CONDITION NUMBER THEOREMS IN LINEAR - QUADRATIC OPTIMIZATION

TULLIO ZOLEZZI *

Department of Mathematics, University of Genova, via Dodecaneso 35, 16146 Genova, Italy.

Abstract. The condition number of a given mathematical problem is often related to the reciprocal of its distance from the set of ill - conditioned problems. Such a property is proved here for linear - quadratic convex optimization problems in the infinite - dimensional setting. A uniform version of such theorem is obtained for suitably equi - bounded classes of optimization problems. An application to the conditioning of a Ritz method is presented.

Key words. Condition numbers, linear - quadratic optimization, condition number theorem.

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1. Introduction. A standard definition of conditioning of a mathematical problem (see e.g. [10, 13]) is the following. Let X, Y be linear normed spaces, with given nonempty subsets $D \subset X, U \subset Y$. D is the set of the data, U is the set of the solutions. Fix any $d^* \in D$. Then the problem corresponding to d^* is called well - conditioned if there exists a unique solution $s(d) \in U$ for each data $d \in D$ sufficiently close to d^*, and the (absolute) condition number of the problem corresponding to d^*, namely

$$\text{cond } (d^*) = \limsup_{d \to d^*} \frac{\|s(d) - s(d^*)\|}{\|d - d^*\|},$$

is finite. Otherwise the problem is called ill - conditioned. Thus cond (d^*) is a measure of the sensitivity of the solution with respect to small changes in problem' s data, as measured in the norms of X and Y.

The notion of conditioning plays, as well known, a basic role in the analysis of the behavior of many numerical solution methods of a given problem. Moreover, a fundamental property of the condition number is that the (appropriately defined) distance of the given problem to the set of ill - conditioned problems is proportional (sometimes equal) to the reciprocal of the condition number. This property, often referred to as the condition number theorem, is linked in a significant way to computational complexity theory and related topics. See e.g. [8, 9, 10, 17, 18], and the book [2] for an interesting survey (especially sections 11, 12, 13, 14 of chapter II there).

Scalar optimization problems are no exceptions to this behavior. For unconstrained convex quadratic optimization, the condition number theorem has been proved in [19]

*E - mail : zolezzi@dima.unige.it

generalizing to the infinite - dimensional setting the classical distance theorem of Eckart - Young in numerical linear algebra. (See also [12] for earlier results about the distance to the set of singular operators in the infinite - dimensional setting.) For generalizations and further properties see [20, 21, 22]. Condition numbers of convex sets are defined in [7].

A different approach for defining conditioning related to (mostly finite - dimensional) linear and convex programming has been introduced in [17, 18] and further developed e.g. in [5, 6, 16]. There the condition number is defined in terms of some measure of the distance to infeasibility, or ill - posedness. Then it is proved that such conditioning concepts are related to the behavior of the optimal solutions under changes of problem's data. In a sense, the approach pursued here is reversed. As in [19, 20] we start with the standard definition of the condition number as the sensitivity of the optimal solution under data perturbations, as described above. Then we show how such a definition is related to the reciprocal of the distance to ill - conditioning.

Aim of this paper is to prove a new condition number theorem for infinite - dimensional optimization problems of the following type: minimize the convex quadratic form

$$x \to \frac{1}{2} < Ax, x >$$

subject to the constraint

$$Sx = 0,$$

where A and S are linear bounded operators with A non negative. The results are new even in the finite - dimensional setting. The problem's data are small perturbations of the objective function and the constraint. Thus we consider the minimizer $m(b, p)$ of

$$x \to \frac{1}{2} < Ax, x > - < p, x >$$

subject to the constraint

$$Sx = b.$$

Then we consider the condition number of the problem corresponding to $p = 0, b = 0$ as defined by

$$\text{lim sup }_{(b,p)\to(0,0)} \frac{\|m(b,p)\|}{\|b\| + \|p\|},$$

under suitable uniqueness requirements.

Compared with the approach of [19, section 4] the present definition takes into account perturbations of the constraint, which were not considered in [19] (so that a direct comparison with those results is not possible).

