $\mu \in \widehat{L}^+$ (large enough with respect to λ) such that, for every pair $x, y \in V_\lambda(x)$ and for every $\omega \in \Omega$,*

$$Q_\mu(\omega) \subset Q_x(\omega) \cap Q_y(\omega).$$

Moreover, for every ν such that $\nu - \mu \in \widehat{L}^+$,

$$\rho^x_\omega(y) = \mu - \sigma(y, z_\mu) = \nu - \sigma(y, z_\nu).$$

Proof. Let $\lambda \in \widehat{L}^+$; consider $Q_{k_\lambda} = \{v \in \mathbb{Q}_0 : \langle v, \alpha_i \rangle > k_\lambda, \ \forall i \in I_0\}$. Choose a special vertex $\mu \in Q_{k_\lambda}$; for every ω consider the special vertex z_μ of $Q_x(\omega)$ such that $\sigma(x, z_\mu) = \mu$, and the sector $Q_\mu(\omega)$ based at z_μ. By Proposition 3.3.6, for every ω, the sector $Q_\mu(\omega)$ lies on $Q_x(\omega) \cap Q_c(\omega)$; hence, by Proposition 3.3.3, $\rho^x_\omega(y) = \mu - \sigma(y, z_\mu)$. The same is true for every ν such that $\nu - \mu \in \widehat{L}^+$; actually, if $\nu - \mu \in \widehat{L}^+$, we have $z_\nu \in Q_\mu(\omega)$. □

We notice that Proposition 3.3.7 holds if $\langle \mu, \alpha_i \rangle \geq k_\lambda, \ \forall i \in I_0$.

As a consequence of Proposition 3.3.7 we obtain the following result.

Theorem 3.3.8. *Let $y \in V_\lambda(x)$ and $z \in V_\mu(x)$. If μ is large enough with respect to λ, then $\Omega(x, z) \subset \Omega(y, z)$. Moreover, for all $\omega \in \Omega(x, z)$, $\rho^x_\omega(y) = \mu - \nu$, if $\sigma(y, z) = \nu$.*

Proof. If $\omega \in \Omega(x, z)$, then $z \in Q_x(\omega)$ and therefore, if μ is large enough, $z \in Q_y(\omega)$, by Proposition 3.3.7, that is $\omega \in \Omega(y, z)$. The second part of the theorem follows immediately from Proposition 3.3.3. □

Corollary 3.3.9. *Let $y \in V_\lambda(x)$ and $z \in V_\mu(x) \cap V_\nu(y)$. If μ is large enough with respect to λ and ν is large enough with respect to $\lambda^\star$, then $\Omega(x, z) = \Omega(y, z)$.*

Let $y \in V_\lambda(x)$ and $\omega \in \Omega$. We know that $\rho^x_\omega(y) = \lambda$, if $y \in Q_x(\omega)$. The following proposition describes the retraction of the vertices of the set $V_\lambda(x)$.

Proposition 3.3.10. *Let $\omega \in \Omega$ and x special; let $\lambda \in \widehat{L}^+$. For every $y \in V_\lambda(x)$, then $\rho^x_\omega(y) \in \Pi_\lambda$.*

Proof. Let f_λ be the type of a minimal gallery connecting 0 to λ; then each vertex $y \in V_\lambda(x)$ is connected to x by a minimal gallery $\gamma(x, y)$ of type $\sigma_i(f_\lambda)$ (see Section 2.12). This implies that $\rho^x_\omega(\gamma(x, y))$ is a gallery of type f_λ (eventually not reduced) on $\mathbb{A}$ joining 0 to $\mu = \rho^x_\omega(y)$; thus there is a reduced gallery from 0 to μ, of type, say, f'. Let $\lambda' = s_{f'} g_l(0)$; since $\lambda = w_\lambda g_l(0)$ and $s_{f'} \leq w_\lambda$, then $\lambda' \in \Pi_\lambda$. On the other hand, if c and d are the chambers of $\gamma(x, y)$ containing x and y respectively, there exists $\mathbf{w} \in \mathbf{W}$ such that $\rho^x_\omega(c) = \mathbf{w}(C_0)$ and hence $\rho^x_\omega(d) = \mathbf{w}(s_{f'}(C_0))$. This implies that $\mu = \mathbf{w}(\lambda')$ belongs to Π_λ. □

It will be useful to determine how many vertices of $V_\lambda(x)$ are mapped by ρ^x_ω onto an element of Π_λ. We shall prove, using Proposition 2.18.2, that this number actually is independent of x and ω.

Theorem 3.3.11. *Let $x \in V_\lambda(x)$ and $\omega \in \Omega$. For $w, w_1 \in W$, then*

$$|\{c \in \mathcal{C}(\Delta) \ : \ \delta(x,c) = w_1, \ \rho^x_\omega(c) = C_0 \cdot w\}|$$

is independent of x and ω.

Proof. Fix $w_1 \in W$; by Proposition 3.3.6, there exists $w_0 \in W$ such that, for every chamber c such that $\delta(x,c) = w_1$, and for every $\omega \in \Omega$, the set $Q_x(\omega) \cap Q_c(\omega)$ contains a chamber c' such that $\delta(x,c') = w_0$. Moreover, by Proposition 3.3.2, $\rho^x_\omega(c) = C_0 \cdot \delta(c^x_\omega, c') \ \delta(c',c) = C_0 \cdot w_0 \ \delta(c',c)$. Hence, for any $w \in W$,

$$\{c : \delta(x,c) = w_1, \rho^x_\omega(c) = C_0 \cdot w\} = \{c : \delta(x,c) = w_1, \ w_0\delta(c',c) = w\} = \{c : \delta(x,c) = w_1, \delta(c',c) = w_0^{-1} \, w\}.$$

On the other hand, Proposition 2.18.2 implies that $|\{c \ : \ \delta(x,c) = w_1, \ \delta(c',c) = w_0^{-1} \ w\}|$ only depends on $\tau(x)$, and $w_0, w_1, w_0^{-1} \ w$. This proves that $|\{c \in \mathcal{C}(\Delta) \ : \ \delta(x,c) = w_1, \ \rho^x_\omega(c) = C_0 \cdot w\}|$ is independent of x and ω. □

Finally we have

Theorem 3.3.12. *Let $x \in V_\lambda(x)$ and $\omega \in \Omega$. For every $\lambda \in \widehat{L}^+$ and $\mu \in \Pi_\lambda$,*

$$|\{y \in V_\lambda(x) \ : \ \rho^x_\omega(y) = \mu\}|$$

is independent of x and ω.

Proof. Let $\lambda \in \widehat{L}^+$ and $\mu \in \Pi_\lambda$; let $\omega \in \Omega$. Consider the set

$$A = \{y \ : \ \sigma(x,y) = \lambda, \ \rho^x_\omega(y) = \mu\}.$$

For any $y \in V_\lambda(x)$, we denote by c_λ the chamber containing y in a minimal gallery $\gamma(x,y)$. Then $y = v_j(c_\lambda)$, if $\tau(y) = j$, and $\delta(x, c_\lambda) = w_\lambda$. Thus

$$A = \{v_j(c), \ \delta(x,c) = w_\lambda, \ v_j(\rho^x_\omega(c)) = \mu\}.$$

Let W_μ be the stabilizer of μ in W; for every $w \in W_\mu$, consider the set of chambers

$$B_w = \{c \ : \ \delta(x,c) = w_\lambda, \ \rho^x_\omega(c) = C_0 \cdot w\}$$

and $B = \cup_{w \in W_\mu} B_w$. We notice that, if $v_j(\rho^x_\omega(c)) = \mu$, then $\rho^x_\omega(c) = C_0 \cdot w$, for some $w \in W_\mu$. Therefore $A = \{v_j(c), \ c \in B\}$, and then $|A| = |B| = \sum_{w \in W_\mu} |B_w|$. Since Theorem 3.3.11 implies that $|B_w|$ is independent of x and ω, the same is true for $|A|$. □

As a consequence of this theorem, we set, for every $x \in V_\lambda(x)$ and $\omega \in \Omega$

$$N(\lambda,\mu) = |\{y \in V_\lambda(x) \ : \ \rho^x_\omega(y) = \mu\}|. \tag{3.3.2}$$

It will be useful to compare, for every $x \in V_\lambda(x)$ and $\omega \in \Omega$, the retraction ρ^x_ω with the retraction ρ_x with respect to x, defined in Section 2.20.

Lemma 3.3.13. *Let c be any chamber and let y be any special vertex of $\widehat{\mathcal{V}}(\Delta)$.*

(i) If c (respectively y) lies on the sector $Q^-_x(\omega)$ opposite to the sector $Q_x(\omega)$, in any apartment $\mathcal{A}(x,\omega)$, then

$$\rho^x_\omega(c) = \rho_x(c), \quad \textit{(respectively} \quad \rho^x_\omega(y) = \rho_x(y)).$$

(ii) If c (respectively y) belongs to the sector $(Q^\alpha_x)^-(\omega)$, $\alpha-$adjacent to $Q^-_x(\omega)$, in a convenient apartment containing c and $Q_x(\omega)$, then

$$\rho^x_\omega(c) = s_\alpha \rho_x(c), \quad \textit{(respectively} \quad \rho^x_\omega(y) = s_\alpha \rho_x(y)).$$

Proof. First assume $\tau(x) = 0$.

(i) We shall prove that $\rho^x_\omega(c) = \rho_x(c)$, for every chamber c of $Q^-_x(\omega)$. Since c lies on the sector $Q^-_x(\omega)$, then $Q_c(\omega) \supset Q_x(\omega)$, and hence c^x_ω belongs to $Q_c(\omega)$. This implies that

$$\rho^x_\omega(c) = C_0 \cdot \delta(c^x(\omega), c).$$

On the other hand $\delta(c^x(\omega), c) = \delta(c^x(\omega), proj_x(c)) \ \delta(proj_x(c), c) = \mathbf{w}_0 \ \delta(proj_x(c), c)$ and therefore

$$\rho^x_\omega(c) = C_0 \cdot \mathbf{w}_0 \ \delta(proj_x(c), c) = C^-_0 \cdot \delta(proj_x(c), c) = \rho^x(c).$$

If $y \in Q^-_x(\omega)$, we may choose $\gamma(x,y)$ in $Q^-_x(\omega)$; hence, if c is the chamber of $\gamma(x,y)$ containing y, we have $\rho^x_\omega(c) = \rho_x(c)$ and hence $\rho^x_\omega(y) = \rho_x(y)$.

(ii) We shall prove that $\rho^x_\omega(c) = s_\alpha \rho_x(c)$, for every chamber c of $(Q^\alpha_x)^-(\omega)$. Since c lies on the sector $(Q^\alpha_x)^-(\omega)$, then $proj_x(c)$ is the base chamber of the sector $(Q^\alpha_x)^-(\omega)$, that is the opposite of the base chamber $c^\alpha_x(\omega)$ of the sector $(Q^\alpha_x)(\omega)$, which is α-adjacent to $(Q_x)^-(\omega)$. This implies that

$\delta(c^x(\omega), proj_x(c)) = s_\alpha \delta(c^\alpha_x(\omega), proj_x(c)) = s_\alpha \mathbf{w}_0$. From this equality it follows that $\delta(c^x(\omega), c) = \delta(c^x(\omega), proj_x(c))\ \delta(proj_x(c), c) = s_\alpha \mathbf{w}_0\ \delta(proj_x(c), c)$, and then

$$\rho^x_\omega(c) = C_0 \cdot s_\alpha \mathbf{w}_0\ \delta(proj_x(c), c) = s_\alpha (C_0 \cdot \mathbf{w}_0 \delta(proj_x(c), c) = s_\alpha \rho^x(c).$$

If $y \in (Q^\alpha_x)^-_x(\omega)$, we may choose $\gamma(x, y)$ in $(Q^\alpha_x)^-_x(\omega)$; hence, if c is the chamber of $\gamma(x, y)$ containing y, we have $\rho^x_\omega(c) = s_\alpha \rho_x(c)$ and hence $\rho^x_\omega(y) = s_\alpha \rho_x(y)$.

If $\tau(x) = i \neq 0$, we only have to change δ with δ_i and the proof is the same. $\square$

3.4. **Topologies on the maximal boundary.** The maximal boundary Ω may be endowed with a totally disconnected compact Hausdorff topology in the following way. Fix a special vertex $x \in \widehat{\mathcal{V}}(\Delta)$, say of type $i = \tau(x)$; consider the family

$$\mathcal{B}_x = \{\ \Omega(x, c),\ c \in \mathcal{C}\}.$$

Then $\mathcal{B}_x$ generates a totally disconnected compact Hausdorff topology on Ω; for every $\omega \in \Omega$, a local base at ω is given by

$$\mathcal{B}_{x,\omega} = \{\ \Omega(x, c),\ c \subset Q_x(\omega)\}.$$

We observe that it suffices to consider, as a local base at ω, only the chambers c lying on $Q_x(\omega)$, such that, for some $\lambda \in \widehat{L}^+$, $\delta(c_x(\omega), c) = \sigma_i(t_\lambda)$, if $c_x(\omega)$ is the base chamber of the sector $Q_x(\omega)$, and $i = \tau(x)$.

Remark 3.4.1. *For every special vertex $y \in \widehat{\mathcal{V}}(\Delta)$, let $\lambda = \sigma(x, y)$; we denote by $\mathcal{C}_y$ the set of all chambers containing y such that $\delta(x, c) = \sigma_i(t_\lambda)$, that is the set of all chambers containing y and opposite to the chamber containing y in a minimal gallery connecting x and y. It is easy to check that*

$$\Omega(x, y) = \bigcup_{c \in \mathcal{C}_y} \Omega(x, c).$$

Moreover, for every chamber c choose $\overline{y} \in \widehat{\mathcal{V}}(\Delta)$ such that c lies on $[x, \overline{y}]$ and let $\lambda = \sigma(x, \overline{y})$. Then

$$\Omega(x, c) = \bigcup_{y \in V_\lambda(x), c \subset [x,y]} \Omega(x, y).$$

Hence the family $\widetilde{\mathcal{B}}_x = \{\ \Omega(x, y),\ y \in \mathcal{V}\}$ generates the same topology on Ω as $\mathcal{B}_x$ and, for every $\omega \in \Omega$, a local base at ω is given by $\widetilde{\mathcal{B}}_{x,\omega} = \{\ \Omega(x, y),\ y \subset Q_x(\omega)\}$.

Proposition 3.4.2. *The topology on Ω does not depend on the particular $x \in \widehat{\mathcal{V}}(\Delta)$.*

Proof. Let x, y special vertices and $\lambda = \sigma(x, y)$. Let $\omega_0 \in \Omega$. We prove that, for every neighborhood $\Omega(y, z)$ of ω_0, there exists a neighborhood $\Omega(x, z')$ of ω_0, such that $\Omega(x, z') \subset \Omega(y, z)$. Actually, if z' is a vertex of $Q_x(\omega_0) \cap Q_y(\omega_0)$, such that $z \in [y, z']$, then $\omega_0 \in \Omega(y, z') \cap \Omega(x, z')$ and $\Omega(y, z') \subset \Omega(y, z)$. On the other hand, if $\sigma(x, z') = \mu$, then, by Theorem 3.3.8, we can choose μ large enough with respect to λ, so that $\Omega(x, z') \subset \Omega(y, z')$. $\square$

3.5. **Probability measures on the maximal boundary.** For each vertex x of $\widehat{\mathcal{V}}(\Delta)$, we denote by ν_x the regular Borel probability measure on Ω, such that, for every $y \in \widehat{\mathcal{V}}(\Delta)$,

$$\nu_x(\Omega(x, y)) = N_\lambda^{-1} = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})} \prod_{\alpha \in R^+} q_\alpha^{-\langle \lambda, \alpha \rangle}\ q_{2\alpha}^{\langle \lambda, \alpha \rangle}, \quad \text{if} \quad y \in V_\lambda(x).$$

We notice that in fact there exists a unique regular Borel probability measure on Ω, satisfying this property; actually ν_x is the measure such that, for every $f \in \mathcal{C}(\Omega)$,

$$J(f) = \int_\Omega f(\omega)\ d\nu_x(\omega),$$

where J denotes the linear functional on $\mathcal{C}(\Omega)$ obtained as extension of the linear functional on the space of all locally constant functions on Ω, defined as

$$J(f) = N_\lambda^{-1} \sum_{\sigma(x,y)=\lambda} f_y,$$

if, for each $y \in V_\lambda(x)$, we set $f_y = f(\omega),\ \forall \omega \in \Omega(x, y)$.

The following property of the measure ν_x is a consequence of Theorem 3.3.6 and Theorem 3.3.11.

Theorem 3.5.1. *Let $x \in \widehat{\mathcal{V}}(\Delta)$ and $w, w_0 \in W$. For each $c \in \mathcal{C}(\Delta)$, such that $\delta(x, c) = w_0$,*

$$\nu_x(\{\omega \in \Omega\ :\ \rho^x_\omega(c) = C_0 \cdot w\})$$

is independent of x and c.

Proof. Fix $w_0 \in W$ and a chamber c such that $\delta(x,c) = w_0$; by Proposition 3.3.6, there exists $w_1 \in W$ such that, for every ω, $Q_{w_1}(\omega) \subset Q_x(\omega) \cap Q_c(\omega)$; moreover $\rho^x_\omega(c) = C_0 \cdot \delta(x,d)\delta(d,c)$, if d is any chamber of $Q_{w_1}(\omega)$. In particular,

$$\rho^x_\omega(c) = C_0 \cdot w_1 \delta(d_{w_1}(\omega), c),$$

if $d_{w_1}(\omega)$ denotes the chamber of $Q_{w_1}(\omega)$ such that $\delta(x, d_{w_1}(\omega)) = w_1$. Therefore, for any $w \in W$, we have $\rho^x_\omega(c) = C_0 \cdot w$ if and only if $w = w_1\delta(d_{w_1}(\omega), c)$, that is if and only if $\delta(c, d_{w_1}(\omega)) = w^{-1}w_1$. Hence, by setting $w^{-1}w_1 = w_2$ and $\mathcal{C}(w_1,w_2) = \{c' \,:\, \delta(x,c') = w_1,\ \delta(c,c') = w_2\}$, we have

$$\{\omega \in \Omega \,:\, \rho^x_\omega(c) = C_0 \cdot w\} = \bigcup_{c' \in \mathcal{C}(w_1,w_2)} \Omega(x,c').$$

This implies that

$$\nu_x(\{\omega \in \Omega \,:\, \rho^x_\omega(c) = C_0 \cdot w\}) = \sum_{c' \in \mathcal{C}(w_1,w_2)} \nu_x(\Omega(x,c')).$$

On the other hand, $\nu_x(\Omega(x,c'))$ has the same value for each chamber c' such that $\delta(x,c') = w_1$; therefore, by fixing any chamber c' such that $\delta(x,c') = w_1$,

$$\nu_x(\{\omega \in \Omega \,:\, \rho^x_\omega(c) = C_0 \cdot w\}) = \nu_x(\Omega(x,c'))\, |\{c' \in \mathcal{C}(\Delta) \,:\, \delta(x,c') = w_1,\ \delta(c,c') = w_2\}|.$$

Thus Theorem 3.3.11 implies that $\nu_x(\{\omega \in \Omega \,:\, \rho^x_\omega(c) = C_0 \cdot w\})$ is independent of the choice of x and c, but only depends on w, w_0. $\square$

A version of this theorem holds for the set of vertices.

Theorem 3.5.2. *Let x be a special vertex of $\widehat{\mathcal{V}}(\Delta)$, let $\lambda \in \widehat{L}^+$ and $\mu \in \Pi_\lambda$. For each $y \in \widehat{\mathcal{V}}(\Delta)$, such that $\sigma(x,y) = \lambda$,*

$$\nu_x(\{\omega \in \Omega \,:\, \rho^x_\omega(y) = \mu\})$$

is independent of x and y.

Proof. Fix $y \in \widehat{\mathcal{V}}(\Delta)$ such that $\sigma(x,y) = \lambda$, and consider, for every $\mu \in \Pi_\lambda$, the set

$$\Omega_\mu = \{\omega \in \Omega \,:\, \rho^x_\omega(y) = \mu\}.$$

If $\tau(x) = i,\ \tau(y) = j$, then $\tau(X_\lambda) = l = \sigma_i^{-1}(j)$. Therefore

$$\Omega_\mu = \{\omega \in \Omega \,:\, v_l(\rho^x_\omega(c_\lambda)) = \mu\},$$

if c_λ denotes, as usual, the chamber containing the vertex y in a minimal gallery connecting x and y. Therefore, $\Omega_\mu = \{\omega \in \Omega \,:\, \rho^x_\omega(y) = C_0 \cdot w,\ w \in W_\mu\} = \bigcup_{w \in W_\mu} \{\omega \in \Omega \,:\, \rho^x_\omega(y) = C_0 \cdot w\}$, if W_μ is the stabilizer of μ in W. Thus Theorem 3.5.1 ends the proof. $\square$

4. The α-boundary Ω_α

4.1. **Walls.** Let Δ be an affine building and let R be its root system. Consider on the fundamental apartment $\mathbb{A} = \mathbb{A}(R)$ the fundamental sector $\mathbb{Q}_0 = Q_0(C_0)$. It is straightforward to call walls of $\mathbb{Q}_0$ the walls of C_0 containing 0 (see Section 2.10). Actually, we slightly change this definition and we shall call *wall* of $\mathbb{Q}_0$ the intersection with $\overline{\mathbb{Q}_0}$ of any hyperplane $H_i = H_{\alpha_i},\ i \in I_0$. Moreover, we say that a wall of $\mathbb{Q}_0$ is the *i-type wall* of $\mathbb{Q}_0$, for each $i \in I_0$, if it lies on H_i. This is the case if and only if it contains the co-type i panel of C_0. For every $i \in I_0$, we denote by $H_{0,i}$ the i-type wall of $\mathbb{Q}_0$.

We extend this definition to each sector of $\mathbb{A}$ by declaring that, for every special vertex X_λ in $\mathbb{A}$, and for every chamber C sharing X_λ, the walls of the sector $Q_\lambda(C)$ based at X_λ are the intersection with $\overline{Q_\lambda(C)}$ of any affine hyperplane $H^k_\alpha,\ \alpha \in R^+,\ k \in \mathbb{Z}$, which is a wall of the chamber C. Moreover we say that a wall of $Q_\lambda(C)$ has type i, for some $i \in I_0$, if there is a type-preserving isomorphism on $\mathbb{A}$ mapping the wall on an affine hyperplane $H^k_i = H^k_{\alpha_i}$, for some $i \in I_0$ and $k \in \mathbb{Z}$.