In section 2 we present the basic setting. In section 3 we characterize well - conditioned problems by coercivity of the linear operator A defining the quadratic form we minimize, and surjectivity of the operator S defining the constraint. An explicit formula for the condition number is obtained in section 4 with the help of the Lagrange multiplier theorem. The main results are presented in section 5. First we show that the distance from ill - conditioning is bounded from above and below by suitable multiples of the reciprocal of the condition number. Then we find classes of linear - quadratic problems which fulfill suitable equi - boundedness properties, yielding uniform versions of the condition number theorem for such optimization problems. In a sense we deal with uniformly well - conditioned problems. The finite - dimensional setting is considered in section 6. The significant example of Ritz - type approximations to a well - conditioned problem is presented. We show that the distance of the finite - dimensional problems to ill - conditioning remains uniformly bounded from below, as an application of the previous results.

2. Notations and basic notions. We consider two real Banach spaces E, F with E reflexive and F non trivial. Let $B(E, F)$ denote the space of all linear bounded operators between E and F equipped with the operator norm. For any normed space X we denote by X^* its dual and by $< u, v >$ the duality pairing between $u \in X^*$ and $v \in X$; S^* denotes the adjoint od the operator S and Y^T is the transpose of the matrix Y. Consider the set $N(E, E^*)$ of the symmetric non negative operators $A \in B(E, E^*)$. Given $A \in N(E, E^*)$ and $S \in B(E, F)$ we deal with the optimization problem $P = (A, S)$, to minimize

$$x \to \frac{1}{2} < Ax, x >$$

subject to the constraint

$$Sx = 0, \ x \in E.$$

Of course $x = 0$ is an optimal solution of P. Along with (A, S) we consider the family of perturbed problems (A, S, b, p) with $p \in E^*$ and $b \in F$, to minimize

$$x \to \frac{1}{2} < Ax, x > - < p, x >$$

subject to the constraint

$$Sx = b, x \in E. \tag{1}$$

Of course $(A, S) = (A, S, 0, 0)$. We posit the following definitions. Problem (A, S) is *well - conditioned* if

I) there exists $\delta > 0$ such that (A, S, b, p) has a unique optimal solution

$$m(b, p) = \text{ arg min } (A, S, b, p)$$

for each $p \in E^*$ and $b \in F$ such that $\|p\| < \delta, \|b\| < \delta$;

II)

$$\text{lim sup }_{(b,p)\to(0,0)} \frac{\|m(b, p)\|}{\|b\| + \|p\|} = \text{ cond } (A, S) < +\infty$$

(strong convergence of $(b,p) \to (0,0)$ in $F \times E^*$). The extended real number cond (A,S) defined by II) is called the *condition number* of (A,S). The set of all well-conditioned problems (A,S) is denoted by WC, while IC denotes the set of *ill - conditioned* problems, namely those (A,S) for which at least one of I), II) fails.

As well known, we have the following Lagrange multipliers theorem (see e.g. [1, theorem 1.8 p. 185].

THEOREM 1. *Let $A \in N(E,E^*)$, let $S \in B(E,F)$ have a closed range, $b \in S(E), p \in E^*$. Then $y = m(b,p)$ iff $y \in E$ and there exists $u \in F^*$ such that*

$$Ay = p + S^* u, \; Sy = b.$$

The distance between two optimization problems

$$(A_1, S_1), (A_2, S_2) \in N(E,E^*) \times B(E,F)$$

is given through the operator norms, namely

$$\text{dist } [(A_1,S_1),(A_2,S_2)] = \|A_1 - A_2\| + \|S_1 - S_2\|,$$

and for a fixed problem (A,S),

$$\text{dist } [(A,S), IC] = \inf \{ \text{ dist } [(A,S),(B,T)] : (B,T) \in IC \}.$$

We write for short

$$P = (A,S) \text{ is well - conditioned}$$

instead of $A \in N(E,E^*), S \in B(E,F)$ and the optimization problem (A,S) is well - conditioned.

3. Characterization of well - conditioned problems. Well - conditioning is characterized in the next result via coercivity of the quadratic form and surjectivity of the operator defining the constraint.