The definition of wall can be extended to each sector of the building; actually, if $Q_x(c)$ is any sector of Δ, and $\mathcal{A}$ is any apartment of the building containing $Q_x(c)$, then the *walls* of $Q_x(c)$ are the inverse images of the walls of the sector $Q_\lambda(C) = \psi_{tp}(Q_x(c))$, under a type-preserving isomorphism $\psi_{tp} : \mathcal{A} \to \mathbb{A}$. Moreover, for every $i \in I_0$, a wall of $Q_x(c)$ has type i, if its image in $\mathbb{A}$ has type i. The previous definition does not depend on the choice of the apartment $\mathcal{A}$ containing the sector and of the type-preserving isomorphism $\psi_{tp} : \mathcal{A} \to \mathbb{A}$. For every sector $Q_x(c)$ and for every $i \in I_0$, we denote by $h_{x,i}(c) = h_{x,i}(Q_x(c))$ the type i wall of the sector. If ω is any element of the maximal boundary Ω, then, for every $x \in \mathcal{V}_{sp}(\Delta)$ and for every $i \in I_0$, we simply denote by $h_{x,i}(\omega)$ the wall of type i of the sector $Q_x(\omega)$. If α is a simple root, that is $\alpha = \alpha_i$, for some $i \in I_0$, for every special vertex x of Δ, and for every $\omega \in \Omega$, we shall denote by $h_{x,\alpha}(\omega)$ the wall of $Q_x(\omega)$ of type i and we simply call it the α-wall of $Q_x(\omega)$. In general, for every simple root α, we shall denote by $h_{x,\alpha}$ the α-wall of any sector based at x.

Definition 4.1.1. *Let $x, y \in \mathcal{V}_{sp}(\Delta)$, $x \neq y$; let $h_{x,\alpha}$ and $h_{y,\alpha}$ be α-walls, based at x and y respectively.*

- *(i) The walls $h_{x,\alpha}$ and $h_{y,\alpha}$ are said to be equivalent if they definitely coincide, i.e. there is $h_{z,\alpha}$ such that $h_{z,\alpha} \subset h_{x,\alpha} \cap h_{y,\alpha}$.*
- *(ii) The walls $h_{x,\alpha}$ and $h_{y,\alpha}$ are said to be parallel if they are not equivalent , but there is an apartment containing them and, through any type-preserving isomorphism ψ_{tp} of this apartment onto $\mathbb{A}$, they correspond to walls of $\mathbb{A}$, lying on parallel affine α-hyperplanes H^k_α, H^j_α, for some $k, j \in \mathbb{Z}$.*
- *(iii) The walls $h_{x,\alpha}$ and $h_{y,\alpha}$ are said to be definitely parallel if there exist $h_{x',\alpha} \subset h_{x,\alpha}$ and $h_{y',\alpha} \subset h_{y,\alpha}$ which are parallel. If $h_{x,\alpha}$ and $h_{y,\alpha}$ are definitely parallel, we call distance between the two walls the usual distance between the two hyperplanes of $\mathbb{A}$, containing the images of their parallel subwalls, that is the positive integer number $|j-k|$, if $\psi_{tp}(h_{x,\alpha}) = H^k_\alpha$ and $\psi_{tr}(h_{y,\alpha}) = H^j_\alpha$.*

We remark that if $h_{x,\alpha}$ and $h_{y,\alpha}$ are definitely parallel, there exists an apartment containing, say, $h_{x,\alpha}$ and a subwall of $h_{y,\alpha}$.

Proposition 4.1.2. *For every $\omega \in \Omega$ and for every pair of special vertices $x, y \in \mathcal{V}_{sp}(\Delta)$, the walls $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$ are equivalent or definitely parallel.*

Proof. Fix $\omega \in \Omega$, $x \neq y$ in $\mathcal{V}_{sp}(\Delta)$ and consider the α-walls $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$. Assume that $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$ are not equivalent and prove that they are definitely parallel. We point out that, if there exists an apartment $\mathcal{A}$ containing $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$, then the two walls are parallel. Actually, if ω' denotes a boundary point α-equivalent to ω and lying onto the apartment $\mathcal{A}$, then $\rho^x_{\omega'}$ is a type-rotating isomorphism from $\mathcal{A}$ onto $\mathbb{A}$, such that $\rho^x_{\omega'}(h_{x,\alpha}(\omega))$ lies on H_α and $\rho^x_{\omega'}(h_{y,\alpha}(\omega))$ lies on H^k_α, for some $k \in \mathbb{Z}$. Hence, in order to prove that $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$ are definitely parallel, we only have to prove that there exists an apartment $\mathcal{A}$ containing subwalls $h_{x',\alpha}(\omega) \subset h_{x,\alpha}(\omega)$ and $h_{y',\alpha}(\omega) \subset h_{y,\alpha}(\omega)$. To this end, we shall use induction with respect to the distance between x and y.

We consider at first the case when $\mathcal{V}_{sp}(\Delta)$ contains vertices of different types. This happens for every building of type different from $\widetilde{G}_2$. If $d(x, y) = 1$, the vertices x and y are adjacent; then there exists a chamber c such that $x, y \in c$; if $\mathcal{A}$ is an apartment containing ω and c, we have $Q_x(\omega), Q_y(\omega) \subset \mathcal{A}$. Thus $h^x_\alpha(\omega), h^y_\alpha(\omega)$ lie on $\mathcal{A}$. Moreover the distance between $h^x_\alpha(\omega)$ and $h^y_\alpha(\omega)$ is zero or one. Now assume that, when $d(x, y) \leq n$, then $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$ have subwalls $h_{x',\alpha}(\omega)$ and $h_{y',\alpha}(\omega)$ lying on an apartment; hence $h_{x',\alpha}(\omega)$ and $h_{y',\alpha}(\omega)$ are parallel and their distance is less than or equal to n. Actually we may assume, without loss of generality, that $d(x', y') \leq n$. Let $d(x, y) = n + 1$ and choose z such that $d(y, z) = 1$ and $d(x, z) = n$. By inductive hypothesis, there exist x', z', with $d(x', z') = n$, such that the subwalls $h_{x',\alpha}(\omega) \subset h_{x,\alpha}(\omega)$ and $h_{z',\alpha}(\omega) \subset h_{z,\alpha}(\omega)$ lie on an apartment $\mathcal{A}_1$ and are parallel, at distance less than or equal to n. Without loss of generality, we may assume, for ease of notation, that $x' = x$ and $z' = z$. Moreover, if c is a chamber such that $y, z \in c$, then there exists an apartment $\mathcal{A}_2$, containing $h_{y,\alpha}(\omega), h_{z,\alpha}(\omega)$ and c. We shall prove that there exists an apartment $\mathcal{A}$ containing $h_{x,\alpha}(\omega), h_{z,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$. If $h_{y,\alpha}(\omega)$ lies on $\mathcal{A}_1$, then $\mathcal{A}_2 = \mathcal{A}_1$, and the required apartment is $\mathcal{A}_1$ and, on this apartment, the distance of the parallel hyperplanes $h_{x,\alpha}(\omega), h_{y,\alpha}(\omega)$ is less than or equal to n. If, on the contrary, $h_{y,\alpha}(\omega)$ does not lie on $\mathcal{A}_1$, we consider two isomorphisms $\psi_1 : \mathcal{A}_1 \to \mathbb{A}$ and $\psi_2 : \mathcal{A}_2 \to \mathbb{A}$ such that $\psi_1(h_{z,\alpha}(\omega)) = \psi_2(h_{z,\alpha}(\omega)) = H_{0,\alpha}$; then,

$$\psi_1(h_{x,\alpha}(\omega)) = H_{h,\alpha}, \quad \psi_2(h_{y,\alpha}(\omega)) = H_{k,\alpha},$$

for some $h, k \in \mathbb{Z}$. When $hk < 0$, then $H_{h,\alpha}$ and $H_{k,\alpha}$ lie on distinct half-apartments $\mathbb{A}^+_{0,\alpha}, \mathbb{A}^-_{0,\alpha}$, say $H_{h,\alpha} \subset \mathbb{A}^+_{0,\alpha}$ and $H_{k,\alpha} \subset \mathbb{A}^-_{0,\alpha}$; in this case consider the apartment $\mathcal{A} = \psi^{-1}(\mathbb{A})$, if $\psi = \psi_1$ on $\mathbb{A}^+_{0,\alpha}$ and $\psi = \psi_2$ on $\mathbb{A}^-_{0,\alpha}$. On the contrary, when $hk > 0$, then $H_{h,\alpha}$ and $H_{k,\alpha}$ lie on a same half-apartment $\mathbb{A}^+_{0,\alpha}$ or $\mathbb{A}^-_{0,\alpha}$, say $H_{h,\alpha}, H_{k,\alpha} \subset \mathbb{A}^+_{0,\alpha}$; in this case consider the apartment $\mathcal{A} = \psi^{-1}(\mathbb{A})$, if $\psi = \psi_1$ on $\mathbb{A}^+_{0,\alpha}$ and $\psi = \psi_2 s_\alpha$ on $\mathbb{A}^-_{0,\alpha}$. In both cases $\mathcal{A}$ is the required apartment, containing $h_{x,\alpha}(\omega), h_{z,\alpha}(\omega)$ and $h^y_\alpha(\omega)$.

Assume now that Δ has type $\widetilde{G}_2$. In this case, all special vertices have type 0 and we can not choose x, y adjacent. However, if we choose as x and y the vertices of type 0 of two adjacent chambers c, c', it is a consequence of the geometry of the building that the walls $h_{x,\alpha}(\omega), h_{y,\alpha}(\omega)$ are definitely parallel and have distance 0 or 1. Hence we can use the same inductive argument as before, to conclude. □

We point out that if Δ has type $\widetilde{C}_n$ or $\widetilde{BC}_n$, a wall of type n of any sector of the building contains special vertices of only one type, that is only of type 0, or only of type n. (The same is true for a wall of type $i, i < n$, of a building of type $\widetilde{B}_n$).

From now on we shall limit to consider walls based at special vertices of the set $\widehat{\mathcal{V}}(\Delta)$.

4.2. **The α-boundary Ω_α.** Let α be a simple root, that is $\alpha = \alpha_i$, for some $i \in I_0$; for every special vertex x of $\widehat{\mathcal{V}}(\Delta)$, and for every $\omega \in \Omega$, we consider the α-wall $h_{x,\alpha}(\omega)$ of $Q_x(\omega)$.

Lemma 4.2.1. *Let $\omega_1, \omega_2 \in \Omega$. If there exists a vertex $x \in \widehat{\mathcal{V}}(\Delta)$ such that $h_{x,\alpha}(\omega_1) = h_{x,\alpha}(\omega_2)$, then $h_{y,\alpha}(\omega_1) = h_{y,\alpha}(\omega_2)$, for every $y \in \widehat{\mathcal{V}}(\Delta)$.*

Proof. (i) At first assume that there exists an apartment $\mathcal{A}$ containing $Q_x(\omega_1)$ and $Q_x(\omega_2)$. Since $h_{x,\alpha}(\omega_1) = h_{x,\alpha}(\omega_2)$, there exists a type-rotating isomorphism $\psi_{tr} : \mathcal{A} \to \mathbb{A}$, mapping $Q_x(\omega_1)$ onto $\mathbb{Q}_0$ and $Q_x(\omega_2)$ onto $s_\alpha \mathbb{Q}_0$. Hence the same property holds for each $y \in \mathcal{A}$. This proves that $h_{y,\alpha}(\omega_1) = h_{y,\alpha}(\omega_2)$, for every $y \in \mathcal{A}$. On the other hand, if $y \notin \mathcal{A}$, the sectors $Q_y(\omega_1)$ and $Q_y(\omega_2)$ do not lie on $\mathcal{A}$, but there exists $z \in \mathcal{A}$, such that $Q_z(\omega_1) \subset Q_y(\omega_1)$, $Q_z(\omega_2) \subset Q_y(\omega_2)$ and $h_{z,\alpha}(\omega_1) = h_{z,\alpha}(\omega_2)$. Hence $Q_y(\omega_1) \cap Q_y(\omega_2)$ contains $h_{z,\alpha}(\omega_1) = h_{z,\alpha}(\omega_2)$, besides y. This implies that $Q_y(\omega_1) \cap Q_y(\omega_2)$ contains the convex hull of y and $h_{z,\alpha}(\omega_1) = h_{z,\alpha}(\omega_2)$, which includes the wall of type α of the two sectors; thus $h_{y,\alpha}(\omega_1) = h_{y,\alpha}(\omega_2)$.

(ii) If there is none apartment containing $Q_x(\omega_1)$ and $Q_x(\omega_2)$, then there exists a vertex z such that $Q_z(\omega_1) \subset Q_x(\omega_1)$ and $Q_z(\omega_2) \subset Q_x(\omega_2)$, and $Q_z(\omega_1)$ and $Q_z(\omega_2)$ lie on some apartment $\mathcal{A}$; moreover $h_{z,\alpha}(\omega_1) = h_{z,\alpha}(\omega_2)$. Hence, using the same argument as in (i), we complete the proof. □

Definition 4.2.2. *Let $\omega, \omega' \in \Omega$. We say that ω is α-equivalent to ω', and we write $\omega \sim_\alpha \omega'$, if, for some x, $h_{\alpha,x}(\omega) = h_{\alpha,x}(\omega')$.*

Lemma 4.2.1 implies that the definition of α-equivalence does not depend on the vertex x such that $h_{\alpha,x}(\omega) = h_{\alpha,x}(\omega')$. Moreover, if ω is α-equivalent to ω', and $\mathcal{A} = \mathcal{A}(\omega, \omega')$ denotes any apartment having ω and ω' as boundary points, then for every $x \in \mathcal{A}$, the sectors $Q_x(\omega)$ and $Q_x(\omega')$ are α-adjacent, that is there exists a type rotating isomorphism $\psi_{tr} : \mathcal{A} \to \mathbb{A}$, mapping $Q_x(\omega)$ onto $\mathbb{Q}_0$ and $Q_x(\omega')$ onto $s_\alpha \mathbb{Q}_0$. On the contrary, if x does not lie on any $\mathcal{A}(\omega, \omega')$, then $Q_x(\omega) \cap Q_x(\omega')$ contains properly their common α-wall.

Definition 4.2.3. *We call α-boundary of the building Δ the set $\Omega_\alpha = \Omega/{\sim_\alpha}$, consisting of all equivalence classes $[\omega]_\alpha$ of boundary points and we denote by η_α any element of Ω_α. Hence $\eta_\alpha = [\omega]_\alpha$, if ω belongs to the equivalence class η_α.*

Fix $\omega \in \Omega$ and consider the set $\mathcal{H}_\alpha(\omega) = \{h_{x,\alpha}(\omega),\ x \in \widehat{\mathcal{V}}(\Delta)\}$. If $\omega' \sim_\alpha \omega$ then, for every x, $h_{x,\alpha}(\omega') = h_{x,\alpha}(\omega)$ and hence $\mathcal{H}_\alpha(\omega) = \mathcal{H}_\alpha(\omega')$. Therefore the set $\mathcal{H}_\alpha(\omega)$ only depends on the equivalence class $\eta_\alpha = [\omega]_\alpha$ represented by ω and we shall denote $\mathcal{H}_\alpha(\eta_\alpha) = \mathcal{H}_\alpha(\omega)$, if $\omega \in \eta_\alpha$. Moreover, if $\omega \not\sim_\alpha \omega'$, then, for every $x \in \widehat{\mathcal{V}}(\Delta)$, $h_{x,\alpha}(\omega) \neq h_{x,\alpha}(\omega')$ and hence $\mathcal{H}_\alpha(\omega) \cap \mathcal{H}_\alpha(\omega') = \emptyset$. This implies that the map

$$\eta_\alpha \to \mathcal{H}_\alpha(\eta_\alpha)$$

is a bijection between the α-boundary Ω_α and the set $\{\mathcal{H}_\alpha(\eta_\alpha)\}$. In particular, for every $x \in \widehat{\mathcal{V}}(\Delta)$, each element η_α of Ω_α determines one α-wall based at x; we shall denote this wall by $h_x(\eta_\alpha)$. Of course, $h_x(\eta_\alpha) = h_{x,\alpha}(\omega)$, for every $\omega \in \eta_\alpha$.

4.3. **Trees at infinity.** Let us consider the α-boundary Ω_α, corresponding to a simple root α of the building. We claim that it is possible to construct a graph associated to each element η_α of Ω_α, and this graph is in fact a tree, whose boundary can be canonically identified with the set of all ω belonging to the class η_α. To this end, we shall examine in details, for any class η_α, the set $\mathcal{H}_\alpha(\eta_\alpha)$ and we prove that the set $\mathcal{H}_\alpha(\eta_\alpha)$ determines a tree. Proposition 4.1.2 implies the following corollary.

Corollary 4.3.1. *For every $\eta_\alpha \in \Omega_\alpha$, the set $\mathcal{H}_\alpha(\eta_\alpha)$ consists of walls equivalent or definitely parallel.*

Let η_α be a fixed element of Ω_α; for every $x \in \widehat{\mathcal{V}}(\Delta)$ consider the wall $h_x(\eta_\alpha)$ of $\mathcal{H}_\alpha(\eta_\alpha)$ and the class of all walls $h_{x'}(\eta_\alpha)$, equivalent to $h_x(\eta_\alpha)$, according to Definition 4.1.1, (i). We simply denote by $\mathbf{x}$ this equivalence class, represented by the wall $h_x(\eta_\alpha)$. Obviously, $\mathbf{x} = \mathbf{y}$ if and only if $h_x(\eta_\alpha)$ and $h_y(\eta_\alpha)$ are equivalent.

Remark 4.3.2. *Consider, on the fundamental apartment $\mathbb{A}$, the α-wall of any sector Q_X equivalent to $\mathbb{Q}_0$. Each of these walls lies on an affine hyperplane H_α^k, for some $k \in \mathbb{Z}$. For every $k \in \mathbb{Z}$, we simply denote by $\mathbf{X}_k$ the class of all walls lying on H_α^k, and we set*

$$\Gamma_0 = \{\mathbf{X}_k,\ k \in \mathbb{Z}\}.$$

For every apartment $\mathcal{A}$ of the building we consider, for any η_α, the walls of $\mathcal{H}_\alpha(\eta_\alpha)$ lying on $\mathcal{A}$, and the equivalence classes $\mathbf{x}$ represented by these walls. By a type-preserving isomorphism $\psi_{tp} : \mathcal{A} \to \mathbb{A}$, each $\mathbf{x}$ maps to an element $\mathbf{X}_k$, of Γ_0, for some $k \in \mathbb{Z}$.

We recall that if the root system R has type C_n or BC_n, and $\alpha = \alpha_n$, then, for every $j \in \mathbb{Z}$, H_α^{2j} only contains special vertices of type 0 and H_α^{2j+1} only contains special vertices of type n. (The same is true if R has type B_n and $\alpha = \alpha_i, i < n$). Hence in this case it is natural to endow the set Γ_0 with a *labelling in the following way: we say that* $\mathbf{X}_k$ *has type 0, if* $k = 2j$ *and has type 1, if* $k = 2j+1$, *for* $j \in \mathbb{Z}$. *This labelling can be extended to all equivalence classes* $\mathbf{x}$ *represented by walls of* $\mathcal{H}_\alpha(\eta_\alpha)$ *lying on any apartment* $\mathcal{A}$, *and hence to all walls of the building; we say that* $\mathbf{x}$ *has type 0 if (through any type-preserving isomorphism) it maps to some* $\mathbf{X}_{2j}$, *and has type 1, if it maps to some* $\mathbf{X}_{2j+1}$.

Definition 4.3.3. *Let* $\eta_\alpha \in \Omega_\alpha$. *We denote by* $T_\alpha(\eta_\alpha)$ *the graph having as vertices the classes* $\mathbf{x}$ *of equivalent walls associated to* η_α, *and as edges the pairs* $[\mathbf{x}, \mathbf{y}]$ *of equivalence classes represented by (definitely parallel) walls* $h_x(\eta_\alpha)$ *and* $h_y(\eta_\alpha)$ *at distance one.*

For every $\omega \in \eta_\alpha$, we can then associate to ω the graph $T_\alpha(\omega) = T_\alpha(\eta_\alpha)$ and, for every $\omega \in \Omega$, we can associate to ω the graph of the element η_α of the α-boundary, represented by ω.

We recall that, according to notation of Section 2.16, the simple root α belongs to R_2 if and only if R is not reduced and $\alpha = \alpha_n = e_n$. In this particular case, for every $k \in \mathbb{Z}$, we have $H^k_\alpha = H^{2k}_{2\alpha}$; hence the parallel hyperplanes of $\mathbb{A}$, orthogonal to α are the hyperplanes $H^h_{2\alpha}$, for all $h \in \mathbb{Z}$. Moreover, for every $k \in \mathbb{Z}$,

$$q_{2\alpha,2k} = q_{\alpha,k} = q_\alpha = r, \qquad q_{2\alpha,2k+1} = q_{2\alpha} = p.$$

In all other cases, that is for all simple root of a reduced building or for all simple root $\alpha_i, i \neq n$, for a building of type $\widetilde{BC_n}$, we always have $\alpha \in R_0$, and hence

$$q_{\alpha,k} = q_\alpha, \quad \text{for every} \quad k \in \mathbb{Z}.$$

Proposition 4.3.4. *For every simple root* α, *and for every* $\eta_\alpha \in \Omega_\alpha$, *the graph* $T_\alpha(\eta_\alpha)$ *is a tree.*

(i) If $\alpha \in R_0$, *the tree is homogeneous, with homogeneity* q_α.

(ii) If $\alpha \in R_2$, *the tree is labelled and semi-homogeneous; each vertex of type 0 shares* $q_{2\alpha} = p$ *edges and each vertex of type 1 shares* $q_\alpha = r$ *edges.*

Proof. We have to prove that $T_\alpha(\eta_\alpha)$ is connected and has no loops.

Let $\mathbf{x}$, $\mathbf{y}$ be two vertices of the graph. If $\omega \in \eta_\alpha$ and $h_{x,\alpha}(\omega)$, $h_{y,\alpha}(\omega)$ are representatives of $\mathbf{x}$ and $\mathbf{y}$ respectively, we may assume, without loss of generality, that the two walls are parallel, and hence they lie on an apartment $\mathcal{A}$. Let n be the distance between the two walls on this apartment. We can choose $x_0, x_1, \dots, x_n$ on $\mathcal{A}$, such that $x_0 \in h_{x,\alpha}(\omega)$, $x_n \in h_{y,\alpha}(\omega)$ and $d(x_{i-1}, x_i) = 1$, for every $i = 1, \dots, n$. The walls $h_{x_0,\alpha}(\omega), h_{x_1,\alpha}(\omega), \dots, h_{x_n,\alpha}(\omega)$ are pairwise adjacent on $\mathcal{A}$ and

$$h_{x_0,\alpha}(\omega) \sim h_{x,\alpha}(\omega), \quad h_{x_n,\alpha}(\omega) \sim h_{y,\alpha}(\omega).$$

Therefore, if $\mathbf{x}_i$ is the vertex of the graph represented by $h_{x_i,\alpha}(\omega)$, for $i = 0, \dots, n$, then $d(\mathbf{x}_{i-1}, \mathbf{x}_i) = 1$, for $i = 0, \dots, n$ and $\mathbf{x} = \mathbf{x}_0$, $\mathbf{y} = \mathbf{x}_n$. This proves that $\mathbf{x}$, $\mathbf{y}$ are connected by a path of length n.

For every $n \geq 2$, let us consider on the graph a path $\mathbf{x}_0, \dots, \mathbf{x}_n$, such that $\mathbf{x}_{i-1} \neq \mathbf{x}_i, \mathbf{x}_{i+1}$, for $i = 1, \dots, n-1$. We shall prove by induction that $\mathbf{x}_0 \neq \mathbf{x}_n$. If $n = 2$, the property holds by definition; assume the property is true for $n-1$ and we show that it is true also for n. Actually, if $h_{x_0,\alpha}(\omega), \dots, h_{x_{n-1},\alpha}(\omega), h_{x_n,\alpha}(\omega)$ are representatives of the vertices $\mathbf{x}_0, \dots, \mathbf{x}_{n-1}, \mathbf{x}_n$ respectively, we know that there exists an apartment $\mathcal{A}$ containing all the walls $h_{x_0,\alpha}(\omega), \dots, h_{x_{n-1},\alpha}(\omega)$ and on this apartment the distance between $h_{x_0,\alpha}(\omega)$ and $h_{x_{n-1},\alpha}(\omega)$ is $n-1$. On the other hand, it is possible to choose the apartment $\mathcal{A}$ in such a way that also the wall $h_{x_n,\alpha}(\omega)$ lies on it. On this apartment, $d(h_{x_0,\alpha}(\omega), h_{x_n,\alpha}(\omega)) = n$, as $h_{x_n,\alpha}(\omega) \neq h_{x_{n-2},\alpha}(\omega)$. This proves that $\mathbf{x}_0 \neq \mathbf{x}_n$.

Finally, if R is not reduced and $\alpha = \alpha_n = e_n$, the parallel hyperplanes of $\mathbb{A}$, orthogonal to α, are the hyperplanes $H^k_{2\alpha}$, for all $k \in \mathbb{Z}$. Moreover, for every $j \in \mathbb{Z}$,

$$q_{2\alpha,2j} = q_{\alpha,k} = q_\alpha = r, \qquad q_{2\alpha,2j+1} = q_{2\alpha} = p.$$

Hence, in this case the number of edges sharing any vertex $\mathbf{x}$ of type 0 is r, while the number of edges sharing the vertex $\mathbf{y}$ is p.

In all other cases, that is for all simple roots of a reduced building or for all simple roots $\alpha_i, i \neq n$, for a building of type $\widetilde{BC_n}$, we always have $\alpha \in R_0$, and hence

$$q_{\alpha,k} = q_\alpha, \quad \text{for every} \quad k \in \mathbb{Z}.$$

Therefore, each wall $h^x_\alpha(\omega)$ is adjacent to q_α walls $h^y_\alpha(\omega)$; hence each vertex $\mathbf{x}$ belongs to q_α edges. $\square$

Remark 4.3.5. *For every apartment* $\mathcal{A}$, *the walls* $h_{x,\alpha}(\omega)$ *of* $\mathcal{H}(\eta_\alpha)$, *lying on* $\mathcal{A}$, *determine a geodesic* $\gamma(\eta_\alpha)$ *of the tree* $T(\eta_\alpha)$, *consisting of all vertices* $\mathbf{x}$ *associated to these walls and of all edges connecting each pair of adjacent vertices* $\mathbf{x}, \mathbf{y}$.

The set Γ_0 *can be seen as the fundamental geodesic of the tree, since each geodesic* $\gamma(\eta_\alpha)$ *of the building is isomorphic to* Γ_0 *through any type-preserving isomorphism* $\psi_{tp} : \mathcal{A} \to \mathbb{A}$, *if* $\mathcal{A}$ *denotes any apartment containing* $\gamma(\eta_\alpha)$.

The tree $T(\eta_\alpha)$, is labelled and semi-homogeneous only when R is not reduced and $\alpha = \alpha_n = e_n$, i.e. only when the building has type $\widetilde{BC}_n$; in this case $\widehat{\mathcal{V}}(\Delta)$ consists only of vertices of type 0. Therefore for such a tree it is straightforward to restrict to consider only its vertices of type 0. Hence, if $\mathbf{x}, \mathbf{y}$ are vertices of type 0, then the geodesic $[\mathbf{x}, \mathbf{y}]$ has length $2n$, for some $n \in \mathbb{N}$. Moreover on the fundamental geodesic Γ_0 we consider only the vertices X_{2n}, for $n \in \mathbb{N}$.

Proposition 4.3.4 shows that, for every element $\eta_\alpha \in \Omega_\alpha$, we may identify the set $\mathcal{H}_\alpha(\eta_\alpha)$ with a tree $T_\alpha(\eta_\alpha)$. Moreover trees $T_\alpha(\eta_{\alpha,1}),\ T_\alpha(\eta_{\alpha,2})$ associated to any two $\eta_{\alpha,1},\ \eta_{\alpha,2}$ in Ω_α are isomorphic. For every $x \in \widehat{\mathcal{V}}(\Delta)$, the vertex $\mathbf{x}$ can be seen as the projection of x onto the tree $T_\alpha(\eta_\alpha)$. In this sense we can refer to $T_\alpha(\eta_\alpha)$ as to *the tree at infinity* associated to the element η_α of the α-boundary.

Proposition 4.3.6. *For every $\eta_\alpha \in \Omega_\alpha$, the set*

$$\{\omega \in \Omega :\ \omega \in \eta_\alpha\}$$

can be identified with the boundary $\partial T_\alpha(\eta_\alpha)$ of the tree $T_\alpha(\eta_\alpha)$.

Proof. We fix $x \in \widehat{\mathcal{V}}(\Delta)$. For every ω in the class $\eta_\alpha = [\omega]_\alpha$, we consider the sector $Q_x(\omega)$ based at x and its wall $h^x_\alpha(\omega)$. Let us denote by $h^{x_j}_\alpha(\omega),\ j \geq 0$, a sequence of walls lying on $Q_x(\omega)$ such that

$$h^{x_0}_\alpha(\omega) = h^x_\alpha(\omega) \quad \text{and} \quad d(h^{x_j}_\alpha(\omega), h^{x_{j+1}}_\alpha(\omega)) = 1, \quad j \geq 0.$$

The sequence $\mathbf{x}_j, j \geq 0$, is a geodesic of the tree $T_\alpha(\eta_\alpha)$ starting from $\mathbf{x}_0 = \mathbf{x}$ and hence it determines, as usual, a boundary point $\overline{\omega}$ of the tree. The map $\omega \to \overline{\omega}$ is a bijection of $\eta_\alpha = [\omega]_\alpha$ onto $\partial T_\alpha(\eta_\alpha)$, since each boundary point of the tree can be obtained from a suitable ω in the class η_α, with the procedure described before, and $\overline{\omega}_1 \neq \overline{\omega}_2$, if $\omega_1 \neq \omega_2$ are two elements of the same class η_α. □

Since the trees $T_\alpha(\eta_{\alpha,1}),\ T_\alpha(\eta_{\alpha,2})$ associated to any two $\eta_{\alpha,1},\ \eta_{\alpha,2}$ in Ω_α are isomorphic, the same is true for their boundaries $\partial T_\alpha(\eta_{\alpha,1}),\ \partial T_\alpha(\eta_{\alpha,2})$. We denote by T_α an abstract tree such that

$$T_\alpha(\eta_\alpha) \sim T_\alpha, \quad \forall \eta_\alpha \in \Omega_\alpha;$$

moreover we denote by $\mathbf{t}$ any element of T_α and by $\mathbf{b}$ any element of its boundary ∂T_α.

As a consequence of Proposition 4.3.6, the maximal boundary Ω of the building can be decomposed as a disjoint union of boundaries of trees, one for each equivalence class $\eta_\alpha = [\omega]_\alpha$:

$$\Omega = \bigcup_{\eta_\alpha \in \Omega_\alpha} \partial T(\eta_\alpha).$$

The previous decomposition implies that each boundary point ω of the building can be seen as a pair $(\eta_\alpha, \mathbf{b}) \in \Omega_\alpha \times \partial T_\alpha$, where η_α is the equivalence class $[\omega]_\alpha$ containing ω and $\mathbf{b}$ is the boundary point of T_α corresponding on $\partial T(\eta_\alpha)$ to $\overline{\omega}$. In this sense we may write, up to isomorphism,

$$\Omega = \Omega_\alpha \times \partial T_\alpha.$$

4.4. **Orthogonal decomposition with respect to a root α.**

Definition 4.4.1. *Let s_α be the reflection with respect to the linear hyperplane H_α of $\mathbb{A}$. For every vector v of the Euclidean space supporting $\mathbb{A}$, we set*

$$P_\alpha(v) = \frac{v - s_\alpha v}{2}, \qquad Q_\alpha(v) = \frac{v + s_\alpha v}{2}.$$

By definition, $P_\alpha(v) + Q_\alpha(v) = v$ and $Q_\alpha(v) - P_\alpha(v) = s_\alpha v$. Moreover

$$P_\alpha(s_\alpha v) = -P_\alpha(v) \quad \text{and} \quad Q_\alpha(s_\alpha v) = Q_\alpha(v).$$

We observe that, for every v, $Q_\alpha(v)$ lies on H_α and $P_\alpha(v)$ is the component of the vector v, in the direction orthogonal to the hyperplane H_α, that is in the direction of the vector α.

Proposition 4.4.2. *Let ω_1, ω_2 be α-equivalent. Then, for every $x, y \in \widehat{\mathcal{V}}(\Delta)$,*

$$Q_\alpha(\rho_{\omega_2}(y) - \rho_{\omega_2}(x)) = Q_\alpha(\rho_{\omega_1}(y) - \rho_{\omega_1}(x)).$$

If x, y belong to an apartment containing both the boundary points ω_1, ω_2, then

$$P_\alpha(\rho_{\omega_2}(y) - \rho_{\omega_2}(x)) = -P_\alpha(\rho_{\omega_1}(y) - \rho_{\omega_1}(x)).$$

PROOF. Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and $\eta_\alpha = [\omega]_\alpha$, for every $\omega \in \Omega$. Consider the tree $T_\alpha(\eta_\alpha)$ and let $\mathbf{x}$ and $\mathbf{y}$ be the vertices of this tree, associated to x and y respectively.

If $\mathbf{x} = \mathbf{y}$, the walls $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$ are equivalent, and hence they intersect in a wall $h_{z,\alpha}(\omega)$. In this case, $Q_\alpha(\rho_\omega(y) - \rho_\omega(x))$ is given by the difference between $\sigma(y, z)$ and $\sigma(x, z)$.

Assume now $\mathbf{x} \neq \mathbf{y}$. If $\mathbf{b}$ is the boundary point of the tree corresponding to ω, we consider the geodesics $[\mathbf{x}, \mathbf{b}]$, $[\mathbf{y}, \mathbf{b}]$ from $\mathbf{x}$ and from $\mathbf{y}$ to $\mathbf{b}$ respectively. We denote by $\mathbf{z}$ the vertex of the tree such that $[\mathbf{z}, \mathbf{b}] = [\mathbf{x}, \mathbf{b}] \cap [\mathbf{y}, \mathbf{b}]$, and by z a vertex of the building corresponding to $\mathbf{z}$, such that $Q_z(\omega) \subset Q_x(\omega) \cap Q_y(\omega)$. In the case when $[\mathbf{y}, \mathbf{b}] \subset [\mathbf{x}, \mathbf{b}]$, then $\mathbf{z} = \mathbf{y}$, and hence $h^z_\alpha(\omega) \subset h^y_\alpha(\omega)$. Otherwise, $h_{z,\alpha}(\omega)$ and $h_{x,\alpha}(\omega)$ are definitely parallel; if $h_{x',\alpha}(\omega)$ is the subwall of $h_{x,\alpha}(\omega)$ parallel to $h_{z,\alpha}(\omega)$, it is easy to check that $Q_\alpha(\rho_\omega(y) - \rho_\omega(x))$ is given by the difference between $\sigma(y, z)$ and $\sigma(x, x')$. In the case when $[\mathbf{x}, \mathbf{b}] \subset [\mathbf{y}, \mathbf{b}]$, a similar argument shows that $Q_\alpha(\rho_\omega(y) - \rho_\omega(x))$ is given by the difference between $\sigma(y, y')$ and $\sigma(x, z)$, if we denote by $h_{y',\alpha}(\omega)$ the subwall of $h_{y,\alpha}(\omega)$ parallel to $h_{z,\alpha}(\omega)$. Finally, if $\mathbf{z} \neq \mathbf{x}$ and $\mathbf{z} \neq \mathbf{y}$, then both the walls $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$ are definitely parallel to $h_{z,\alpha}(\omega)$. If we denote by $h_{x',\alpha}(\omega)$ and by $h_{y',\alpha}(\omega)$ the subwall of $h_{x,\alpha}(\omega)$ and of $h_{y,\alpha}(\omega)$ respectively, which are parallel to $h_{z,\alpha}(\omega)$, then $Q_\alpha(\rho_\omega(y) - \rho_\omega(x))$ is given by the difference between $\sigma(y, y')$ and $\sigma(x, x')$. In every case $Q_\alpha(\rho_\omega(y) - \rho_\omega(x))$ is a vector lying on the hyperplane H_α and it is the same for all boundary points α-equivalent to ω. Assume now that there exists an apartment containing x, y and both the boundary points ω_1, ω_2. In this particular case, $\rho_{\omega_2}(y) - \rho_{\omega_2}(x) = s_\alpha(\rho_{\omega_1}(y) - \rho_{\omega_1}(x))$. Therefore in this case

$$P_\alpha(\rho_{\omega_2}(y) - \rho_{\omega_2}(x)) = -P_\alpha(\rho_{\omega_1}(y) - \rho_{\omega_1}(x)).$$

□

4.5. **Topologies on Ω_α.** As the maximal boundary, also each α-boundary Ω_α may be endowed with a totally disconnected compact Hausdorff topology. Let x, y be special vertices in $\widehat{\mathcal{V}}(\Delta)$; consider the set $\Omega(x, y)$, defined in Section 3. We define a set of Ω_α in the following way:

$$\Omega_\alpha(x, y) = \{\eta_\alpha = [\omega]_\alpha,\ \omega \in \Omega(x, y)\}.$$

Let $x \in \widehat{\mathcal{V}}(\Delta)$; the family

$$\widetilde{\mathcal{B}}^x_\alpha = \{\ \Omega_\alpha(x, y),\ y \in \widehat{\mathcal{V}}(\Delta),\ y \in \cup\ h^x_\alpha\}$$

generates a (totally disconnected compact Hausdorff) topology on Ω_α; for every $\eta_\alpha \in \Omega_\alpha$, say $\eta_\alpha = [\omega]_\alpha$, a local base at η_α is given by

$$\widetilde{\mathcal{B}}_{x,\eta_\alpha} = \{\ \Omega_\alpha(x, y),\ y \in Q_x(\omega)\}.$$

We observe that there exists a α-wall based at x containing y, if and only if $y \in V_\lambda(x)$, with $\lambda \in H_{0,\alpha}$. Then, for every pair of vertices $x, y \in \widehat{\mathcal{V}}(\Delta)$, such that $y \in V_\lambda(x)$, with $\lambda \in H_{0,\alpha}$, we have

$$\Omega_\alpha(x, y) = \{\eta_\alpha \in \Omega_\alpha\ :\ y \in h^x_\alpha(\eta_\alpha)\}.$$

Moreover the family

$$\mathcal{B}^x_\alpha = \{\ \Omega_\alpha(x, y),\ y \in \widehat{\mathcal{V}}(\Delta),\ y \in \cup h^x_\alpha\}$$

generates the same topology on Ω_α as before; hence, for every $\eta_\alpha \in \Omega_\alpha$, a local base at η_α is given by

$$\mathcal{B}_{x,\eta_\alpha} = \{\ \Omega_\alpha(x, y),\ y \subset h_x(\eta_\alpha)\}.$$

By the same argument used for the maximal boundary, we can prove that the topology on Ω_α does not depend on the particular $x \in \widehat{\mathcal{V}}(\Delta)$.

4.6. **Probability measures on the α- boundary.** For every x of $\widehat{\mathcal{V}}(\Delta)$, we define a regular Borel measure ν^α_x on Ω_α, in the following way. For every $y \in \widehat{\mathcal{V}}(\Delta)$, let $\lambda = \sigma(x, y)$; then $\sigma(\mathbf{x}, \mathbf{y}) = P_\alpha\lambda$, if $\mathbf{x}$ and $\mathbf{y}$ are the projection of x and y on the tree at infinity associated with any $\omega \in \Omega(x, y)$. Thus define

$$\nu^\alpha_x(\Omega_\alpha(x, y)) = \frac{N^\alpha_{P_\alpha\lambda}}{N_\lambda},$$

if $N^\alpha_{P_\alpha\lambda} = |\{\mathbf{z} : \sigma(\mathbf{x}, \mathbf{z}) = P_\alpha\lambda\}|$. By the same argument used on the maximal boundary we can in fact prove that there exists a unique regular Borel probability measure ν^α_x on Ω, satisfying this property. We notice that if $\lambda \in H_{0,\alpha}$, then $\mathbf{y} = \mathbf{x}$ and then $P_\alpha\lambda = \lambda$. Therefore in this case

$$\nu^\alpha_x(\Omega_\alpha(x, y)) = \nu_x(\Omega(x, y)).$$

Define

$$R^+_\alpha = \{\beta \in R^+,\ \beta \neq \alpha, 2\alpha\};$$

then, recalling the formula for N_λ given in Corollary 2.16.2, we have

$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})} \prod_{\beta\in R_\alpha^+} q_\beta^{-<\lambda,\beta>}\ q_{2\beta}^{<\lambda,\beta>}, \qquad \text{if } \lambda \in H_{0,\alpha},$$
$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \frac{\mathbf{W}_\lambda(q^{-1})(1+q_\alpha^{-1})}{\mathbf{W}(q^{-1})} \prod_{\beta\in R_\alpha^+} q_\beta^{-<\lambda,\beta>}\ q_{2\beta}^{<\lambda,\beta>}, \qquad \text{otherwise.}$$

4.7. **Topologies and probability measures on the trees at infinity.** Let T_α be the abstract tree isomorphic to each tree at infinity $T_\alpha(\eta_\alpha)$ and let ∂T_α be its boundary. As usual, we denote by $\widehat{\mathcal{V}}(T_\alpha)$ the set of all vertices of T_α, when the tree is homogeneous, or the set of all vertices of type 0, when the tree is semi-homogeneous. For every $\mathbf{t} \in \widehat{\mathcal{V}}(T_\alpha)$ and every $\mathbf{b} \in \partial T_\alpha$, we denote by $\gamma(\mathbf{t},\mathbf{b})$ the geodesic from $\mathbf{t}$ to $\mathbf{b}$. It is well known that, for every $\mathbf{t} \in \widehat{\mathcal{V}}(T_\alpha)$, the family

$$\mathcal{B}_\mathbf{t} = \{\ B(\mathbf{t},\mathbf{t}'),\ \mathbf{t}' \in \widehat{\mathcal{V}}(T_\alpha)\},$$

where, for every $\mathbf{t},\mathbf{t}' \in \widehat{\mathcal{V}}(T_\alpha)$, $B(\mathbf{t},\mathbf{t}') = \{\mathbf{b} \in \partial T_\alpha\ :\ \mathbf{t}' \in \gamma(\mathbf{t},\mathbf{b})\}$, generates a totally disconnected compact Hausdorff topology on ∂T_α; moreover for every element $\mathbf{b}$, a local base at $\mathbf{b}$ is given by

$$\mathcal{B}_{\mathbf{t},\mathbf{b}} = \{\ B(\mathbf{t},\mathbf{t}'),\ \mathbf{t}' \in \gamma_\mathbf{t}(\mathbf{b})\}.$$

We shall denote by $\mu_\mathbf{t}$ the usual probability measure on ∂T_α associated with the isotropic random walk on T_α starting from the vertex $\mathbf{t}$. We refer the reader to [5] and to [1] for the definition of this measure. We recall that, in the homogeneous case, with homogeneity q_α, we have, for every vertex $\mathbf{t}'$,

$$\mu_\mathbf{t}(B(\mathbf{t},\mathbf{t}')) = \frac{1}{q_\alpha+1}\ q_\alpha^{1-n},$$

if n is the length of the finite geodesic $[\mathbf{t},\mathbf{t}']$. Otherwise, in the semi-homogeneous case, with homogeneities p,r, we have, for every vertex $\mathbf{t}'$, at distance $2n$ from $\mathbf{t}$,

$$\mu_\mathbf{t}(B(\mathbf{t},\mathbf{t}')) = \frac{1}{p(1+r)}\ (pr)^{1-n}.$$

Since, for every element $\eta_\alpha \in \Omega_\alpha$, the tree $T(\eta_\alpha)$ is isomorphic to the abstract tree T_α, all previous arguments apply to $\partial T(\eta_\alpha)$, if $\mathbf{t}$ is replaced by the projection $\mathbf{x}$ on $T(\eta_\alpha)$ of some $x \in \widehat{\mathcal{V}}(\Delta)$, and in particular $\mathbf{e}$ is the projection on $T(\eta_\alpha)$ of the fundamental vertex e of the building. We point out that, for every $x \in \widehat{\mathcal{V}}$, the measure $\mu_\mathbf{x}$ on $\partial T_\alpha(\eta_\alpha)$ defined before can be seen as a measure on Ω, supported on $[\omega]_\alpha$, if $\eta_\alpha = [\omega]_\alpha$. Actually, it is easy to check that, if $\eta_\alpha = [\omega]_\alpha$, then, through the identification of $\partial T_\alpha(\eta_\alpha)$ with the subset $[\omega]_\alpha$ of the maximal boundary, the measure $\mu_\mathbf{x}$ coincides with the measure $\nu_{x,\omega}^\alpha$ on Ω, obtained as restriction to $[\omega]_\alpha$ of the probability measure ν_x on Ω.