THEOREM 2. *Let $A \in N(E,E^*)$ and $S \in B(E,F)$. Then the following properties are equivalent:*

$$(A,S) \textit{ is well - conditioned }; \tag{2}$$

$$S \textit{ is onto and for some constant } \alpha > 0 \textit{ we have } < Ax, x > \geq \alpha \|x\|^2 \textit{ if } Sx = 0. \tag{3}$$

Proof. Let (A,S) be well - conditioned. Then, by condition I), the constraint (1) must be feasible for all b sufficiently small, hence S is onto. By condition I) and theorem 1, for every sufficiently small $b \in F, p \in E^*$ there exist solutions $x \in E, u \in F^*$ of

$$Ax = p + S^* u, \; Sx = b. \tag{4}$$

If (x_1, u_1) and (x_2, u_2) are solutions of (4), then $x_1 = x_2$ by I), hence $u_1 = u_2$ by injectivity of S^*. Then standard reasoning shows that (4) has a unique solution (x, u) for every $b \in F, p \in E^*$. Thus (3) follows by [3, lemma 4.124]. Conversely, let (3) hold. For all sufficiently small $b \in F$ there exists $\bar{x} \in E$ such that $S\bar{x} = b$ and

$$\|\bar{x}\| \leq k\|b\| \tag{5}$$

for a suitable constant k, see [11, lemma 1 p. 487]. Then every x fulfilling the feasible constraint (1) can be written as

$$x = \bar{x} + y \text{ with } Sy = 0. \tag{6}$$

Fix any $p \in E^*, b \in F$ (sufficiently small) and a corresponding $\bar{x}$ fulfilling (5). Then, by (3) and (6), for every $x \in E$ with $Sx = b$ we have

$$\frac{1}{2} < Ax, x > - < p, x >= \frac{1}{2} < A(\bar{x} + y), \bar{x} + y > - < p, \bar{x} + y > \geq \frac{\alpha}{2}\|y\|^2 + C_1\|y\| + C_2$$

with suitable constants C_1, C_2. By standard reasoning based on boundedness of the minimizing sequences, lower semicontinuity and reflexivity of E, it follows that problem (A, S, b, p) has optimal solutions. Let x_1, x_2 be two of them, then by theorem 1 for suitable $u_1, u_2 \in F^*$ we have

$$Ax_1 = p + S^*u_1,\ Ax_2 = p + S^*u_2,\ Sx_1 = Sx_2 = b$$

hence

$$\alpha\|x_1 - x_2\|^2 \leq < A(x_1 - x_2), x_1 - x_2 >=< S^*(u_1 - u_2), x_1 - x_2 >=< u_1 - u_2, S(x_1 - x_2) >= 0$$

whence $x_1 = x_2$. It follows that (A, S, b, p) has a unique optimal solution

$$x = \bar{x} + y,\ Sy = 0. \tag{7}$$

Then by theorem 1 and a suitable $u \in F^*$,

$$Ax = A\bar{x} + Ay = p + S^*u$$

hence by (3), (5) and (7)

$$\alpha\|y\|^2 \leq < Ay, y >=< u, Sy > + < p - A\bar{x}, y >=< p - A\bar{x}, y > \leq \|p\|\|y\| + k\|A\|\|y\|\|b\|.$$

Then

$$\|x\| \leq \|\bar{x}\| + \|y\| \leq L(\|b\| + \|p\|)$$

for a suitable constant L, so that

$$\limsup_{(b,p)\to(0,0)} \frac{\|x\|}{\|b\| + \|p\|} \leq L,$$

and the proof is completed.

4. A formula for the condition number. In order to prove a form of the condition number theorem we shall rely on an explicit formula. We need two preliminary results, as follows.

LEMMA 3. *Let* $P \in B(E^*, E), Q \in B(F, E)$. *Then*

$$\textit{lim sup}_{\ (x,y)\to(0,0)} \frac{\|Px+Qy\|}{\|x\|+\|y\|} = \ \textit{max}\ \{\|P\|, \|Q\|\}$$

(where $(x, y) \to (0, 0)$ *in the strong convergence of* $E^* \times F$*).*

Proof. For all x and y

$$\|Px + Qy\| \le (\|x\| + \|y\|) \max \{\|P\|, \|Q\|\}. \tag{8}$$

Let e.g. $\|P\| \ge \|Q\|$. There exists a sequence $x_n \in E^*, \|x_n\| \le 1$ such that $\|Px_n\| \to \|P\|$. Thus by (8)

$$\sup \{\|Px + Qy\| : \|x\| + \|y\| \le 1\} = \|P\| = \ \max \{\|P\|, \|Q\|\}.$$

The conclusion follows because

$$(x, y) \to \frac{\|Px+Qy\|}{\|x\|+\|y\|}$$

is positively homogeneous of degree 0.