4.8. **Decomposition of the measure ν_x.** Let $x \in \widehat{\mathcal{V}}(\Delta)$; let $\mathbf{x}$ be its projection on the tree $T(\eta_\alpha)$ associated with an assigned $\omega \in \Omega$ and let $\mathbf{t}$ be the element of the abstract tree T_α, which corresponds to the vertex $\mathbf{x}$. For ease of notation, from now on, we identify $\mathbf{t}$ with $\mathbf{x}$. If we identify the maximal boundary Ω with $\Omega_\alpha \times \partial T_\alpha$, according to Section 4.3, we claim that each probability measure ν_x splits as product of the probability measure ν_x^α on the α-boundary Ω_α and the canonical probability measure $\mu_\mathbf{x}$ on the boundary of the tree T_α. In order to prove this decomposition we consider, for $x, y \in \widehat{\mathcal{V}}(\Delta)$, the set $\Omega(x,y)$. If $\omega \in \Omega(x,y)$ and $\omega = (\eta_\alpha, \mathbf{b})$, then $\eta_\alpha \in \Omega_\alpha(x,y)$ and $\mathbf{b} \in B(\mathbf{x},\mathbf{y})$. Hence

$$\Omega(x,y) = \Omega_\alpha(x,y) \times B(\mathbf{x},\mathbf{y}).$$

Proposition 4.8.1. *For every $x \in \widehat{\mathcal{V}}(\Delta)$, then $\nu_x = \nu_x^\alpha \times \mu_\mathbf{x}$.*

Proof. Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and $y \in V_\lambda(x)$. Let $\mathbf{x}$ and $\mathbf{y}$ be the projection of x and y on the tree at infinity associated with any $\omega \in \Omega(x,y)$. We prove that

$$\nu_x(\Omega(x,y)) = \nu_x^\alpha(\Omega_\alpha(x,y))\ \mu_\mathbf{x}(B(\mathbf{x},\mathbf{y})).$$

If $\lambda \in H_{0,\alpha}$, we proved that $\nu_x(\Omega(x,y)) = \nu_x^\alpha(\Omega_\alpha(x,y))$; on the other hand, in this case $\mathbf{y} = \mathbf{x}$, and therefore $B(\mathbf{x},\mathbf{y}) = \partial T_\alpha$. Hence $\mu_\mathbf{x}(B(\mathbf{x},\mathbf{y})) = 1$ and the required statement is proved. Assume now $\lambda \notin H_{0,\alpha}$; in this case $\mu_\mathbf{x}(B(\mathbf{x},\mathbf{y})) = N^\alpha_{P_\alpha\lambda}$. Then the required formula is a direct consequence of the definition of $\nu_x^\alpha(\Omega_\alpha(x,y))$. □

5. Characters and Poisson kernels

5.1. **Characters of $\mathbb{A}$.** Consider in the fundamental apartment $\mathbb{A}$ the co-weight lattice $\widehat{L}$. We call *character* of $\mathbb{A}$ any multiplicative complex-valued function χ acting on $\widehat{L}$:

$$\chi(\lambda_1+\lambda_2)=\chi(\lambda_1)\,\chi(\lambda_2),\quad \forall\lambda_1,\lambda_2\in\widehat{L}.$$

We assume, without loss of generality, that a character of $\mathbb{A}$ is the restriction to $\widehat{L}$ of a multiplicative complex-valued function acting on $\mathbb{V}$. We denote by $\mathbf{X}(\widehat{L})$ the group of all characters of $\mathbb{A}$. If $n=\dim\mathbb{V}$, then $\mathbf{X}(\widehat{L})\cong(\mathbb{C}^\times)^n$, and the group $\mathbf{X}(\widehat{L})$ can be endowed with the weak topology and also with the usual measure of $\mathbb{C}^n$.

The Weyl group $\mathbf{W}$ acts on $\mathbf{X}(\widehat{L})$ in the following way: for every $\mathbf{w}\in\mathbf{W}$ and for every $\chi\in\mathbf{X}(\widehat{L})$,

$$(\mathbf{w}\chi)(\lambda)=\chi(\mathbf{w}^{-1}(\lambda)),\qquad\text{for all}\quad\lambda\in\widehat{L}.$$

It is immediate to observe that $\mathbf{w}\chi$ is a character and we simply denote $\chi^{\mathbf{w}}=\mathbf{w}\chi$.

5.2. **The fundamental character χ_0 of $\mathbb{A}$.** We shall be interested in a particular character of $\mathbb{A}$.

Definition 5.2.1. *We denote by χ_0 the following function on $\widehat{L}$:*

$$\chi_0(\lambda)=\prod_{\alpha\in R^+}q_\alpha^{\langle\lambda,\alpha\rangle}\;q_{2\alpha}^{-\langle\lambda,\alpha\rangle},\quad\forall\lambda\in\widehat{L}.$$

Being α a linear functional on the vector space $\mathbb{V}$ supporting $\mathbb{A}$, the function χ_0 is a character of $\mathbb{A}$, called the *fundamental* character of $\mathbb{A}$. Since each α in the previous formula is a positive root (with respect to $\mathbb{Q}_0$) then $\chi_0(\lambda)>1$, for all $\lambda\in\widehat{L}^+$.

If R is reduced, then $2\alpha\notin R$ and therefore $q_{2\alpha}=1$, for every $\alpha\in R$; hence

$$\chi_0(\lambda)=\prod_{\alpha\in R^+}q_\alpha^{\langle\lambda,\alpha\rangle}.$$

In particular if R is reduced and all roots have the same length, that is for buildings of type $\widetilde{A}_n,\ \widetilde{D}_n,\ \widetilde{E}_6,\ \widetilde{E}_7$ and $\widetilde{E}_8$, then $q_\alpha=q$, for every $\alpha\in R^+$ and

$$\chi_0(\lambda)=q^{\sum_{\alpha\in R^+}\langle\lambda,\alpha\rangle}=q^{2\langle\lambda,\delta\rangle},$$

if $\delta=\frac{1}{2}(\sum_{\alpha\in R^+}\alpha)$. Instead, if R is reduced but it contains long and short roots, then, denoting by α any long root and by β any short root and setting $\delta_l=\frac12(\sum\alpha),\ \delta_s=\frac12(\sum\beta)$, it follows that

$$\chi_0(\lambda)=q^{2\langle\lambda,\delta_l\rangle}\;p^{2\langle\lambda,\delta_s\rangle}.$$

This happens for buildings of type $\widetilde{B}_n,\ \widetilde{C}_n,\ \widetilde{F}_4$ and $\widetilde{G}_2$.

Assume now that R is not reduced, that is the building is of type $\widetilde{(BC)}_n$. In this case $R=R_0\cup R_1\cup R_2$. We denote by α,β and γ any root of R_0,R_1 and R_2 respectively. Then, keeping in mind that $R_2=\{\beta/2,\ \beta\in R_1\}$, it follows that

$$\begin{aligned}\chi_0(\lambda)&=\prod_{\alpha\in R_0^+}q_\alpha^{\langle\lambda,\alpha\rangle}\prod_{\beta\in R_1^+}q_\beta^{\langle\lambda,\beta\rangle}\prod_{\gamma\in R_2^+}q_\gamma^{\langle\lambda,\gamma\rangle}q_{2\gamma}^{-\langle\lambda,\gamma\rangle}=\prod_{\alpha\in R_0^+}q_\alpha^{\langle\lambda,\alpha\rangle}\prod_{\beta\in R_1^+}q_\beta^{\langle\lambda,\beta\rangle}\prod_{\beta\in R_1^+}q_{\beta/2}^{\langle\lambda,\beta/2\rangle}q_\beta^{-\langle\lambda,\beta/2\rangle}\\&=\prod_{\alpha\in R_0^+}q_\alpha^{\langle\lambda,\alpha\rangle}\prod_{\beta\in R_1^+}(q_{\beta/2}\,q_\beta)^{\langle\lambda,\beta/2\rangle}=q^{2\langle\lambda,\delta_0\rangle}(pr)^{\langle\lambda,\delta_1\rangle}\end{aligned}$$

if $\delta_0=\frac12(\sum\alpha),\ \delta_1=\frac12\sum\beta$.

We notice that, by Proposition 2.16.1, then, for every $\lambda\in\widehat{L}^+$,

$$\chi_0(\lambda)=q_{t_\lambda}.$$

More generally, if λ is any element of $\widehat{L}$, and $t_\lambda=u_\lambda g_l$, with $u_\lambda=s_{i_1}\cdots s_{i_r}$, then the same argument used in Proposition 2.16.1 shows that,

$$\chi_0(\lambda)=\prod_{j\in J^+}q_{i_j}\cdot\prod_{j\in J^-}q_{i_j}^{-1},$$

where

$$\begin{aligned}J^+&=\{j\ :\ s_{i_1}\cdots s_{i_{j-1}}(C_0)\prec s_{i_1}\cdots s_{i_j}(C_0)\}\\J^-&=\{j\ :\ s_{i_1}\cdots s_{i_j}(C_0)\prec s_{i_1}\cdots s_{i_{j-1}}(C_0)\}.\end{aligned}$$

Actually, we notice that, when λ is dominant, then $J^-=\emptyset$ and thus $J^+=\{1,\cdots,r\}$; so we get the previous formula for $\chi_0(\lambda)$.

We can easily compute the fundamental character in each simple co-root $\alpha^\vee$. We consider separately the reduced and non-reduced case.

Proposition 5.2.2. *Let R be a reduced root system; for every simple root α, then*

$$\chi_0(\alpha^\vee) = q_\alpha^2.$$

PROOF. We notice that, for every simple α, we have $\langle \alpha^\vee, \delta\rangle = 1$. This is a consequence of (13.3) in [6]. $\square$

Proposition 5.2.3. *Let R be a non-reduced root system; then*

(i) $\chi_0(\alpha^\vee) = q^2$, *for every* $\alpha = e_i - e_{i+1}$, $i = 1, \cdots, n-1$;
(ii) $\chi_0(\beta^\vee) = pr$, *for* $\beta = 2e_n$.

PROOF. We compute $\chi_0(\alpha^\vee)$ and $\chi_0(\beta^\vee)$ by using the formula of $\chi_0(\lambda)$ given above.

(i) If $\alpha = \alpha_i = e_i - e_{i+1}$, for some $i = 1, \ldots, n-1$, then $\alpha_i^\vee = \alpha_i$, and, by definition,

$$\begin{aligned}\chi_0(\alpha_i^\vee) = \chi_0(\alpha_i) &= \left(\prod_{\alpha\in R_0^+} q^{\langle \alpha_i, \alpha\rangle}\right) \left(\prod_{\beta\in R_1^+} p^{\langle \alpha_i, \beta\rangle} \left(\frac{r}{p}\right)^{\langle \alpha_i, \beta/2\rangle}\right) \\ &= q^{\langle \alpha_i, \sum_{\alpha\in R_0^+}\alpha\rangle}\, p^{\langle \alpha_i, \sum_{\beta\in R_1^+}\beta\rangle} \left(\frac{r}{p}\right)^{\langle \alpha_i, \sum_{\beta\in R_1^+}\beta/2\rangle}.\end{aligned}$$

We notice that

$$\sum_{\alpha\in R_0^+} \alpha = 2[(n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1}] \quad \text{and} \quad \sum_{\beta\in R_1^+}\beta = 2\sum_{k=1}^n e_k.$$

Hence, for every $i = 1, \cdots, n-1$,

$$\langle \alpha_i, \sum_{\alpha\in R_0^+}\alpha\rangle = 2[(n-i) - (n-i-1)] = 2 \quad \text{and} \quad \langle \alpha_i, \sum_{\beta\in R_1^+}\beta\rangle = 0,$$

since $\langle e_i - e_{i+1}, 2e_k\rangle = 2, -2, 0$, if $k = i; k = i+1$ or $k \neq i, i+1$ respectively. Therefore

$$\prod_{\alpha\in R_0^+} q^{\langle \alpha_i, \alpha\rangle} = q^2 \quad \text{and} \quad \prod_{\beta\in R_1^+} p^{\langle \alpha_i, \beta\rangle} = \prod_{\beta\in R_1^+} \left(\frac{r}{p}\right)^{\langle \alpha_i, \beta/2\rangle} = 1$$

and we conclude that $\chi_0(\alpha_i^\vee) = q^2$, for every i.

(ii) If $\beta = \beta_n = 2e_n$, then $\beta^\vee = e_n$; therefore

$$\begin{aligned}\chi_0(\beta_n^\vee) &= \left(\prod_{\alpha\in R_0^+} q^{\langle \beta^\vee, \alpha\rangle}\right) \left(\prod_{\beta\in R_1^+} p^{\langle \beta_n^\vee, \beta\rangle} \left(\frac{r}{p}\right)^{\langle \beta_n^\vee, \beta/2\rangle}\right) \\ &= q^{\langle \beta_n^\vee, \sum_{\alpha\in R_0^+}\alpha\rangle}\, p^{\langle \beta_n^\vee, \sum_{\beta\in R_1^+}\beta\rangle} \left(\frac{r}{p}\right)^{\langle \beta_n^\vee, \sum_{\beta\in R_1^+}\beta/2\rangle}.\end{aligned}$$

On the other hand

$$\langle \beta_n^\vee, \sum_{\alpha\in R_0^+}\alpha\rangle = 0 \quad \text{and} \quad \langle \beta_n^\vee, \sum_{\beta\in R_1^+}\beta\rangle = 2,$$

since $\langle \beta_n^\vee, e_k\rangle = \langle e_n, 2e_k\rangle = 2$ or 0, according if $k = n$ or $k \neq n$. Therefore

$$\prod_{\alpha\in R_0^+} q^{\langle \beta_n^\vee, \alpha\rangle} = 1, \qquad \prod_{\beta\in R_1^+} p^{\langle \beta_n^\vee, \beta\rangle} = p^2, \qquad \prod_{\beta\in R_1^+} \left(\frac{r}{p}\right)^{\langle \beta_n^\vee, \frac{\beta}{2}\rangle} = \frac{r}{p}$$

and we conclude that $\chi_0(\beta^\vee) = pr$.

$\square$

For every simple root α we define, for every $\lambda \in \widehat{L}$,

$$\chi_0^\alpha(\lambda) = \prod_{\beta\in R_\alpha^+} q_\beta^{\langle \lambda, \beta\rangle}\; q_{2\beta}^{-\langle \lambda, \beta\rangle}.$$

Obviously χ_0^α is a character on $\mathbb{A}$; moreover it is easy to check that, if $\lambda \in H_{0,\alpha}$, then

$$\chi_0^\alpha(\lambda) = \chi_0(\lambda),$$

since for every $\lambda \in H_{0,\alpha}$, we have $\langle\lambda,\alpha\rangle = \langle\lambda, 2\alpha\rangle = 0$ and therefore

$$\prod_{\beta\in R_\alpha^+} q_\beta^{\langle\lambda,\beta\rangle}\ q_{2\beta}^{-\langle\lambda,\beta\rangle} = \prod_{\beta\in R^+} q_\beta^{\langle\lambda,\beta\rangle}\ q_{2\beta}^{-\langle\lambda,\beta\rangle} = \chi_0(\lambda).$$

Let T_α be the abstract tree isomorphic to each tree at infinity $T_\alpha(\eta_\alpha)$. We denote by Γ_0 the fundamental geodesic of the tree and by Γ_0^+ the fundamental geodesic based at 0. We define a character $\overline{\chi}_0$ on Γ_0 in the following way:
$\overline{\chi}_0(X_n) = q_\alpha^n$, if X_n is the vertex of Γ_0^+ at distance n from 0, in the homogeneous case;
$\overline{\chi}_0(X_{2n}) = (pr)^n$, if X_{2n} is the vertex of Γ_0^+ at distance $2n$ from 0, otherwise.

The characters χ_0, χ_0^α and $\overline{\chi}_0$ are related through the operators P_α and Q_α defined in Section 4.4, as the following lemma shows.

Lemma 5.2.4. *Let $\lambda \in \widehat{L}$; assume $\lambda \in H_{n,\alpha}$, if $\alpha \in R_0$, and $\lambda \in H_{2n,\alpha}$, if $\alpha \in R_2$. Then*

(i) $\chi_0(Q_\alpha(\lambda)) = \chi_0^\alpha(Q_\alpha(\lambda)) = \chi_0^\alpha(\lambda)$,

(ii) $\chi_0(P_\alpha(\lambda)) = \begin{cases} \overline{\chi}_0(\mathbf{X}_n) = q_\alpha^n, & \text{if } \alpha \in R_0, \\ \overline{\chi}_0(\mathbf{X}_{2n}) = (pr)^n, & \text{if } \alpha \in R_2. \end{cases}$

Proof. (i) We notice at first that $\langle Q_\alpha(\lambda),\alpha\rangle = 0$, for every α. Hence

$$\chi_0^\alpha(Q_\alpha(\lambda)) = \prod_{\beta\in R_\alpha^+} q_\beta^{\langle Q_\alpha(\lambda),\beta\rangle}\ q_{2\beta}^{-\langle Q_\alpha(\lambda),\beta\rangle} = \prod_{\beta\in R^+} q_\beta^{\langle Q_\alpha(\lambda),\beta\rangle}\ q_{2\beta}^{-\langle Q_\alpha(\lambda),\beta\rangle} = \chi_0(Q_\alpha(\lambda)).$$

Moreover it is easy to prove that

$$\prod_{\beta\in R_\alpha^+} q_\beta^{\langle P_\alpha(\lambda),\beta\rangle}\ q_{2\beta}^{-\langle P_\alpha(\lambda),\beta\rangle} = 1.$$

Actually, for every $\beta \in R_\alpha^+$ the root $s_\alpha\beta$ belongs to R_α^+, and $\langle P_\alpha(\lambda),\beta\rangle = -\langle P_\alpha(\lambda),\sigma_\alpha\beta\rangle$. Therefore,

$$\chi_0^\alpha(\lambda) = \prod_{\beta\in R_\alpha^+} q_\beta^{\langle\lambda,\beta\rangle}\ q_{2\beta}^{-\langle\lambda,\beta\rangle} = \prod_{\beta\in R_\alpha^+} q_\beta^{\langle Q_\alpha(\lambda),\beta\rangle}\ q_{2\beta}^{-\langle Q_\alpha(\lambda),\beta\rangle} \prod_{\beta\in R_\alpha^+} q_\beta^{\langle P_\alpha(\lambda),\beta\rangle}\ q_{2\beta}^{-\langle P_\alpha(\lambda),\beta\rangle} = \chi_0^\alpha(Q_\alpha(\lambda)).$$

(ii) By the same argument of (i), we have

$$\chi_0(P_\alpha(\lambda)) = q_\alpha^{\langle P_\alpha(\lambda),\alpha\rangle} q_{2\alpha}^{-\langle P_\alpha(\lambda),\alpha\rangle} \prod_{\beta\in R_\alpha^+} q_\beta^{\langle P_\alpha(\lambda),\beta\rangle} q_{2\beta}^{-\langle P_\alpha(\lambda),\beta\rangle} = q_\alpha^{\langle P_\alpha(\lambda),\alpha\rangle} q_{2\alpha}^{-\langle P_\alpha(\lambda),\alpha\rangle} = q_\alpha^{\langle\lambda,\alpha\rangle} q_{2\alpha}^{-\langle\lambda,\alpha\rangle};$$

therefore (ii) is proved, because

$$q_\alpha^{\langle\lambda,\alpha\rangle}\ q_{2\alpha}^{-\langle\lambda,\alpha\rangle} = \begin{cases} \overline{\chi}_0(\mathbf{X}_n) & \text{if } \alpha \in R_0, \\ \overline{\chi}_0(\mathbf{X}_{2n}) & \text{if } \alpha \in R_2. \end{cases}$$

□

Corollary 5.2.5. *For every $\lambda \in \widehat{L}$, $\chi_0(\lambda) = \chi_0^\alpha(Q_\alpha(\lambda))\ \overline{\chi}_0(\mathbf{X}_\lambda)$, if $\mathbf{X}_\lambda$ is the vertex of Γ_0 corresponding to $P_\alpha(\lambda)$.*

Let $\rho_{\mathbf{b}}$ be the retraction of the tree on Γ_0, with respect to the boundary point $\mathbf{b}$, such that $\rho_{\mathbf{b}}(\gamma(\mathbf{e},\mathbf{b})) = \Gamma_0^+$. (Here $\mathbf{e}$ denotes the fundamental vertex of the tree). An immediate consequence of Lemma 5.2.4 is the following proposition.

Proposition 5.2.6. *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$. Let $\mathbf{x}$ and $\mathbf{y}$ be the projection of x and y on the tree at infinity $T_\alpha(\eta_\alpha)$ associated with ω. Then*

(i) $\chi_0(Q_\alpha(\rho_\omega(y) - \rho_\omega(x)) = \chi_0^\alpha(\rho_\omega(y) - \rho_\omega(x))$,

(ii) $\chi_0(P_\alpha(\rho_\omega(y) - \rho_\omega(x)) = \overline{\chi}_0(\rho_{\mathbf{b}}(\mathbf{y}) - \rho_{\mathbf{b}}(\mathbf{x}))$.

Proof. Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$. If $\lambda = \rho_\omega(y) - \rho_\omega(x)$, (i) follows from Lemma 5.2.4, (i).

Let $\eta_\alpha = [\omega]_\alpha$, and consider the vertices $\mathbf{x}, \mathbf{y}$ of the tree $T(\eta_\alpha)$, corresponding to x, y. If $\mathbf{b}$ is the boundary point of this tree, corresponding to ω, then $\mathbf{b} \in B(\mathbf{x},\mathbf{y})$; this implies that $\rho_{\mathbf{b}}(\mathbf{y}) - \rho_{\mathbf{b}}(\mathbf{x}) = n$, if $\langle\lambda,\alpha\rangle = n$. Hence (ii) follows from Lemma 5.2.4, (ii). □

5.3. **Probability measures on the boundaries.** The measure ν_x defined, for any $x \in \widehat{\mathcal{V}}(\Delta)$, on the maximal boundary Ω can be characterized in terms of the character χ_0.

Proposition 5.3.1. *Let x and y be vertices of $\widehat{\mathcal{V}}(\Delta)$; then, for every $\omega \in \Omega(x,y)$,*

$$\nu_x(\Omega(x,y)) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\ \chi_0^{-1}(\rho_\omega^x(y)) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\ \chi_0^{-1}(\rho_\omega(y) - \rho_\omega(x)).$$

Proof. Since $\chi_0(\lambda) = q_{t_\lambda}$, for every $\lambda \in \hat{L}^+$, then, by definition of ν_x, we have, for each $y \in V_\lambda(x)$,

$$\nu_x(\Omega(x,y)) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\ \chi_0^{-1}(\lambda).$$

On the other hand, in Section 3.3 we have proved that, if $y \in \Omega_x(\omega)$, then $\rho_\omega^x(y) = \sigma(x,y)$, and that $\rho_\omega^x(y) = \rho_\omega(y) - \rho_\omega(x)$. Therefore the required formula is proved. □

Let α be any simple root of the root system R associated with Δ. The measure ν_x^α defined in Section 4.6 on the α-boundary can be characterized in terms of the character χ_0^α.

Proposition 5.3.2. *Let $\lambda \in \hat{L}^+$, and $y \in V_\lambda(x)$; then, for every $\eta_\alpha \in \Omega_\alpha(x,y)$ and for every ω in the class η_α,*

$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\ (\chi_0^\alpha)^{-1}(\rho_\omega(y) - \rho_\omega(x)), \qquad \textit{if} \quad \lambda \in H_{0,\alpha}$$
$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \frac{\mathbf{W}_\lambda(q^{-1})(1+q_\alpha^{-1})}{\mathbf{W}(q^{-1})}\ (\chi_0^\alpha)^{-1}(\rho_\omega(y) - \rho_\omega(x)), \qquad \textit{otherwise.}$$

Proof. Recalling the definition of the character χ_0^α we have

$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\ (\chi_0^\alpha)^{-1}(\lambda), \qquad \text{if } \lambda \in H_{0,\alpha},$$
$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \frac{\mathbf{W}_\lambda(q^{-1})(1+q_\alpha^{-1})}{\mathbf{W}(q^{-1})}\ (\chi_0^\alpha)^{-1}(\lambda), \qquad \text{otherwise.}$$

On the other hand, for every $\eta_\alpha \in \Omega_\alpha(x,y)$ and for every ω in the class η_α,

$$\rho_\omega(y) - \rho_\omega(x) = \lambda, \quad \text{if } \sigma(x,y) = \lambda.$$

In particular, if we assume $y \in V_\lambda(x)$, with $\lambda \in H_{0,\alpha}$, then the vector $\rho_\omega(y) - \rho_\omega(x)$ belongs to $H_{0,\alpha}$. □

Taking in account Proposition 5.2.6, we can express the measures ν_x^α and $\mu_{\mathbf{x}}$ in terms of the character χ_0 and the operators P_α and Q_α.

Corollary 5.3.3. *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and $y \in V_\lambda(x)$. Let $\mathbf{x}$ and $\mathbf{y}$ be the projection of x and y on the tree at infinity $T_\alpha(\eta_\alpha)$ associated with any $\omega \in \Omega(x,y)$. Then*

$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \begin{cases} \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\ (\chi_0)^{-1}(\rho_\omega(y) - \rho_\omega(x)), & \lambda \in H_{0,\alpha}, \\ \\ \frac{\mathbf{W}_\lambda(q^{-1})(1+q_\alpha^{-1})}{\mathbf{W}(q^{-1})}\ (\chi_0)^{-1}(Q_\alpha(\rho_\omega(y) - \rho_\omega(x))) & \textit{otherwise.} \end{cases}$$

Moreover

$$\mu_{\mathbf{x}}(B(\mathbf{x},\mathbf{y})) = \begin{cases} 1, & \textit{if } \lambda \in H_{0,\alpha}, \\ \frac{q_\alpha}{1+q_\alpha}(\chi_0)^{-1}(P_\alpha(\rho_\omega(y) - \rho_\omega(x))), & \textit{otherwise.} \end{cases}$$

Therefore, in view of Corollaries 5.3.3, the decomposition of the measure ν_x for the maximal boundary, stated in Section 4.8, is a direct consequence of the orthogonal decomposition $\chi_0(\lambda) = \chi_0(P_\alpha(\lambda))\,\chi_0(Q_\alpha(\lambda))$.

5.4. **Poisson kernel and Poisson transform.**

Proposition 5.4.1. *For $x, y \in \widehat{\mathcal{V}}(\Delta)$ the measures ν_x, ν_y are mutually absolutely continuous and the Radon-Nikodym derivative of ν_y with respect to ν_x is given by*

$$\frac{d\nu_y}{d\nu_x}(\omega) = \chi_0(\rho_\omega^x(y)) = \chi_0(\rho_\omega(y) - \rho_\omega(x)), \quad \forall \omega \in \Omega.$$

Proof. We fix x, y and ω; by Corollary 3.3.9, we can choose a special vertex z lying into $Q_y(\omega)\cap Q_x(\omega)$, so that $\Omega(x,z) = \Omega(y,z)$. We set $\Omega_z = \Omega(x,z) = \Omega(y,z)$. Of course ω belongs to Ω_z. We have, by Proposition 5.3.1,

$$\nu_x(\Omega_z) = \nu_x(\Omega(x,z)) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\ \chi_0^{-1}(\rho_\omega(z) - \rho_\omega(x)),$$
$$\nu_y(\Omega_z) = \nu_y(\Omega(y,z)) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\ \chi_0^{-1}(\rho_\omega(z) - \rho_\omega(y)).$$

So we conclude that

$$\frac{\nu_y(\Omega_z)}{\nu_x(\Omega_z)} = \frac{\chi_0^{-1}(\rho_\omega(z) - \rho_\omega(y))}{\chi_0^{-1}(\rho_\omega(z) - \rho_\omega(x))} = \chi_0(\rho_\omega(y) - \rho_\omega(x)).$$

This proves that ν_y is absolutely continuous with respect to ν_x and shows the required formula for the Radon-Nikodym derivative of ν_y with respect to ν_x. □

Definition 5.4.2. *We call Poisson kernel of the building Δ the function*

$$P(x,y,\omega) = \chi_0(\rho_\omega(y) - \rho_\omega(x)) = \chi_0(\rho^x_\omega(y)), \quad \forall x, y \in \widehat{\mathcal{V}}(\Delta) \textit{ and } \forall \omega \in \Omega.$$

This definition does not depend on the choice of the special vertex e. By Proposition 5.4.1, for every choice of x, y in $\widehat{\mathcal{V}}(\Delta)$, the function $P(x, y, \cdot)$ is the Radon-Nikodym derivative of ν_y with respect to ν_x :

$$\frac{d\nu_y}{d\nu_x}(\omega) = P(x,y,\omega), \quad \forall \omega \in \Omega.$$

Using the same argument of Proposition 5.4.1, we can prove the following proposition.

Proposition 5.4.3. *For $x, y \in \widehat{\mathcal{V}}(\Delta)$, the measures ν_x^α, ν_y^α are mutually absolutely continuous and*

$$\frac{d\nu_y^\alpha}{d\nu_x^\alpha}(\eta_\alpha) = \chi_0^\alpha(\rho_\omega(y) - \rho_\omega(x)), \quad \forall \omega \in \eta_\alpha, \quad \forall \eta_\alpha \in \Omega_\alpha.$$

We shall denote, for every $x, y \in \widehat{\mathcal{V}}(\Delta)$ and for every $\eta_\alpha \in \Omega_\alpha$,

$$P^\alpha(x,y,\eta_\alpha) = \frac{d\nu_y^\alpha}{d\nu_x^\alpha}(\eta_\alpha) = \chi_0^\alpha(\rho_\omega(y) - \rho_\omega(x)), \quad \forall \omega \in \eta_\alpha.$$

It is known that, for every pair of vertices $\mathbf{t}, \mathbf{t}'$ in $\widehat{\mathcal{V}}(T_\alpha)$, the measure $\mu_{\mathbf{t}'}$ is absolutely continuous with respect to $\mu_{\mathbf{t}}$, and the Radon-Nikodym derivative $d\mu_{\mathbf{t}'}/d\mu_{\mathbf{t}}(\mathbf{b})$ is the Poisson kernel $P(\mathbf{t}, \mathbf{t}', \mathbf{b})$, where

$P(\mathbf{t}, \mathbf{t}', \mathbf{b}) = q_\alpha^{n-1}$, if $d(\mathbf{t}, \mathbf{t}') = n$, in the homogeneous case

$P(\mathbf{t}, \mathbf{t}', \mathbf{b}) = (pr)^{n-1}$, if $d(\mathbf{t}, \mathbf{t}') = 2n$, in the semi-homogeneous case.

In both cases, as a straightforward consequence of the definition,

$$P(\mathbf{t}, \mathbf{t}', \mathbf{b}) = \overline{\chi}_0(\rho_{\mathbf{b}}(\mathbf{t}') - \rho_{\mathbf{b}}(\mathbf{t})), \quad \forall \mathbf{b} \in \partial T_\alpha.$$

Since , for every pair of vertices $x, y \in \widehat{\mathcal{V}}(\Delta)$, the measure ν_y on Ω is absolutely continuous with respect to ν_x, the measure ν_y^α on Ω_α is absolutely continuous with respect to ν_x^α and the measure $\mu_{\mathbf{y}}$ on ∂T_α is absolutely continuous with respect to $\mu_{\mathbf{x}}$; actually we have the following result.

Corollary 5.4.4. *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$, and $\omega \in \Omega$. If $\omega = (\eta_\alpha, \mathbf{b})$, and $\mathbf{x}$ and $\mathbf{y}$ are the projection of x and y on the tree at infinity $T_\alpha(\eta_\alpha)$, then*

$$P(x,y,\omega) = P^\alpha(x,y,\eta_\alpha)\ P(\mathbf{x}, \mathbf{y}, \mathbf{b}).$$

Proof. By Proposition 5.2.6, for every $x, y \in \widehat{\mathcal{V}}(\Delta)$, and every $\omega \in \Omega$,

$$P^\alpha(x,y,\eta_\alpha) = \chi_0(Q_\alpha(\rho_\omega(y) - \rho_\omega(x)) \quad \text{and} \quad P(\mathbf{x}, \mathbf{y}, \mathbf{b}) = \chi_0(P_\alpha(\rho_\omega(y) - \rho_\omega(x)).$$

Therefore, the decomposition of the Poisson kernel $P(x,y,\omega)$ is a direct consequence of the orthogonal decomposition $\chi_0(\lambda) = \chi_0(P_\alpha(\lambda))\ \chi_0(Q_\alpha(\lambda))$.

□

Definition 5.4.2 can be generalized, if the character χ_0 is replaced by any character χ.

Definition 5.4.5. *We call generalized Poisson kernel of the building Δ associated with the character χ the function*

$$P^\chi(x,y,\omega) = \chi(\rho_\omega(y) - \rho_\omega(x)), \quad \forall x, y \in \widehat{\mathcal{V}}(\Delta) \textit{ and } \forall \omega \in \Omega.$$

It is obvious that also this definition does not depend on the choice of the vertex e. According to this definition, $P(x,y,\omega) = P^{\chi_0}(x,y,\omega)$.

The following proposition shows the properties of any function $P^{\chi}(x,y,\omega)$.

Proposition 5.4.6. *Let χ be a character on $\mathbb{A}$; then,*

(i) $P^{\chi}(x,x,\omega) = 1$, *for every x and every ω; moreover, for every x, y and every ω,*

$$P^{\chi}(y,x,\omega) = (P^{\chi}(x,y,\omega))^{-1} = P^{\chi^{-1}}(x,y,\omega);$$

(ii) for every x and every ω, the function $P^{\chi}(x,\cdot,\omega)$ is constant on the set of vertices

$$\{y \in \widehat{\mathcal{V}}(\Delta) : \sigma(x,y) = \lambda,\ \rho^x_\omega(y) = \mu\},$$

for any $\lambda \in \widehat{L}^+$ and $\mu \in \Pi_\lambda$.

(iii) for every x, y, the function $P^{\chi}(x,y,\cdot)$ is locally constant on Ω, and, if $\sigma(x,y) = \lambda$, then $P^{\chi}(x,y,\omega) = \chi(\lambda)$, for all $\omega \in \Omega(x,y)$.

Proof. (i) and (ii) follow immediately from the definition. Moreover (iii) is a consequence of the properties of the retraction ρ^x_ω, proved in Section 3.3. Actually, if $\sigma(x,y) = \lambda$, and we choose μ big enough with respect to λ, then $\Omega = \cup_{z \in V_\mu(x)} \Omega(x,z)$ and $\rho^x_\omega(y)$ does not depend on the choice of ω in each set $\Omega(x,z)$. In particular, $\rho^x_\omega(y) = \lambda$, for all $\omega \in \Omega(x,y)$. □

Definition 5.4.7. *Let $x_0 \in \widehat{\mathcal{V}}(\Delta)$ and let χ be a character on $\mathbb{A}$. For any complex valued function f on Ω, we call generalized Poisson transform of f of initial point x_0, associated with the character χ, the function on $\widehat{\mathcal{V}}(\Delta)$ defined by*

$$\mathcal{P}^{\chi}_{x_0} f(x) = \int_\Omega P^{\chi}(x_0,x,\omega) f(\omega) d\nu_x(\omega) = \int_\Omega \chi(\rho_\omega(x) - \rho_\omega(x_0)) f(\omega) d\nu_{x_0}(\omega), \quad \forall x \in \widehat{\mathcal{V}}(\Delta),$$

whenever the integral exists.

In particular, we set $\mathcal{P}_{x_0} = \mathcal{P}^{\chi_0}_{x_0}$ and $\mathcal{P} = \mathcal{P}_e$.

6. The algebra $\mathcal{H}(\Delta)$ and its eigenvalues

6.1. **The algebra $\mathcal{H}(\Delta)$.** For every $\lambda \in \widehat{L}^+$, we define an operator A_λ, acting on the space of complex valued functions f on $\widehat{\mathcal{V}}(\Delta)$, by

$$(A_\lambda f)(x) = \sum_{y \in V_\lambda(x)} f(y) = \sum_{y \in \widehat{\mathcal{V}}(\Delta)} \mathbb{I}_{V_\lambda(x)}(y) f(y), \quad \text{for all} \quad x \in \widehat{\mathcal{V}}(\Delta).$$

The operators A_λ are linear; moreover, for each y, the coefficient $\mathbb{I}_{V_\lambda(x)}(y)$ only depends on λ. We notice that the operators $\{A_\lambda, \lambda \in \widehat{L}^+\}$ are linearly independent. Actually, if assume $\sum_{\lambda \in \widehat{L}^+} a_\lambda A_\lambda = 0$, then

$$\sum_{\lambda \in \widehat{L}^+} a_\lambda (A_\lambda \delta_y)(x) = 0, \qquad \forall x, y \in \widehat{\mathcal{V}}(\Delta).$$

On the other hand $\sum_{\lambda \in \widehat{L}^+} a_\lambda (A_\lambda \delta_y)(x) = a_\mu$, if $\sigma(x,y) = \mu$. Hence we get $a_\mu = 0$, for every $\mu \in \widehat{L}^+$.

We denote by $\mathcal{H}(\Delta)$ the linear span of $\{A_\lambda, \lambda \in \hat{L}^+\}$ over $\mathbb{C}$.

Proposition 6.1.1. *The space $\mathcal{H}(\Delta)$ is a commutative $\mathbb{C}$-algebra.*

Proof. We shall prove that, for every λ, μ the operator $A_\lambda \circ A_\mu$ is a finite linear combination of operators A_ν, for convenient ν. Actually, recalling (2.18.1), for every function f and for every $x \in \widehat{\mathcal{V}}(\Delta)$,

$$\begin{aligned}
A_\lambda \circ A_\mu f(x) &= \sum_{y \in \widehat{\mathcal{V}}(\Delta)} \mathbb{I}_{V_\lambda(x)}(y)\ A_\mu f(y) = \sum_{y \in \widehat{\mathcal{V}}(\Delta)} \mathbb{I}_{V_\lambda(x)}(y) \sum_{z \in \widehat{\mathcal{V}}(\Delta)} \mathbb{I}_{V_\mu(y)}(z) f(z) \\
&= \sum_{z \in \widehat{\mathcal{V}}(\Delta)} \left(\sum_{y \in \widehat{\mathcal{V}}(\Delta)} \mathbb{I}_{V_\lambda(x)}(y) \mathbb{I}_{V_\mu(y)}(z) \right) f(z) \\
&= \sum_{z \in \widehat{\mathcal{V}}(\Delta)} \left| \{y \in \widehat{\mathcal{V}}(\Delta) : \sigma(x,y) = \lambda,\ \sigma(y,z) = \nu\} \right| f(z) \\
&= \sum_{\nu \in \widehat{L}^+} \sum_{z \in V_\nu(x)} N(\nu, \lambda, \mu^\star) f(z) = \sum_{\nu \in \widehat{L}^+} N(\nu, \lambda, \mu^\star)(A_\nu f)(x)
\end{aligned}$$

and $N(\nu,\lambda,\mu^\star)$ is different from zero only for finitely many ν. Moreover

$$A_\mu \circ A_\lambda f(x) = \sum_{\nu\in\widehat{L}^+} N(\nu,\mu,\lambda^\star)(A_\nu f)(x) = \sum_{\nu\in\widehat{L}^+} N(\nu,\lambda,\mu^\star)(A_\nu f)(x) = A_\lambda \circ A_\mu f(x)$$

and this complete the proof. $\square$

We refer to the numbers $N(\nu,\lambda,\mu^\star)$ in Proposition 6.1.1 as the *structure constants* of $\mathcal{H}(\Delta)$.

6.2. **Eigenvalue of the algebra $\mathcal{H}(\Delta)$ associated with a character χ.** In this section we study the eigenvalues of the algebra $\mathcal{H}(\Delta)$.

Let χ be a character on $\mathbb{A}$; consider the generalized Poisson kernel $P^\chi(x,y,\omega)$ associated with χ.

Lemma 6.2.1. *Let $z \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$. For every $\lambda \in \widehat{L}^+$, the sum $\sum_{y\in V_\lambda(z)} \chi(\rho_\omega(y)-\rho_\omega(z))$ is independent of z and*

$$\sum_{y\in V_\lambda(z)} \chi(\rho_\omega(y)-\rho_\omega(z)) = \sum_{\mu\in\Pi_\lambda} N(\lambda,\mu)\chi(\mu),$$

where $N(\lambda,\mu) = |\{y : \sigma(e,y)=\lambda,\ \rho_\omega(y)=\mu\}|$.

Proof. For every $z \in \widehat{\mathcal{V}}(\Delta)$, $\omega\in\Omega$ and $\lambda\in\widehat{L}^+$, we have

$$\sum_{y\in V_\lambda(z)} \chi(\rho_\omega(y)-\rho_\omega(z)) = \sum_{\mu\in\Pi_\lambda} \left|\{y\in\widehat{\mathcal{V}}(\Delta)\ :\ \sigma(z,y)=\lambda,\ \rho_\omega(y)-\rho_\omega(z)=\mu\}\right|\ \chi(\mu).$$

By Theorem 3.3.12, for every $\mu\in\Pi_\lambda$,

$$\left|\{y\in\widehat{\mathcal{V}}(\Delta)\ :\ \sigma(z,y)=\lambda,\ \rho_\omega(y)-\rho_\omega(z)=\mu\}\right| = \left|\{y\in\widehat{\mathcal{V}}(\Delta)\ :\ \sigma(e,y)=\lambda,\ \rho_\omega(y)=\mu\}\right| = N(\lambda,\mu).$$

Hence the lemma is proved. $\square$

For every $\lambda\in\widehat{L}^+$, we define

$$\Lambda^\chi(\lambda) = \sum_{\mu\in\Pi_\lambda} N(\lambda,\mu)\chi(\mu).$$

Proposition 6.2.2. *For every $\lambda\in\widehat{L}^+$, $\Lambda^\chi(\lambda)$ is an eigenvalue of the operator A_λ and, for every $x\in\widehat{\mathcal{V}}(\Delta)$ and $\omega\in\Omega$, the function $P^\chi(x,\cdot,\omega)$ is an eigenfunction of A_λ, associated with the eigenvalue $\Lambda^\chi(\lambda)$:*

$$A_\lambda P^\chi(x,\cdot,\omega) = \Lambda^\chi(\lambda)\ P^\chi(x,\cdot,\omega).$$

Proof. For every $z\in\widehat{\mathcal{V}}(\Delta)$, we can write

$$\begin{aligned} A_\lambda P^\chi(x,\cdot,\omega)(z) &= \sum_{y\in V_\lambda(z)} P^\chi(x,y,\omega) = \sum_{y\in V_\lambda(z)} \chi(\rho_\omega(y)-\rho_\omega(x)) = \sum_{y\in V_\lambda(z)} \chi(\rho_\omega(y))\chi(-\rho_\omega(x)) \\ &= \chi(\rho_\omega(z)-\rho_\omega(x)) \sum_{y\in V_\lambda(z)} \chi(\rho_\omega(y)-\rho_\omega(z)) = P^\chi(x,z,\omega) \sum_{y\in V_\lambda(z)} \chi(\rho_\omega(y)-\rho_\omega(z)). \end{aligned}$$

Hence, by Lemma 6.2.1, we conclude that

$$A_\lambda P^\chi(x,\cdot,\omega) = \Lambda^\chi(\lambda)\ P^\chi(x,\cdot,\omega).$$

$\square$

Since $\{A_\lambda,\ \lambda\in\widehat{L}^+\}$ generates $\mathcal{H}(\Delta)$, then $\{\Lambda^\chi(\lambda),\ \lambda\in\widehat{L}^+\}$ generates an algebra homomorphism Λ^χ from $\mathcal{H}(\Delta)$ to $\mathbb{C}$, such that $\Lambda^\chi(A_\lambda)=\Lambda^\chi(\lambda)$, for every $\lambda\in\widehat{L}^+$. Moreover, for every $x\in\widehat{\mathcal{V}}(\Delta)$ and $\omega\in\Omega$, the function $P^\chi(x,\cdot,\omega)$ is an eigenfunction of $\mathcal{H}(\Delta)$, associated with the eigenvalue Λ^χ.