We shall now employ, for a given $A \in N(E, E^*)$, the coercivity condition

$$\alpha\|x\|^2 \ \le <Ax, x> \le \ \omega\|x\|^2, x \in E \tag{9}$$

for fixed positive constants α, ω.

LEMMA 4. *Let* (A, S) *be well - conditioned and assume (9). Then*

$$D = SA^{-1}S^* : F^* \to F \tag{10}$$

is an isomorphism.

Proof. As easily checked, D is a linear bounded symmetric operator. Taking the Fenchel conjugates in (9) we have

$$\frac{\|u\|^2}{\omega} \ \le <u, A^{-1}u> \le \ \frac{\|u\|^2}{\alpha}, u \in E^* \tag{11}$$

hence for all $x \in F^*$

$$< x, Dx > \geq \frac{\|S^* x\|^2}{\omega}. \tag{12}$$

S is onto by theorem 2, hence by [4, theorem II.19]

$$\|x\| \leq \text{ (const.) } \|S^* x\|, x \in F^*$$

whence by (12)

$$< x, Dx > \geq \text{ (const.) } \|x\|^2 \tag{13}$$

showing that D is one - to - one. Symmetry of D and (13) show that D is onto (again by [4, theorem II.19]), ending the proof.

The explicit formula we need for the condition number is obtained as follows.

THEOREM 5. *Let* (A, S) *be well - conditioned and assume (9). Let* D *be given by (10). Then*

$$\textit{cond } (A, S) = \textit{ max } \{\|A^{-1} S^* D^{-1}\|, \|A^{-1} - A^{-1} S^* D^{-1} S A^{-1}\|\}.$$

Proof. Given $p \in E^*$ and $b \in E$, by theorem 1, if $x = m(b, p)$ there exists $u \in F^*$ such that

$$x = A^{-1} p + A^{-1} S^* u, \; Sx = b,$$

thus

$$Du = b - SA^{-1} p,$$

and by lemma 4

$$x = A^{-1} p + A^{-1} S^* D^{-1} (b - SA^{-1} p).$$

The conclusion follows by lemma 3.

5. Condition number theorems. Here we relate the distance of well - conditioned problems from IC to their condition numbers. First we prove a condition number formula for a fixed optimization problem. Next we prove a uniform version of the condition number theorem dealing with suitable classes of linear - quadratic problems. This is motivated, for example, by applications to optimization problems depending on parameters, and to the conditioning of finite - dimensional approximations of a given infinite - dimensional problem as considered in section 6. In order to obtain such a result, we shall need to properly define classes of well - conditioned problems fulfilling appropriate uniform boundedness conditions.

Let $S \in B(E, F)$ be onto, and consider

$$c^* = c^*(S) = \text{ inf } \{c > 0 : \|x\| \leq c \|S^* x\| \text{ for all } x \in F^*\}, \tag{14}$$

$$k = k(S) = \sup \{h > 0 : T \in B(E,F), \|S - T\| \leq h \text{ imply } T \text{ onto } \}. \tag{15}$$

Since S is onto, k and c^* are well - defined (see [4, theorem II. 19]), and k is not $+\infty$ since F is nontrivial. Of course k is the distance of S to the set of non surjective operators. We need the following (possibly known)

LEMMA 6. *If S is onto, then*

$$k = \frac{1}{c^*}.$$

Proof. Given $T \in B(E,F)$ consider the reduced minimum modulus of T (see [15, p. 231]) given by

$$\gamma(T) = \sup \{m > 0 : \|Tx\| \geq m \text{ dist } (x, \ker T) \text{ for every } x \in E\},$$

and

$$\sigma(S) = \inf \{\alpha > 0 : \text{ for every } y \in S(E) \text{ there exists } x \in E \text{ such that } Sx = y$$

$$\text{and } \|x\| \leq \alpha \|y\|\}$$

as defined in [14]. Then

$$k = \frac{1}{\sigma(S)} \tag{16}$$

by [14, proposition 1.5]. Since S is onto, then $+\infty > \gamma(S) > 0$ by [15, theorem 5.2 p. 231]. Now we prove that

$$\frac{1}{\sigma(S)} = \gamma(S). \tag{17}$$

Consider

$$U_1 = \{\alpha > 0 : \text{ for every } y \in F \text{ there exists } x \in E \text{ such that } Sx = y \text{ and } \|x\| \leq \alpha \|y\|\},$$

and

$$U_2 = \{m > 0 : \|Sx\| \geq m \text{ dist } (x, \ker S) \text{ for every } x \in E\}.$$

Let $\alpha \in U_1$. Given $x \in E$ let $y = Sx$. Then there exists $w \in E$ with $Sw = y$ and $\|w\| \leq \alpha \|y\|$. Since $Sw = Sx$ we have

$$\text{dist } (w, \ker S) = \text{ dist } (x, \ker S),$$

hence

$$\text{dist } (x, \ker S) \leq \|w\| \leq \alpha \|Sx\|$$

whence $1/\alpha \in U_2$. Conversely, let $m \in U_2$. Let $y \in F, u \in E$ be such that $Su = y$. By reflexivity of E, there exists $u_0 \in \ker S$ such that

$$\text{dist } (u, \ker S) = \|u - u_0\|.$$

Thus

$$\|S(u-u_0)\| \geq m\|u-u_0\|,\ S(u-u_0)=y$$

hence $1/m \in U_1$. It follows that (17) is proved. By [15, theorem 5.13 p. 234] we have $\gamma(S)=\gamma(S^*)$. Moreover, since S^* is one - to - one,

$$c^* = \ \sup\ \{\frac{1}{\|S^*x\|} : \|x\|=1\} = \frac{1}{\gamma(S^*)}$$

and the conclusion follows by (16) and (17).

Now we are in position to get a version of the condition number theorem for a fixed optimization problem (A,S), by proving an upper and a lower bound of its distance to the set IC of ill - conditioned problems in terms of the condition number cond (A,S).

LEMMA 7. *Let $A \in N(E,E^*)$ fulfill (9). If (A,S) is well - conditioned, then*

$$\frac{c_1}{cond\ (A,S)} \leq \ dist\ [(A,S), IC]$$

for every constant c_1 such that

$$0 < c_1 < \ min\ \{\frac{cond\ (A,S)}{c^*}, \frac{cond\ (A,S)}{\|A^{-1}\|}\},$$

c^* given by (14).

Proof. Write $L=$ cond (A,S). Then $L>0$ by theorem 5. Let $T \in B(E,F)$ have a closed range and $B \in N(E,E^*)$ be such that

$$\text{dist}\ [(A,S),(B,T)] \leq \frac{c_1}{L}$$

so that

$$\|A-B\| \leq \frac{c_1}{L},\ \|S-T\| \leq \frac{c_1}{L}.$$

Then we need to prove that $(B,T) \in WC$. By theorem 2, suffices to prove that B is an isomorphism between E and E^*, and T is onto. Since $c_1 < L/\|A^{-1}\|$, the conclusion about B comes from a known property (see [15, p. 196]). The conclusion about T comes from lemma 6 since $c_1 < L/c^*$.

LEMMA 8. *Let $A \in N(E,E^*)$ fulfill (9). If (A,S) is well - conditioned, then*

$$dist\ [(A,S), IC] \leq \frac{c_2}{cond\ (A,S)}$$

for every constant c_2 such that

$$c_2 > \frac{cond\ (A,S)}{c^*},$$

c^* given by (14).

Proof. Again write $L = \text{cond}\,(A,S)$. Since $c_2/L > 1/c^*$, by lemma 6 there exists some $T_0 \in B(E,F)$ which is not onto with $\|S - T_0\| \le c_2/L$. It follows that the optimization problem $(A, T_0) \in IC$ by theorem 2, hence

$$\text{dist}\ [(A,S), IC] \le \|T_0 - S\| \le \frac{c_2}{L},$$

ending the proof.