In the particular case when $\chi=\chi_0$, then, for every $x\in\widehat{\mathcal{V}}(\Delta)$ and for every $\omega\in\Omega$, the Poisson kernel $P(x,\cdot,\omega)$ is an eigenfunction of all operators A_λ, with associated eigenvalue $\Lambda^{\chi_0}(\lambda)$. Since $P(x,y,\omega)$ is the Radon-Nikodym derivative of the measure ν_y with respect to the measure ν_x, this implies that

$$\sum_{y\in V_\lambda(x)} \nu_y = \Lambda^{\chi_0}(\lambda)\ \nu_x.$$

On the other hand, since ν_y and ν_x are probability measures on Ω, then

$$\sum_{y\in V_\lambda(x)} \nu_y = \left|\{y\in\widehat{\mathcal{V}}(\Delta)\ :\ \sigma(x,y)=\lambda\}\right|\ \nu_x.$$

This implies that

$$\Lambda^{\chi_0}(\lambda) = \left|\{y\in\widehat{\mathcal{V}}(\Delta)\ :\ \sigma(x,y)=\lambda\}\right|,$$

and hence

$$\sum_{\mu\in\Pi_\lambda} N(\lambda,\mu)\ \chi_0(\mu) = \left|\{y\in\widehat{\mathcal{V}}(\Delta)\ :\ \sigma(x,y)=\lambda\}\right| = N_\lambda.$$

Corollary 6.2.3. *For every $f\in L^1(\Omega,\nu_x)$, the Poisson transform $\mathcal{P}^\chi_x(f)$ of f, of initial point x, associated with the character χ, is an eigenfunction of the algebra $\mathcal{H}(\Delta)$, associated with the eigenvalue Λ^χ.*

Proof. Actually, for every $\lambda\in\widehat{L}^+$,

$$\begin{aligned}
A_\lambda\mathcal{P}^\chi_x(f)(z) &= \sum_{y\in V_\lambda(x)} \mathcal{P}^\chi_x(f)(y) = \sum_{y\in V_\lambda(x)}\int_\Omega P^\chi(x,y,\omega)f(\omega)\ d\nu_x(\omega)\\
&= \int_\Omega\left(\sum_{y\in V_\lambda(x)} P^\chi(x,y,\omega)\right) f(\omega)\ d\nu_x(\omega) = \int_\Omega \Lambda^\chi(\lambda)\ P^\chi(x,z,\omega)\ f(\omega)\ d\nu_x(\omega) = \Lambda^\chi(\lambda)\ \ \mathcal{P}^\chi_x(f)(z).
\end{aligned}$$

□

Since the Weyl group $\mathbf{W}$ acts on the characters χ, according to definition given in Section 5.1, then $\mathbf{W}$ acts also on the eigenvalues Λ^χ of the algebra $\mathcal{H}(\Delta)$. We shall prove that in fact these eigenvalues are invariant with respect to the action of $\mathbf{W}$, in the sense that, for every character χ,

$$\Lambda^{\chi\chi_0^{1/2}} = \Lambda^{\chi^{\mathbf{w}}\chi_0^{1/2}},\ \ \forall\mathbf{w}\in\mathbf{W}.$$

6.3. **Preliminary results.** Let χ be a fixed character on $\mathbb{A}$; let α be a fixed simple root and let η_α be an element of the α-boundary Ω_α.

Definition 6.3.1. *Let $x\in\widehat{\mathcal{V}}(\Delta)$; for each pair ω_1,ω_2 in the class $\eta_\alpha\in\Omega_\alpha$, we fix a vertex of $\widehat{\mathcal{V}}(\Delta)$, say $e=e_{\omega_1,\omega_2}$, in any apartment $\mathcal{A}(\omega_1,\omega_2)$ containing both the boundary points. We set*

$$j^\alpha_{x,\chi}(\omega_1,\omega_2) = \chi\chi_0^{1/2}(P_\alpha(\rho_{\omega_1}(e)+\rho_{\omega_2}(e)-\rho_{\omega_1}(x)-\rho_{\omega_2}(x))).$$

Remark 6.3.2. *The function $j^\alpha_{x,\chi}(\omega_1,\omega_2)$ does not depend on the choice of the vertex e_{ω_1,ω_2} on any apartment $\mathcal{A}(\omega_1,\omega_2)$. Actually, if e and e' are two vertices on this apartment, then, for every $x\in\widehat{\mathcal{V}}(\Delta)$,*

$$\begin{aligned}
&P_\alpha(\rho_{\omega_1}(x)-\rho_{\omega_1}(e)+\rho_{\omega_2}(x)-\rho_{\omega_2}(e)) - P_\alpha(\rho_{\omega_1}(x)-\rho_{\omega_1}(e')+\rho_{\omega_2}(x)-\rho_{\omega_2}(e'))\\
&\quad= P_\alpha((\rho_{\omega_1}(e')-\rho_{\omega_1}(e))+(\rho_{\omega_2}(e')-\rho_{\omega_2}(e))) = P_\alpha((\rho_{\omega_1}(e')-\rho_{\omega_1}(e)))+P_\alpha((\rho_{\omega_2}(e')-\rho_{\omega_2}(e)))=0,
\end{aligned}$$

since $P_\alpha((\rho_{\omega_1}(e')-\rho_{\omega_1}(e))) = -P_\alpha((\rho_{\omega_2}(e')-\rho_{\omega_2}(e)))$, as we proved in Proposition 4.4.2,

For every $\omega\in\Omega$, let η_α be the element of the α-boundary Ω_α, such that $\omega\in\eta_\alpha$. We denote by $\nu^\alpha_{x,\omega}$ the restriction of the measure ν_x to the set $\{\omega'\in\Omega\ :\ \omega'\in\eta_\alpha\}$. Since the set $\{\omega'\in\Omega\ :\ \omega'\in\eta_\alpha\}$ can be identified with the boundary of the tree $T(\eta_\alpha)$, then $\nu^\alpha_{x,\omega}$ can be seen as the usual measure $\mu_{\mathbf{x}}$ on $\partial T(\eta_\alpha)$.

Definition 6.3.3. *Let $x\in\widehat{\mathcal{V}}(\Delta)$; we denote by $J^\alpha_{x,\chi}$ the following operator acting on the complex valued functions f defined on Ω :*

$$J^\alpha_{x,\chi}(f)(\omega_0) = \int_\Omega j^\alpha_{x,\chi}(\omega_0,\omega)\ f(\omega)\ d\nu^\alpha_{x,\omega_0}(\omega),\quad \forall\omega_0\in\Omega.$$

Theorem 6.3.4. *Assume that $|\chi(\alpha^\vee)|<1$; then*

(i) $J^\alpha_{x,\chi}\mathbf{1}=c(\chi)\mathbf{1}$, where $c(\chi)$ is a non zero complex number.
(ii) $J^\alpha_{x,\chi}: L^\infty(\Omega)\to L^\infty(\Omega)$ is a bounded operator.

Proof. (i) Fix ω_0 in Ω and let $\eta_\alpha=[\omega_0]_\alpha$. By Definitions 6.3.1 and 6.3.3, we have

$$J^\alpha_{x,\chi}\mathbf{1}(\omega_0) = \int_\Omega j^\alpha_{x,\chi}(\omega_0,\omega)\ \ d\nu^\alpha_{x,\omega_0}(\omega) = \int_{[\omega_0]_\alpha} \chi\chi_0^{1/2}(P_\alpha(\rho_{\omega_0}(e)+\rho_\omega(e)-\rho_{\omega_0}(x)-\rho_\omega(x)))\ \ d\nu^\alpha_{x,\omega_0}(\omega),$$

if e is a vertex in any apartment containing ω_0 and ω.

Consider the tree $T(\eta_\alpha)$ and its boundary $\partial T(\eta_\alpha)$. According to notation of Section 5.2, we simply denote by $\overline{\chi}$ the character on the fundamental geodesic Γ_0 of the tree, such that, for every $n\in\mathbb{Z}$,

$$\overline{\chi}(\mathbf{X}_n)=\chi(P_\alpha(\lambda)),\ \text{if}\ \ \alpha\in R_0,$$
$$\overline{\chi}(\mathbf{X}_{2n})=\chi(P_\alpha(\lambda)),\ \ \text{if}\ \ \alpha\in R_2,$$

if $\lambda \in \widehat{L}$ satisfies $\langle \lambda, \alpha \rangle = n$. Since we can identify the set $[\omega_0]_\alpha$ with the boundary of the tree $T(\eta_\alpha)$ and the measure ν^α_{x,ω_0} can be seen as the usual measure $\mu_{\mathbf{x}}$ on $\partial T(\eta_\alpha)$, we can write

$$J^\alpha_{x,\chi}\mathbf{1}(\omega_0) = \int_{\partial T(\eta_\alpha)} \overline{\chi}\ \overline{\chi}_0^{1/2}(\rho_{\mathbf{b}_0}(\mathbf{e}) + \rho_{\mathbf{b}}(\mathbf{e}) - \rho_{\mathbf{b}_0}(\mathbf{x}) - \rho_{\mathbf{b}}(\mathbf{x}))\ \ d\mu_{\mathbf{x}}(\mathbf{b}),$$

if $\mathbf{b}_0$ is the boundary point of the tree corresponding to ω_0, $\mathbf{b}$ is the boundary point of the tree corresponding to ω, for every $\omega \in [\omega_0]_\alpha$, and $\mathbf{e}$ is the vertex of the geodesic $\gamma(\mathbf{b}_0, \mathbf{b})$ obtained as projection of e on the tree $T(\eta_\alpha)$. For every $x \in \widehat{\mathcal{V}}(\Delta)$, let $\mathbf{x}$ be the vertex of the tree corresponding to x and denote by $N_{\mathbf{x}}(\mathbf{b}_0, \mathbf{b})$ the distance of $\mathbf{x}$ from the geodesic $[\mathbf{b}_0, \mathbf{b}]$, that is the minimal distance of $\mathbf{x}$ from the set $\{\mathbf{y} \in \mathcal{V}(T(\eta_\alpha)) \ :\ \mathbf{y} \in [\mathbf{b}_0, \mathbf{b}]\}$. For every $j \geq 0$, we set

$$B_j(\mathbf{x}, \mathbf{b}_0) = \{\mathbf{b} \in \partial T(\eta_\alpha) \ :\ N_{\mathbf{x}}(\mathbf{b}_0, \mathbf{b}) = j\}.$$

Then, we can decompose $\partial T(\eta_\alpha)$, as a disjoint union, in the following way

$$\partial T(\eta_\alpha) = \cup_j B_j(\mathbf{x}, \mathbf{b}_0).$$

We can easily compute $\mu_{\mathbf{x}}(B_j(\mathbf{x}, \mathbf{b}_0))$, for every $j \geq 0$. If $\alpha \in R_0$, the tree $T(\eta_\alpha)$ is homogeneous and

$$\mu_{\mathbf{x}}(B_0(\mathbf{x}, \mathbf{b}_0)) = \frac{q_\alpha}{q_\alpha + 1} \quad \text{and} \quad \mu_{\mathbf{x}}(B_j(\mathbf{x}, \mathbf{b}_0)) = \frac{q_\alpha - 1}{q_\alpha + 1} q_\alpha^{-j} \quad \text{for all } j > 0.$$

Otherwise, if $\alpha \in R_2$, the tree $T(\eta_\alpha)$ is semi-homogeneous and we have

$$\begin{aligned} \mu_{\mathbf{x}}(B_0(\mathbf{x}, \mathbf{b}_0)) &= \frac{r}{r+1} \\ \mu_{\mathbf{x}}(B_{2j}(\mathbf{x}, \mathbf{b}_0)) &= \frac{r-1}{(r+1)}(pr)^{-j}, \quad \text{for all } j > 0 \\ \mu_{\mathbf{x}}(B_{2j+1}(\mathbf{x}, \mathbf{b}_0)) &= \frac{p-1}{p(r+1)}(pr)^{-j}, \quad \text{for all } j \geq 0. \end{aligned}$$

It is easy to see that, for every $j \geq 0$,

$$\rho_{\mathbf{b}_0}(\mathbf{e}) + \rho_{\mathbf{b}}(\mathbf{e}) - \rho_{\mathbf{b}_0}(\mathbf{x}) - \rho_{\mathbf{b}}(\mathbf{x}) = X_{2j}, \quad \text{for all } \mathbf{b} \in B_j(\mathbf{x}, \mathbf{b}_0).$$

Thus

$$J^\alpha_{x,\chi}\mathbf{1}(\omega_0) = \sum_{j=0}^{\infty} \mu_{\mathbf{x}}(B_j(\mathbf{x}, \mathbf{b}_0)\ \overline{\chi}\ \overline{\chi}_0^{1/2}(\mathbf{X}_{2j}).$$

Therefore, if $\alpha \in R_0$, then

$$\begin{aligned} J^\alpha_{x,\chi}\mathbf{1}(\omega_0) &= \frac{q_\alpha}{q_\alpha + 1}\ \overline{\chi}\ \overline{\chi}_0^{1/2}(0) + \sum_{j\geq 1} \frac{q_\alpha - 1}{q_\alpha + 1} q_\alpha^{-j}\ \overline{\chi}\ \overline{\chi}_0^{1/2}(\mathbf{X}_{2j}) \\ &= \frac{q_\alpha}{q_\alpha + 1} + \frac{q_\alpha - 1}{q_\alpha + 1} \sum_{j\geq 1} q_\alpha^{-j} q_\alpha^{j}\ \ \overline{\chi}(2j\mathbf{X}_1) = \frac{q_\alpha}{q_\alpha + 1} + \frac{q_\alpha - 1}{q_\alpha + 1} \sum_{j\geq 1} (\overline{\chi}(\mathbf{X}_1))^{2j}. \end{aligned}$$

Analogously, if $\alpha \in R_2$, then

$$\begin{aligned} J^\alpha_{x,\chi}\mathbf{1}(\omega_0) &= \frac{r}{r+1}\ \overline{\chi}\ \overline{\chi}_0^{1/2}(\mathbf{X}_0) + \sum_{j\geq 1} \frac{r-1}{r+1}(pr)^{-j}\ \overline{\chi}\ \overline{\chi}_0^{1/2}(\mathbf{X}_{4j}) + \sum_{j\geq 1} \frac{r(p-1)}{r+1}(pr)^{-j}\ \overline{\chi}\ \overline{\chi}_0^{1/2}(\mathbf{X}_{4j-2}) \\ &= \frac{r}{(r+1)} + \frac{r-1}{r+1} \sum_{j\geq 1} (pr)^{-j}(pr)^{j}\overline{\chi}(2j\mathbf{X}_2) + \frac{r(p-1)}{r+1} \frac{1}{\sqrt{pr}} \sum_{j\geq 1} (pr)^{-j}(pr)^{j}\overline{\chi}((2j-1)\mathbf{X}_2) \\ &= \frac{r}{(r+1)} + \left[\frac{r-1}{r+1} + \frac{r(p-1)}{r+1} \frac{1}{\sqrt{pr}} \overline{\chi}(-\mathbf{X}_2)\right] \sum_{j\geq 1} (\overline{\chi}(\mathbf{X}_2))^{2j}. \end{aligned}$$

Since $\overline{\chi}(\mathbf{X}_2) = \chi(\alpha^\vee)$, and $\overline{\chi}(\mathbf{X}_1) = \chi^{1/2}(\alpha^\vee)$, then, if we assume $|\chi(\alpha^\vee)| < 1$, it follows that $|\overline{\chi}(\mathbf{X}_1)| < 1$, if $\alpha \in R_0$, and that $|\overline{\chi}(\mathbf{X}_2)| < 1$, if $\alpha \in R_2$; hence the geometric series $\sum_{j\geq 1} (\overline{\chi}(\mathbf{X}_1))^{2j}$ and $\sum_{j\geq 1} (\overline{\chi}(\mathbf{X}_2))^{2j}$ converge. Since the sum of these series does not depend on the choice of x and ω_0, we have proved (i) by setting

$$\begin{aligned} c(\chi) &= \frac{q_\alpha}{q_\alpha + 1} + \frac{q_\alpha - 1}{q_\alpha + 1} \sum_{j\geq 1} (\overline{\chi}(\mathbf{X}_1))^{2j}, \ \text{ if } \alpha \in R_0\ , \\ c(\chi) &= \frac{r}{(r+1)} + \left[\frac{r-1}{r+1} + \frac{r(p-1)}{r+1} \frac{1}{\sqrt{pr}} \overline{\chi}^2(-\mathbf{X}_2)\right] \sum_{j\geq 1} (\overline{\chi}(\mathbf{X}_2))^{2j}, \ \text{ if } \alpha \in R_2\ . \end{aligned}$$

(ii) The same argument, applied to the real character $|\chi|$, shows that

$$\int_\Omega \left| j^\alpha_{x,\chi}(\omega_0,\omega)\right| \quad d\nu^\alpha_{x,\omega_0}(\omega) = k(\chi),$$

being $k(\chi)$ a real positive number. Hence, for any $f \in L^\infty(\Omega)$, and for every $\omega_0 \in \Omega$,

$$\left|J^\alpha_{x,\chi} f(\omega_0)\right| \le ||f||_\infty \int_\Omega \left| j^\alpha_{x,\chi}(\omega_0,\omega)\right| \quad d\nu^\alpha_{x,\omega_0}(\omega) = k(\chi)\ ||f||_\infty.$$

This proves that $J^\alpha_{x,\chi} f$ belongs to $L^\infty(\Omega)$ and that $J^\alpha_{x,\chi}$ is a bounded operator. □

Remark 6.3.5. *The constant $c(\chi)$ is different from 1 except in the case when $\chi = \chi_0^{-1}$.*

Definition 6.3.6. *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$; we denote by $T^\chi_{x,y}$ the following operator acting on the complex valued functions f defined on Ω :*

$$T^\chi_{x,y}(f)(\omega) = P^{\chi\chi_0^{-1}}(x,y,\omega)\ f(\omega), \quad \forall \omega \in \Omega.$$

For every $x, y \in \widehat{\mathcal{V}}(\Delta)$, the operator $T^\chi_{x,y}$ is bounded on the space $L^\infty(\Omega)$, because $P^{\chi\chi_0^{-1}}(x,y,\cdot)$ is a locally constant function on Ω.

Proposition 6.3.7. *Assume $|\chi(\alpha^\vee)| < 1$. For every pair of vertices $x, y \in \widehat{\mathcal{V}}(\Delta)$,*

$$J^\alpha_{y,\chi} \circ T^{\chi\chi_0^{1/2}}_{x,y} = T^{\chi^{s_\alpha}\chi_0^{1/2}}_{x,y} \circ J^\alpha_{x,\chi}.$$

Proof. By Theorem 6.3.4, the assumption $|\chi(\alpha^\vee)| < 1$ assures that, for every pair $x, y \in \widehat{\mathcal{V}}(\Delta)$, the operators $J^\alpha_{x,\chi},\ J^\alpha_{y,\chi}$ are bounded on the space $L^\infty(\Omega)$. By Definitions 6.3.1, 6.3.3 and 6.3.6, for every function f and for every boundary point ω_0, we have

$$\begin{aligned}
&\left(T^{\chi^{s_\alpha}\chi_0^{1/2}}_{x,y} \circ J^\alpha_{x,\chi}\right) f(\omega_0) = P^{\chi^{s_\alpha}\chi_0^{-1/2}}(x,y,\omega_0) \int_\Omega j^\alpha_{x,\chi}(\omega_0,\omega)\ f(\omega)\ d\nu^\alpha_{x,\omega_0}(\omega)\\
&= \int_\Omega j^\alpha_{x,\chi}(\omega_0,\omega)\ P^{\chi^{s_\alpha}\chi_0^{-1/2}}(x,y,\omega_0)\quad f(\omega)\ d\nu^\alpha_{x,\omega_0}(\omega)\\
&= \int_\Omega \frac{j^\alpha_{x,\chi}(\omega_0,\omega)}{j^\alpha_{y,\chi}(\omega_0,\omega)}\ j^\alpha_{y,\chi}(\omega_0,\omega)\ P^{\chi^{s_\alpha}\chi_0^{-1/2}}(x,y,\omega_0)\quad f(\omega)\ \frac{d\nu^\alpha_{x,\omega_0}(\omega)}{d\nu^\alpha_{y,\omega_0}(\omega)}\ d\nu^\alpha_{y,\omega_0}(\omega).
\end{aligned}$$

Definition 6.3.1 implies that, for any vertex e lying on any apartment containing ω_0 and ω,

$$\frac{j^\alpha_{x,\chi}(\omega_0,\omega)}{j^\alpha_{y,\chi}(\omega_0,\omega)} = \frac{\chi\chi_0^{1/2}(P_\alpha(\rho_{\omega_0}(e)+\rho_\omega(e)-\rho_{\omega_0}(x)-\rho_\omega(x))}{\chi\chi_0^{1/2}(P_\alpha(\rho_{\omega_0}(e)+\rho_\omega(e)-\rho_{\omega_0}(y)-\rho_\omega(y))} = \frac{\chi\chi_0^{1/2}(P_\alpha(-\rho_{\omega_0}(x)-\rho_\omega(x))}{\chi\chi_0^{1/2}(P_\alpha(-\rho_{\omega_0}(y)-\rho_\omega(y))}.$$

Moreover, according to definition of measure ν^α_{x,ω_0},

$$\frac{d\nu^\alpha_{x,\omega_0}(\omega)}{d\nu^\alpha_{y,\omega_0}(\omega)} = \chi_0(P_\alpha(\rho_\omega(x)-\rho_\omega(y)).$$

Therefore

$$\begin{aligned}
&\frac{j^\alpha_{x,\chi}(\omega_0,\omega)}{j^\alpha_{y,\chi}(\omega_0,\omega)}\ \frac{d\nu^\alpha_{x,\omega_0}(\omega)}{d\nu^\alpha_{y,\omega_0}(\omega)} = \frac{\chi\chi_0^{1/2}(P_\alpha(-\rho_{\omega_0}(x)-\rho_\omega(x))}{\chi\chi_0^{1/2}(P_\alpha(-\rho_{\omega_0}(y)-\rho_\omega(y))}\ \chi_0(P_\alpha(\rho_\omega(x)-\rho_\omega(y)))\\
&= \frac{\chi(P_\alpha(\rho_\omega(y)-\rho_\omega(x)))}{\chi(P_\alpha(\rho_{\omega_0}(x)-\rho_{\omega_0}(y)))}\ \chi_0^{1/2}(P_\alpha(\rho_{\omega_0}(y)-\rho_{\omega_0}(x))\chi_0^{-1/2}(P_\alpha(\rho_\omega(y)-\rho_\omega(x))\\
&= \frac{\chi\chi_0^{-1/2}(P_\alpha(\rho_\omega(y)-\rho_\omega(x)))}{\chi^{s_\alpha}\chi_0^{-1/2}(P_\alpha(\rho_{\omega_0}(y)-\rho_{\omega_0}(x)))}.
\end{aligned}$$

Moreover, if we recall that $Q_\alpha(\rho_{\omega_0}(y)-\rho_{\omega_0}(x)) = Q_\alpha(\rho_\omega(y)-\rho_\omega(x))$ (see Proposition 4.4.2), we have

$$\frac{j^\alpha_{x,\chi}(\omega_0,\omega)}{j^\alpha_{y,\chi}(\omega_0,\omega)}\ \frac{d\nu^\alpha_{x,\omega_0}(\omega)}{d\nu^\alpha_{y,\omega_0}(\omega)} = \frac{\chi\chi_0^{-1/2}(\rho_\omega(y)-\rho_\omega(x))}{\chi^{s_\alpha}\chi_0^{-1/2}(\rho_{\omega_0}(y)-\rho_{\omega_0}(x))} = \frac{P^{\chi\chi_0^{-1/2}}(x,y,\omega)}{P^{\chi^{s_\alpha}\chi_0^{-1/2}}(x,y,\omega_0)}.$$

So we can conclude that

$$\left(T_{x,y}^{\chi^{s_\alpha}\chi_0^{1/2}}\circ J_{x,\chi}^{\alpha}\right)f(\omega_0)=\int_\Omega j_{y,\chi}^{\alpha}(\omega_0,\omega)\ \frac{P^{\chi\chi_0^{-1/2}}(x,y,\omega)}{P^{\chi^{s_\alpha}\chi_0^{-1/2}}(x,y,\omega_0)}\ P^{\chi^{s_\alpha}\chi_0^{-1/2}}(x,y,\omega_0)\ f(\omega)\ \ d\nu_{y,\omega_0}^{\alpha}(\omega)$$
$$=\int_\Omega j_{y,\chi}^{\alpha}(\omega_0,\omega)\ P^{\chi\chi_0^{-1/2}}(x,y,\omega)\ f(\omega)\ d\nu_{y,\omega_0}^{\alpha}(\omega)=\int_\Omega j_{y,\chi}^{\alpha}(\omega_0,\omega)\ T_{x,y}^{\chi\chi_0^{1/2}}(f)(\omega)\ d\nu_{y,\omega_0}^{\alpha}(\omega)$$
$$=\left(J_{y,\chi}^{\alpha}\circ T_{x,y}^{\chi\chi_0^{1/2}}\right)f(\omega_0).$$

□

6.4. **W-invariance of the eigenvalues.**

Theorem 6.4.1. *For every character χ and for for every simple root α,*

$$\Lambda^{\chi\chi_0^{1/2}}=\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}. \tag{6.4.1}$$

Proof. (i) At first assume $|\chi(\alpha^\vee)|>1$. Then $|\chi^{-1}(\alpha^\vee)|<1$ and hence Theorem 6.3.4 implies that, for every $x,y\in\widehat{\mathcal{V}}(\Delta)$, $J_{x,\chi^{-1}}^{\alpha}$ and $J_{y,\chi^{-1}}^{\alpha}$ are bounded operators on $L^\infty(\Omega)$. Therefore, applying Proposition 6.3.7, we get, for every $x,y\in\widehat{\mathcal{V}}(\Delta)$,

$$J_{y,\chi^{-1}}^{\alpha}\circ T_{x,y}^{\chi^{-1}\chi_0^{1/2}}\mathbf{1}(\omega)=T_{x,y}^{(\chi^{s_\alpha})^{-1}\chi_0^{1/2}}\circ J_{x,\chi^{-1}}^{\alpha}\mathbf{1}(\omega),\quad\forall\omega\in\Omega,$$

since $(\chi^{s_\alpha})^{-1}=(\chi^{-1})^{s_\alpha}$. Thus if we fix $y\in\widehat{\mathcal{V}}(\Delta)$ and, for every $\lambda\in\widehat{L}$, sum on all x such that $\sigma(y,x)=\lambda$, we get, by linearity,

$$\sum_{x\in V_\lambda(y)}J_{y,\chi^{-1}}^{\alpha}\circ T_{x,y}^{\chi^{-1}\chi_0^{1/2}}\mathbf{1}(\omega)=J_{y,\chi^{-1}}^{\alpha}\circ\sum_{x\in V_\lambda(y)}T_{x,y}^{\chi^{-1}\chi_0^{1/2}}\mathbf{1}(\omega)=J_{y,\chi^{-1}}^{\alpha}\left(\sum_{x\in V_\lambda(y)}P^{\chi^{-1}\chi_0^{-1/2}}(x,y,\cdot)\right)(\omega)$$

and, if we recall that $\sum_{x\in V_\lambda(y)}P^{\chi^{-1}\chi_0^{-1/2}}(x,y,\omega)=\sum_{x\in V_\lambda(y)}P^{\chi\chi_0^{1/2}}(y,x,\omega)=\Lambda^{\chi\chi_0^{1/2}}(\lambda)$, , for every $\omega\in\Omega$, then

$$\sum_{x\in V_\lambda(y)}J_{y,\chi^{-1}}^{\alpha}\circ T_{x,y}^{\chi^{-1}\chi_0^{1/2}}\mathbf{1}(\omega)=J_{y,\chi^{-1}}^{\alpha}(\Lambda^{\chi\chi_0^{1/2}}(\lambda)\mathbf{1})(\omega)=\Lambda^{\chi\chi_0^{1/2}}(\lambda)\ J_{y,\chi^{-1}}^{\alpha}\mathbf{1}(\omega)=\Lambda^{\chi\chi_0^{1/2}}(\lambda)\ c(\chi^{-1}).$$

On the other hand,

$$\sum_{x\in V_\lambda(y)}T_{x,y}^{(\chi^{s_\alpha})^{-1}\chi_0^{1/2}}\circ J_{x,\chi^{-1}}^{\alpha}\mathbf{1}(\omega)=\sum_{x\in V_\lambda(y)}T_{x,y}^{(\chi^{s_\alpha})^{-1}\chi_0^{1/2}}(c(\chi^{-1})\mathbf{1})(\omega)=c(\chi^{-1})\sum_{x\in V_\lambda(y)}T_{x,y}^{(\chi^{s_\alpha})^{-1}\chi_0^{1/2}}\mathbf{1}(\omega)$$
$$=c(\chi^{-1})\sum_{x\in V_\lambda(y)}P^{(\chi^{s_\alpha})^{-1}\chi_0^{-1/2}}(x,y,\omega)=c(\chi^{-1})\sum_{x\in V_\lambda(y)}P^{\chi^{s_\alpha}\chi_0^{1/2}}(y,x,\omega)=c(\chi^{-1})\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}(\lambda).$$

Since $c(\chi^{-1})$ is a real number different from zero, the identity

$$c(\chi^{-1})\ \Lambda^{\chi\chi_0^{1/2}}(\lambda)=c(\chi^{-1})\ \Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}(\lambda)$$

implies $\Lambda^{\chi\chi_0^{1/2}}(\lambda)=\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}(\lambda)$, for every $\lambda\in\widehat{L}$.

(ii) Assume now $|\chi(\alpha^\vee)|<1$. In this case $|\chi^{s_\alpha}(\alpha^\vee)|>1$ and therefore, by (i),

$$\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}=\Lambda^{\chi^{s_\alpha^2}\chi_0^{1/2}}=\Lambda^{\chi\chi_0^{1/2}}.$$

(iii) Finally, if $|\chi(\alpha^\vee)|=1$, the required identity can be proved by a standard argument of continuity, as the eigenvalue associated with a character χ depends continuously on χ, with respect to the weak topology on the space $Hom(\hat{L},\mathbb{C})$; actually, there exists a character χ', with $|\chi'(\alpha^\vee)|<1$, arbitrarily closed to χ. □

Since the reflections $s_\alpha,\ \alpha=\alpha_i,\ i=1,\dots,n$, generate $\mathbf{W}$, we have the following

Corollary 6.4.2. *For every character χ and for every $\mathbf{w}\in\mathbf{W}$,*

$$\Lambda^{\chi\chi_0^{1/2}}=\Lambda^{\chi^{\mathbf{w}}\chi_0^{1/2}}.$$

6.5. **Technical results about the Poisson transform.** According to Definition 5.4.7, we denote by $\mathcal{P}_x^\chi$ the generalized Poisson transform of initial point x associated with the character χ. It will be useful to analyze the relationship between the Poisson transform and the operators defined in Sections 6.3.

Proposition 6.5.1. *For every pair $x, y \in \widehat{\mathcal{V}}(\Delta)$, and for every $f \in L^\infty(\Omega)$,*

$$\mathcal{P}_y^\chi(T_{x,y}^\chi f) = \mathcal{P}_x^\chi(f).$$

Proof. For every vertex $z \in \widehat{\mathcal{V}}(\Delta)$,

$$\begin{aligned}\mathcal{P}_y^\chi(T_{x,y}^\chi f)(z) &= \int_\Omega P^\chi(y,z,\omega)P^{\chi\chi_0^{-1}}(x,y,\omega)f(\omega)d\nu_y(\omega)\\ &= \int_\Omega \chi(\rho_\omega(z)-\rho_\omega(y))\ \chi(\rho_\omega(y)-\rho_\omega(x))\ f(\omega)\ \chi_0(\rho_\omega(x)-\rho_\omega(y))\ d\nu_y(\omega)\\ &= \int_\Omega \chi(\rho_\omega(z)-\rho_\omega(x))\ f(\omega)\ \frac{d\nu_x(\omega)}{d\nu_y(\omega)}\ d\nu_y(\omega) = \int_\Omega P^\chi(x,z,\omega)\ f(\omega)\ d\nu_x(\omega) = \mathcal{P}_x^\chi f(z).\end{aligned}$$

□

By Corollary 6.2.3, for every $f \in L^\infty(\Omega)$, $\mathcal{P}_x^{\chi\chi_0^{1/2}}(f)$ and $\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(f)$ are eigenfunctions of the algebra $\mathcal{H}(\Delta)$, associated with eigenvalues $\Lambda^{\chi\chi_0^{1/2}}$ and $\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}$ respectively. On the other hand, by Theorem 6.4.1, $\Lambda^{\chi\chi_0^{1/2}} = \Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}$. Therefore, for every $f \in L^\infty(\Omega)$, $\mathcal{P}_x^{\chi\chi_0^{1/2}}(f)$ and $\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(f)$ are eigenfunctions associated to the same eigenvalue. If $|\chi(\alpha^\vee)| < 1$, the following theorem exhibits, for every $f \in L^\infty(\Omega)$, a function $g \in L^\infty(\Omega)$ such that

$$\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(g) = c(\chi)\mathcal{P}_x^{\chi\chi_0^{1/2}}(f),$$

where $c(\chi)$ is the real non zero constant defined in Theorem 6.3.4.

Theorem 6.5.2. *Assume that $|\chi(\alpha^\vee)| < 1$; then, for every $x \in \widehat{\mathcal{V}}(\Delta)$ and for every $f \in L^\infty(\Omega)$,*

$$\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J_{x,\chi}^\alpha f) = c(\chi)\mathcal{P}_x^{\chi\chi_0^{1/2}}(f).$$

Proof. (i) First of all we prove that

$$\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J_{x,\chi}^\alpha f)(x) = c(\chi)\ \mathcal{P}_x^{\chi\chi_0^{1/2}}(f)(x). \tag{6.5.1}$$

We notice that, being $P^{\chi^{s_\alpha}\chi_0^{1/2}}(x,x,\omega) = 1$,

$$\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J_{x,\chi}^\alpha f)(x) = \int_\Omega J_{x,\chi}^\alpha f(\omega_0)\ d\nu_x(\omega_0);$$

so, by Definition 6.3.3,

$$\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J_{x,\chi}^\alpha f)(x) = \int_\Omega \left(\int_\Omega j_{x,\chi}^\alpha(\omega_0,\omega)\ f(\omega)\ d\nu_{x,\omega_0}^\alpha(\omega)\right)\ d\nu_x(\omega_0)$$

and taking into account that, for every ω, the measure $\nu_{x,\omega}^\alpha$ is the restriction of the measure ν_x to the subset $\{\omega' \in \Omega\ :\ \omega' \in [\omega]_\alpha\}$, we obtain

$$\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J_{x,\chi}^\alpha f)(x) = \int_\Omega \left(\int_\Omega j_{x,\chi}^\alpha(\omega_0,\omega)\ f(\omega)\ d\nu_x(\omega)\right)\ d\nu_x(\omega_0),$$

if we set $j_{x,\chi}^\alpha(\omega_0,\omega) = 0$, for $\omega \notin [\omega_0]_\alpha$. On the other hand,

$$\int_\Omega \left(\int_\Omega j_{x,\chi}^\alpha(\omega_0,\omega)\ f(\omega)\ d\nu_x(\omega)\right)\ d\nu_x(\omega_0) = \int_\Omega \left(\int_\Omega j_{x,\chi}^\alpha(\omega_0,\omega)\ d\nu_x(\omega_0)\right)\ f(\omega)\ d\nu_x(\omega),$$

since the integral is absolutely convergent. Therefore

$$\begin{aligned}\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J_{x,\chi}^\alpha f)(x) &= \int_\Omega \left(\int_\Omega j_{x,\chi}^\alpha(\omega_0,\omega)\ d\nu_x(\omega_0)\right)\ f(\omega)\ d\nu_x(\omega)\\ &= \int_\Omega \left(\int_\Omega j_{x,\chi}^\alpha(\omega,\omega_0)\ d\nu_x(\omega_0)\right)\ f(\omega)\ d\nu_x(\omega) = \int_\Omega J_{x,\chi}^\alpha \mathbf{1}(\omega)\ f(\omega)\ d\nu_x(\omega)\\ &= c(\chi)\ \int_\Omega f(\omega)\ d\nu_x(\omega) = c(\chi)\ \mathcal{P}_x^{\chi\chi_0^{1/2}}(f)(x).\end{aligned}$$

(ii) Now assume $y \neq x$; by Proposition 6.5.1, we have

$$\mathcal{P}_x^\chi f(y) = \mathcal{P}_y^\chi(T_{x,y}^\chi f)(y).$$

Hence, if we apply (i), replacing x with y and f with $T^{\chi}_{x,y}f$, we obtain

$$\mathcal{P}_y^{\chi^{s_\alpha}\chi_0^{1/2}}(J^\alpha_{y,\chi}(T^{\chi\chi_0^{1/2}}_{x,y}f))(y) = c(\chi)\,\mathcal{P}_y^{\chi\chi_0^{1/2}}(T^{\chi\chi_0^{1/2}}_{x,y}f)(y) = c(\chi)\,\mathcal{P}_x^{\chi\chi_0^{1/2}}f(y).$$

On the other hand, by Proposition 6.3.7,

$$\mathcal{P}_y^{\chi^{s_\alpha}\chi_0^{1/2}}(J^\alpha_{y,\chi}(T^{\chi\chi_0^{1/2}}_{x,y}f))(y) = \mathcal{P}_y^{\chi^{s_\alpha}\chi_0^{1/2}}(T^{\chi^{s_\alpha}\chi_0^{1/2}}_{x,y}(J^\alpha_{x,\chi}f))(y),$$

and applying again Proposition 6.5.1, we conclude that

$$\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J^\alpha_{x,\chi}f)(y) = c(\chi)\,\mathcal{P}_x^{\chi}f(y).$$

□

Remark 6.5.3. *Theorem 6.5.2 provides a different proof of the identity* $\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}} = \Lambda^{\chi\chi_0^{1/2}}$, *when* $|\chi(\alpha^\vee)| < 1$. *Actually, for every* $f \in L^\infty(\Omega)$, *the function* $\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(f)$ *is an eigenfunction of the algebra* $\mathcal{H}(\Delta)$ *associated with the eigenvalue* $\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}$ *and, when* $|\chi(\alpha^\vee)| < 1$, $J^\alpha_{x,\chi}f$ *belongs to* $L^\infty(\Omega)$. *Then*

$$A_\lambda\,(\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J^\alpha_{x,\chi}f)) = \Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}\,\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J^\alpha_{x,\chi}f), \quad \forall\lambda \in \widehat{L}.$$

On the other hand, for every $f \in L^\infty(\Omega)$, $\mathcal{P}_x^{\chi\chi_0^{1/2}}(f)$ *is an eigenfunction of the algebra* $\mathcal{H}(\Delta)$ *associated with the eigenvalue* $\Lambda^{\chi\chi_0^{1/2}}$, *and therefore*

$$A_\lambda\,(c(\chi)\,\mathcal{P}_x^{\chi\chi_0^{1/2}}(f)) = \Lambda^{\chi\chi_0^{1/2}}\,c(\chi)\,\mathcal{P}_x^{\chi\chi_0^{1/2}}(f), \quad \forall\lambda \in \widehat{L};$$

hence, by Theorem 6.5.2,

$$A_\lambda(\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J^\alpha_{x,\chi}f)) = \Lambda^{\chi\chi_0^{1/2}}\,\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J^\alpha_{x,\chi}f), \quad \forall\lambda \in \widehat{L}.$$

So we have proved that, if $|\chi(\alpha^\vee)| < 1$, *then, for every* $f \in L^\infty(\Omega)$, $\mathcal{P}_x^{\chi^{s_\alpha}\chi_0^{1/2}}(J^\alpha_{x,\chi}f)$ *belongs to the eigenspaces associated to both the eigenvalues* $\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}}$ *and* $\Lambda^{\chi\chi_0^{1/2}}$. *This implies that* $\Lambda^{\chi^{s_\alpha}\chi_0^{1/2}} = \Lambda^{\chi\chi_0^{1/2}}$.

7. Satake isomorphism

7.1. **Convolution operators on $\mathbb{A}$.** In this section we consider the fundamental apartment $\mathbb{A}$. The set $\widehat{\mathcal{V}}(\mathbb{A}) = \widehat{L}$ can be identified with $\mathbb{Z}^n$, if $n = |I_0|$; actually the $\mathbb{Z}$-span of the vectors $\{\lambda_i,\ i \in I_0\}$ coincides with $\mathbb{Z}^n$; then each $\lambda \in \widehat{L}$ can be identified with the element $(m_1, \cdots, m_n)$ of $\mathbb{Z}^n$, if $\lambda = \sum_{i=1}^n m_i\,\lambda_i$. Hence $\widehat{L}$ inherits the structure of finitely generated free abelian group of $\mathbb{Z}^n$. We denote by $\mathcal{L}(\widehat{L})$ the $\mathbb{C}$-algebra of all complex-valued functions on $\widehat{L}$, with finite support. Each function h in $\mathcal{L}(\widehat{L})$ determines a convolution operator on all functions on $\widehat{L}$; as usual, we set, for every function on $\widehat{L}$,

$$\tau_h(F) = h \star F.$$

Proposition 7.1.1. *Every character* χ *on* $\mathbb{A}$ *is an eigenfunction of all operators* $\tau_h,\ h \in \mathcal{L}(\widehat{L})$:

$$(\tau_h\chi) = \Theta^\chi(h)\,\chi, \quad \forall h \in \mathcal{L}(\widehat{L}),$$

with associated eigenvalue $\Theta^\chi(h) = \sum_{\mu\in\widehat{L}} h(\mu)\chi(\mu)$.

Proof. For every $\lambda \in \widehat{L}$, we can write

$$(\tau_h\chi)(\lambda) = \sum_{\mu\in\widehat{L}} h(\mu)\chi(\lambda+\mu) = \left(\sum_{\mu\in\widehat{L}} h(\mu)\chi(\mu)\right)\,\chi(\lambda).$$

□

Proposition 7.1.2. *Let* $h \in \mathcal{L}(\widehat{L})$; *then*

$$h = 0 \Longleftrightarrow \Theta^\chi(h) = 0 \ \textit{for all}\ \chi \in Hom(\widehat{L}, \mathbb{C}^\times).$$

Proof. There is a natural identification of $\widehat{L}$ with the group T of all translations $t_\lambda,\ \lambda \in \widehat{L}$. Hence $\mathcal{L}(\widehat{L})$ is the algebra $\mathcal{L}(T)$ defined by (1.1) of [8]. Using this identification and following notation of [8], the mapping

$$h \mapsto \sum_{\lambda \in \widehat{L}} h(\lambda)\ \lambda,$$

is a $\mathbb{C}$-algebra isomorphism of $\mathcal{L}(\widehat{L})$ onto the group algebra $\mathbb{C}[\widehat{L}]$ of $\widehat{L}$ over $\mathbb{C}$. Since $\widehat{L}$ is a free abelian group generated by the finite set $\{\lambda_1, \cdots, \lambda_n\}$, it follows that

$$\mathbb{C}[\widehat{L}] = \mathbb{C}[\pm\lambda_i, i = 1, \cdots, n],$$

hence it is a commutative integral domain. Consequently $\mathbb{C}[\widehat{L}]$ is the coordinate ring of an affine algebraic variety, say S, whose points are the $\mathbb{C}$-algebra homomorphisms $s : \mathbb{C}[\widehat{L}] \to \mathbb{C}$. The restriction of these homomorphisms to $\widehat{L}$ gives a bijection of S onto $\mathbf{X}(\widehat{L}) = Hom(\widehat{L}, \mathbb{C}^\times)$, and we shall identify $\mathbf{X}(\widehat{L})$ with S in this way. The elements of $\mathbb{C}[\hat{L}]$ can therefore be regarded as functions on $\mathbf{X}(\widehat{L})$. Hence, by the Nullstellensatz, if $\eta \in \mathbb{C}[\widehat{L}]$,

$$\eta = 0 \Longleftrightarrow \chi(\eta) = 0 \text{ for all } \chi \in \mathbf{X}(\widehat{L}).$$

Keeping in mind the $\mathbb{C}$-algebra isomorphism of $\mathcal{L}(\widehat{L})$ onto $\mathbb{C}[\widehat{L}]$, each χ defines a homomorphism $\mathcal{L}(\widehat{L}) \to \mathbb{C}$, namely

$$\chi(h) = \sum_{\lambda \in \widehat{L}} h(\lambda)\chi(\lambda),$$

and we have

$$h = 0 \Longleftrightarrow \chi(h) = 0, \text{ for all } \chi \in \mathbf{X}(\widehat{L}).$$

On the other hand, for every h in $\mathcal{L}(\widehat{L})$, $\chi(h) = \Theta^\chi(h)$, according to Proposition 7.1.1; hence

$$h = 0 \Longleftrightarrow \Theta^\chi(h) = 0, \text{ for all } \chi \in \mathbf{X}(\widehat{L}).$$

□

7.2. **The Hecke algebra on $\mathbb{A}$.** The group $\mathbf{W}$ acts on $\mathcal{L}(\widehat{L})$ in the following way: for every $h \in \mathcal{L}(\widehat{L})$,

$$h^{\mathbf{w}}(\lambda) = (\mathbf{w}h)(\lambda) = h(\mathbf{w}^{-1}(\lambda)), \quad \forall \lambda \in \widehat{L}.$$

We denote by $\mathcal{L}(\widehat{L})^{\mathbf{W}}$ the subring of $\mathcal{L}(\widehat{L})$, consisting of all $\mathbf{W}$-invariant functions in $\mathcal{L}(\widehat{L})$, that is the functions h in $\mathcal{L}(\widehat{L})$ such that $h^{\mathbf{w}} = h$, for every $\mathbf{w} \in \mathbf{W}$.

Proposition 7.2.1. *For every h in $\mathcal{L}(\widehat{L})^{\mathbf{W}}$, the operator τ_h is $\mathbf{W}$-invariant, i. e. for every $\mathbf{w} \in \mathbf{W}$, and for every function F on $\widehat{L}$,*

$$\tau_h(F^{\mathbf{w}}) = (\tau_h F)^{\mathbf{w}}.$$

Proof. Fix any $\mathbf{w} \in \mathbf{W}$. For every function F, and for every λ, we write, using the $\mathbf{W}$-invariance of h,

$$(\tau_h F)(\mathbf{w}^{-1}(\lambda)) = \sum_{\mu \in \widehat{L}} h(\mu)\ F(\mathbf{w}^{-1}(\lambda) + \mu) = \sum_{\mu \in \widehat{L}} h(\mathbf{w}(\mu))\ F(\mathbf{w}^{-1}(\lambda) + \mu),$$

and by setting $\mathbf{w}(\mu) = \mu'$,

$$\begin{aligned}(\tau_h F)(\mathbf{w}^{-1}(\lambda)) &= \sum_{\mu' \in \widehat{L}} h(\mu')\ F(\mathbf{w}^{-1}(\lambda) + \mathbf{w}^{-1}(\mu')) = \sum_{\mu' \in \widehat{L}} h(\mu')\ F(\mathbf{w}^{-1}(\lambda + \mu')) \\ &= \sum_{\mu' \in \widehat{L}} h(\mu')\ F^{\mathbf{w}}(\lambda + \mu') = (\tau_h F^{\mathbf{w}})(\lambda).\end{aligned}$$

□

We set

$$\mathcal{H}(\mathbb{A}) = \{\tau_h\ , h \in \mathcal{L}(\hat{L})^{\mathbf{W}}\}.$$

Obviously, $\mathcal{H}(\mathbb{A})$ is a $\mathbb{C}$- algebra; following Humphreys ([6]), we call $\mathcal{H}(\mathbb{A})$ the *Hecke algebra on $\mathbb{A}$.*

Proposition 7.1.1 implies that every character χ on $\widehat{L}$ is an eigenfunction of the whole algebra $\mathcal{H}(\mathbb{A})$. We denote by Θ^χ the associated eigenvalue, that is the homomorphism from the algebra $\mathcal{H}(\mathbb{A})$ to $\mathbb{C}^\times$ such that, for every operator $\tau_h \in \mathcal{H}(\mathbb{A})$, $\Theta^\chi(\tau_h)$ is the eigenvalue associated to the eigenfunction χ of the operator τ_h. Then, for every $h \in \mathcal{L}(\widehat{L})^{\mathbf{W}}$,

$$\Theta^\chi(\tau_h) = \Theta^\chi(h) = \sum_{\mu \in \widehat{L}} h(\mu)\chi(\mu).$$

We notice that the restriction to $\widehat{L}$ of Θ^χ is the character χ. Keeping in mind this fact, we easily obtain the following proposition.

Proposition 7.2.2. *For every eigenvalue Θ of the Hecke algebra of $\mathbb{A}$ there exists a character χ on $\widehat{L}$ such that*

$$\Theta = \Theta^\chi.$$

Proof. For every $\lambda \in \widehat{L}$, let δ_λ be the function on $\widehat{L}$ such that $\delta_\lambda(\lambda) = 1$ and $\delta_\lambda(\mu) = 0$, for every $\mu \neq \lambda$. Then each $h \in \mathcal{L}(\widehat{L})^{\mathbf{W}}$ can be written as $h = \sum_\lambda h(\lambda)\delta_\lambda$. Let Θ be any eigenvalue of $\mathcal{H}(\mathbb{A})$ and let χ be its restriction to $\widehat{L}$, that is

$$\chi(\lambda) = \Theta(\delta_\lambda), \ \forall \lambda \in \widehat{L}.$$

It is immediate to observe that χ belongs to $\mathbf{X}(\widehat{L})$ and, for every $h \in \mathcal{L}(\widehat{L})^{\mathbf{W}}$, we have

$$\Theta(h) = \sum_\lambda h(\lambda)\Theta(\delta_\lambda) = \sum_\lambda h(\lambda)\chi(\lambda) = \Theta^\chi(h).$$

This implies that $\Theta = \Theta^\chi$. □

7.3. **Operators $\widetilde{A}_\lambda$.** Assume that ω is a fixed boundary point of the building. For every $\lambda \in \widehat{L}^+$ and for every vertex $\mu \in \widehat{L}$, the number $N(\lambda, \mu)$, defined in (3.3.2) with respect to ω, does not depend on the choice of ω.

For every $\lambda \in \widehat{L}^+$, let h_λ be the following function on $\widehat{L}$:

$$h_\lambda(\mu) = \chi_0^{1/2}(\mu)\ N(\lambda, \mu), \quad \forall \mu \in \widehat{L}.$$

Since $N(\lambda, \mu) = 0$ but for finitely many $\mu \in \hat{L}$, then $h_\lambda \in \mathcal{L}(\widehat{L})$.

Definition 7.3.1. *For every $\lambda \in \widehat{L}^+$, we denote by $\widetilde{A}_\lambda$ the convolution operator associated with the function h_λ, that is*

$$\widetilde{A}_\lambda F(\mu) = h_\lambda \star F(\mu) = \sum_{\mu' \in \widehat{L}} N(\lambda, \mu')\ \chi_0^{1/2}(\mu')\ F(\mu + \mu'), \quad \forall \mu \in \widehat{L},$$

for every function F on $\widehat{L}$.

Proposition 7.1.1 implies that every character χ on $\widehat{L}$ is an eigenfunction of the operator $\widetilde{A}_\lambda$, with associated eigenvalue

$$\Theta^\chi(\lambda) = \Theta^\chi(h_\lambda) = \sum_{\mu \in \widehat{L}} h_\lambda(\mu)\ \chi(\mu) = \sum_{\mu \in \widehat{L}} N(\lambda, \mu)\ \chi_0^{1/2}(\mu)\ \chi(\mu).$$

If we recall the expression of the eigenvalue $\Lambda^\chi(\lambda)$ of the operator $A_\lambda \in \mathcal{H}(\Delta)$ given in Section 6, it is obvious that

$$\Theta^\chi(\lambda) = \Lambda^{\chi\chi_0^{1/2}}(\lambda). \tag{7.3.1}$$

Now we can prove that, for every $\lambda \in \widehat{L}^+$, the function h_λ belongs to $\mathcal{L}(\widehat{L})^{\mathbf{W}}$.

Proposition 7.3.2. *For every $\mathbf{w} \in \mathbf{W}$, then $h_\lambda = h_\lambda^{\mathbf{w}}$.*

Proof. Since the Weyl group $\mathbf{W}$ is generated by reflections s_α, $\alpha \in B$, we only need to prove that $h_\lambda = h_\lambda^{s_\alpha}$, for every simple root α. Fix any s_α and consider, for every $\mu \in \widehat{L}$, the function

$$h_\lambda^{s_\alpha}(\mu) = \chi_0^{1/2}(s_\alpha(\mu))N(\lambda, s_\alpha(\mu)), \quad \forall \mu \in \widehat{L}.$$

For every character χ and every $\mu \in \widehat{L}$, we have

$$h_\lambda \star \chi(\mu) = \Theta^\chi(h_\lambda)\ \chi(\mu), \qquad h_\lambda^{s_\alpha} \star \chi(\mu) = \Theta^\chi(h_\lambda^{s_\alpha})\ \chi(\mu).$$

On the other hand, as we have noticed before,

$$\Theta^\chi(h_\lambda) = \sum_{\mu \in \hat{L}} N(\lambda, \mu)\ \chi_0^{1/2}(\mu)\ \chi(\mu) = \Lambda^{\chi\chi_0^{1/2}}(\lambda)$$

and, by setting $\mu' = s_\alpha(\mu)$,

$$\Theta^\chi(h_\lambda^{s_\alpha}) = \sum_{\mu \in \widehat{L}} N(\lambda, s_\alpha(\mu))\ \chi_0^{1/2}(s_\alpha(\mu))\ \chi(\mu) = \sum_{\mu' \in \widehat{L}} N(\lambda, \mu')\chi_0^{1/2}(\mu')\chi^{s_\alpha}(\mu') = \Lambda_\lambda^{(\chi^{s_\alpha})\chi_0^{1/2}}.$$

Thus, Theorem 6.4.1 implies $\Theta^\chi(h_\lambda^{s_\alpha}) = \Theta^\chi(h_\lambda)$, for every χ. So $h_\lambda = h_\lambda^{s_\alpha}$, by Proposition 7.1.2. □

As an obvious consequence of Proposition 7.2.1 and Proposition 7.3.2, we obtain

Corollary 7.3.3. *For every $\lambda \in \widehat{L}^+$, the operator $\widetilde{A}_\lambda$ belongs to the Hecke algebra $\mathcal{H}(\mathbb{A})$.*

Proposition 7.3.4. *The operators $\widetilde{A}_\lambda$, $\lambda \in \widehat{L}^+$, form a $\mathbb{C}$-basis of $\mathcal{H}(\mathbb{A})$.*

Proof. We only need to show that the functions h_λ, $\lambda \in \widehat{L}^+$, form a $\mathbb{C}$-basis of $\mathcal{L}(\widehat{L})^{\mathbf{W}}$. For each $\lambda \in \widehat{L}^+$, let ξ_λ be the characteristic function of the $\mathbf{W}$-orbit of λ. Then the functions ξ_λ, as λ runs through $\widehat{L}^+$, form a $\mathbb{C}$-basis of $\mathcal{L}(\widehat{L})^{\mathbf{W}}$. Hence, we can write, summing on all λ' in $\widehat{L}^+$,

$$h_\lambda = \sum_{\lambda'} h_\lambda(\lambda')\ \xi_{\lambda'}.$$

Since $N(\lambda, \lambda) = 1$, then $h_\lambda(\lambda) = \chi_0^{1/2}(\lambda)$. Consequently the previous sum takes the form

$$h_\lambda = \chi_0^{1/2}(\lambda)\ \xi_\lambda + \sum_{\lambda' \neq \lambda} h_\lambda(\lambda')\ \xi_{\lambda'}$$

and in this sum $h_\lambda(\lambda') = 0$, but for $\lambda' \in \Pi_\lambda$. Since $\chi_0^{1/2}(\lambda) \neq 0$, we conclude that the h_λ form a $\mathbb{C}$-basis of $\mathcal{L}(\hat{L})^{\mathbf{W}}$. $\square$

Definition 7.3.5. *For every $\lambda \in \widehat{L}^+$, let g_λ be the function of $\mathcal{L}(\widehat{L})$, defined as $g_\lambda(\mu) = N(\lambda, \mu)$, for every $\mu \in \widehat{L}$. We denote by B_λ the following operator acting on the complex-valued functions F on $\widehat{L}$:*

$$B_\lambda F(\mu) = g_\lambda \star F(\mu) = \sum_{\mu' \in \widehat{L}} N(\lambda, \mu')\ F(\mu + \mu'), \quad \forall \mu \in \widehat{L}.$$

We notice that the operator B_λ is linear and invariant with respect to any translation in $\mathbb{A}$, as their coefficients $N(\lambda, \mu')$ do not depend on μ. However, B_λ is not $\mathbf{W}$-invariant, because g_λ does not belong to $\mathcal{L}(\hat{L})^{\mathbf{W}}$, as $N(\lambda, \mu) \neq N(\lambda, \mathbf{w}^{-1}\mu)$ for $\mathbf{w} \in \mathbf{W}$ different from the identity. The following proposition relates the operator B_λ to the operator A_λ.

Proposition 7.3.6. *For every function F on $\widehat{L}$, let*

$$f(x) = F(\rho_\omega(x)), \quad \textit{for every} \quad x \in \widehat{\mathcal{V}}(\Delta).$$

Then, for every $\lambda \in \widehat{L}^+$,

$$A_\lambda f(x) = B_\lambda F(\mu), \quad \textit{if} \quad \mu = \rho_\omega(x).$$

Proof. By definition of A_λ, we can write, for every function f,

$$A_\lambda(f)(x) = \sum_{y \in V_\lambda(x)} f(y) = \sum_{\nu \in \widehat{L}} \left(\sum_{\{y : \sigma(x,y) = \lambda,\ \rho_\omega(y) = \nu\}} f(y) \right).$$

In the case when $f(x) = F(\rho_\omega(x))$, then, for every $\nu \in \widehat{L}$, $f(y) = F(\nu)$, for all y such that $\rho_\omega(y) = \nu$. Hence, by setting $\mu = \rho_\omega(x)$ and $\mu + \mu' = \nu$, we have

$$A_\lambda(f)(x) = \sum_{\mu' \in \widehat{L}} N(\lambda, \mu') F(\mu + \mu') = B_\lambda F(\mu).$$

$\square$

The operators B_λ and $\widetilde{A}_\lambda$ are related by simple relations, as the following proposition states.

Proposition 7.3.7. *For every $\lambda \in \widehat{L}^+$ and every function F,*

$$\widetilde{A}_\lambda F = \chi_0^{-1/2}\ B_\lambda(\chi_0^{1/2} F), \qquad B_\lambda F = \chi_0^{1/2}\ \widetilde{A}_\lambda(\chi_0^{-1/2} F).$$

Proof. For every $\mu \in \widehat{L}$, we have, by Definitions 7.3.1 and 7.3.5,

$$\begin{aligned}(\widetilde{A}_\lambda F)(\mu) &= \sum_{\mu' \in \widehat{L}} N(\lambda, \mu')\ \chi_0^{1/2}(\mu')\ F(\mu + \mu') = \chi_0^{-1/2}(\mu) \sum_{\mu' \in \widehat{L}} N(\lambda, \mu')\ \chi_0^{1/2}(\mu + \mu')\ F(\mu + \mu') \\ &= \chi_0^{-1/2}(\mu)\ B_\lambda(\chi_0^{1/2} F)(\mu).\end{aligned}$$

Moreover

$$\begin{aligned}(B_\lambda F)(\mu) &= \sum_{\mu' \in \widehat{L}} N(\lambda, \mu')\ F(\mu + \mu') = \chi_0^{1/2}(\mu) \sum_{\mu' \in \widehat{L}} N(\lambda, \mu')\ \chi_0^{1/2}(\mu')\chi_0^{-1/2}(\mu + \mu')\ F(\mu + \mu') \\ &= \chi_0^{1/2}(\mu) \sum_{\mu' \in \widehat{L}} N(\lambda, \mu')\ \chi_0^{1/2}(\mu')(\chi_0^{-1/2} F)(\mu + \mu') = \chi_0^{1/2}(\mu)\ \widetilde{A}_\lambda(\chi_0^{-1/2} F)(\mu).\end{aligned}$$

$\square$

7.4. **Satake isomorphism.** Consider the mapping

$$i : A_\lambda \to \widetilde{A}_\lambda, \quad \text{for all } \lambda \in \widehat{L}^+.$$

Since $\{A_\lambda,\ \lambda \in \widehat{L}^+\}$ is a basis for the algebra $\mathcal{H}(\mathbb{A})$, we extend this map to the whole Hecke algebra $\mathcal{H}(\Delta)$ by linearity. We shall prove that $i : \mathcal{H}(\Delta) \to \mathcal{H}(\mathbb{A})$ is a $\mathbb{C}$-algebra isomomorphism.

Theorem 7.4.1. *The mapping $i : A_\lambda \to \widetilde{A}_\lambda$ is a $\mathbb{C}$-algebra isomorphism of $\mathcal{H}(\Delta)$ onto $\mathcal{H}(\mathbb{A})$.*

Proof. First of all, we prove that i is a $\mathbb{C}$-algebra homomorphism from $\mathcal{H}(\Delta)$ to $\mathcal{H}(\mathbb{A})$. By definition, if $A = \sum_{j=1}^k c_j\ A_{\lambda_j}$, then

$$i(A) = \sum_{j=1}^k c_j i(A_{\lambda_j}) = \sum_{j=1}^k c_j \widetilde{A}_{\lambda_j}.$$

Consider now, for any pair $\lambda, \lambda' \in \widehat{L}^+$, the operator $A_\lambda \circ A_{\lambda'}$ and prove that

$$i(A_\lambda \circ A_{\lambda'}) = i(A_\lambda) \circ i(A_{\lambda'}).$$

We know that $A_\lambda \circ A_{\lambda'}$ is a linear combination of operators $A_{\lambda_1}, \cdots, A_{\lambda_k}$, for convenient $\lambda_1, \cdots, \lambda_k$:

$$(A_\lambda \circ A_{\lambda'})f = \sum_{j=1}^k\ c_j A_{\lambda_j} f.$$

Hence, $i(A_\lambda \circ A_{\lambda'}) = \tau_{h_{\lambda,\lambda'}}$, if $h_{\lambda,\lambda'}$ is the $\mathbf{W}$-invariant function on $\widehat{L}$, defined as

$$h_{\lambda,\lambda'} = \sum_{j=1}^k\ c_j h_{\lambda_j}.$$

This proves that $i(A_\lambda \circ A_{\lambda'})$ belongs to the algebra $\mathcal{H}(\mathbb{A})$.

Now we prove that, for every pair λ, λ',

$$i(A_\lambda \circ A_{\lambda'}) = i(A_\lambda) \circ i(A_{\lambda'}).$$

To this end, we consider, for every character χ, the eigenvalue $\Theta^\chi(h_{\lambda,\lambda'})$; for ease of notation, we set $\Theta^\chi(\lambda, \lambda') = \Theta^\chi(h_{\lambda,\lambda'})$. Since $\tau_{h_{\lambda,\lambda'}} = \sum_{j=1}^k\ c_j \tau_{h_{\lambda_j}}$, we have

$$\Theta^\chi(\lambda, \lambda') = \sum_{j=1}^k\ c_j \Theta^\chi(\lambda_j).$$

Therefore, keeping in mind (7.3.1),

$$\Theta^\chi(\lambda, \lambda') = \sum_{j=1}^k\ c_j \Lambda^{\chi\chi_0^{1/2}}(\lambda_j) = \Lambda^{\chi\chi_0^{1/2}}(A_\lambda \circ A_{\lambda'}) = \Lambda^{\chi\chi_0^{1/2}}(\lambda)\ \Lambda^{\chi\chi_0^{1/2}}(\lambda') = \Theta^\chi(\lambda)\ \Theta^\chi(\lambda').$$

So we have

$$\Theta^\chi(i(A_\lambda \circ A_{\lambda'})) = \Theta^\chi(i(A_\lambda))\Theta^\chi(i(A_{\lambda'})) = \Theta^\chi(i(A_\lambda) \circ i(A_{\lambda'})),$$

for every χ. Thus Proposition 7.1.2 implies that $i(A_\lambda \circ A_{\lambda'})) = i(A_\lambda) \circ i(A_{\lambda'})$. This proves that i is a $\mathbb{C}$-algebra homomorphism from $\mathcal{H}(\Delta)$ to $\mathcal{H}(\mathbb{A})$.

Since the operators A_λ form a $\mathbb{C}$-basis of $\mathcal{H}(\Delta)$ and, according to Proposition 7.3.4, the operators $\widetilde{A}_\lambda = i(A_\lambda)$ form a $\mathbb{C}$-basis of $\mathcal{H}(\mathbb{A})$, it follows immediately that the operator i is a bijection from the algebra $\mathcal{H}(\Delta)$ onto the algebra $\mathcal{H}(\mathbb{A})$. □

We shall call the operator i the *Satake isomorphism* between $\mathcal{H}(\Delta)$ and $\mathcal{H}(\mathbb{A})$.

7.5. **Characterization of the eigenvalues of the algebra $\mathcal{H}(\Delta)$.** We proved in Section 7.1 that, for every eigenvalue Θ of the algebra $\mathcal{H}(\mathbb{A})$ there exists a character χ, such that $\Theta = \Theta^\chi$. The Satake isomorphism between $\mathcal{H}(\Delta)$ and $\mathcal{H}(\mathbb{A})$ allows us to extend this characterization to the eigenvalues of the algebra $\mathcal{H}(\Delta)$.

Corollary 7.5.1. *For every eigenvalue Λ of the algebra $\mathcal{H}(\Delta)$ there exists a character χ on $\widehat{L}$ such that $\Lambda = \Lambda^{\chi\chi_0^{1/2}}$.*

Proof. Let Λ be an eigenvalue of the algebra $\mathcal{H}(\Delta)$. By Theorem 7.4.1, there exists a unique eigenvalue $\Theta \in Hom(\mathcal{H}(\mathbb{A}), \mathbb{C})$, such that

$$\Theta(\lambda) = \Lambda(\lambda), \quad \text{for every} \quad \lambda \in \widehat{L}^{+}.$$

Since, by Proposition 7.2.2, there exists a character χ such that $\Theta = \Theta^{\chi}$, and taking in account the identity (7.3.1), we conclude that $\Lambda = \Lambda^{\chi\chi_0^{1/2}}$. $\square$

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