From lemmas 7, 8 and theorem 5 we obtain

COROLLARY 9. *If $P = (A,S) \in WC$ and A fulfills (9), then*

$$\frac{c_1}{cond\ (P)} \le\ dist\ (P, IC) \le \frac{c_2}{cond\ (P)}$$

for suitable positive constants c_1, c_2 depending only on the norms of $A^{-1}S^*D^{-1}, A^{-1} - A^{-1}S^*D^{-1}SA^{-1}$ and on α, ω, c^*.

The following definition isolates properties of a set of well - conditioned optimization problems for which a uniform estimate holds of their distance to ill - conditioning by the reciprocal of their condition numbers. Let V be a nonempty set of optimization problems $P = (A,S) \in WC$ with $A \in N(E, E^*)$ an isomorphism. Let $L = \text{cond}\ (P)$. Then V is *admissible* if there exist positive constants a_1, a_2, a_3 such that for every $P \in V$ we have

$$a_1 \le \frac{L}{c^*} \le a_3,\ a_2 \le \frac{L}{\|A^{-1}\|}, \tag{18}$$

where c^* is given by (14) (we see from lemma 6 that $c^* > 0$ for every $P \in V$). As a direct consequence of lemmas 7 and 8 we get

THEOREM 10. *For every admissible set $V \subset WC$ there exist positive constants C_1, C_2 such that*

$$\frac{C_1}{cond\ (P)} \le\ dist\ (P, IC) \le \frac{C_2}{cond\ (P)}$$

for every $P \in V$.

Proof. As already remarked (proof of lemma 7), cond $P > 0$ for every $P \in V$, so that the conclusion makes sense. By lemmas 7 and 8 suffices to consider the constants a_1, a_2, a_3 in (18) and take

$$0 < C_1 <\ \min\ \{a_1, a_2\},\ C_2 > a_3. \tag{19}$$

We end this section with a sufficient condition for admissibility. We need

LEMMA 11. *Let $S \in B(E,F)$ be onto. Then $k \le \|S\|$ where k is given by (15).*

Proof. Let

$$U = \{h > 0 : T \in B(E,F), \|S - T\| \le h \text{ imply } T \text{ onto }\}.$$

Then U is a nonempty interval contained in $[0,k]$. Since the operator $T = 0$ is not onto (being F non trivial), it follows that $\|S\| \notin U$, hence $\|S\| \ge k = \sup U$ as required.

THEOREM 12. *Given positive constants C, α, ω, k_0, let V be the set of all pairs $(A,S) \in N(E,E^*) \times B(E,F)$ such that A fulfills (9) and*

$$\|S\| \le C, \quad \mathit{inf}\,\{\|S^* x\| : \|x\| = 1\} \ge k_0. \tag{20}$$

Then V is admissible.

Proof. $V \subset WC$ by (9) and (20) due to theorem 2. We check conditions (18). Let $(A,S) \in V$ and $L = \text{cond }(A,S)$. By (11) and (20)

$$\|A^{-1}S^* D^{-1}u\| \ge \frac{1}{\omega}\|S^* D^{-1}u\| \ge \frac{k_0}{\omega}\|D^{-1}u\|, u \in F. \tag{21}$$

Moreover, again by (11),

$$< x, Dx > \le \frac{C^2}{\alpha}\|x\|^2, x \in F^*,$$

and taking the Fenchel conjugates

$$\frac{\alpha}{C^2}\|u\|^2 \le < D^{-1}u, u >, u \in F$$

hence $(\alpha/C^2)\|u\| \le \|D^{-1}u\|$. Then by (21)

$$\|A^{-1}S^* D^{-1}u\| \ge L_1\|u\|, u \in F,$$

where

$$L_1 = \frac{k_0\alpha}{\omega C^2}, \tag{22}$$

hence by theorem 5

$$L \ge \|A^{-1}S^* D^{-1}\| \ge L_1, \ (A,S) \in V.$$

By (20) and taking the Fenchel conjugates in (12) we obtain

$$< D^{-1}u, u > \le \frac{\omega}{k_0^2}\|u\|^2, u \in F.$$

It follows that

$$\|A^{-1}S^* D^{-1}\| \le \|A^{-1}\|\|S\|\|D^{-1}\| \le \frac{C\omega}{\alpha k_0^2}$$

for every $(A, S) \in V$. Thus by theorem 5 we have

$$L_1 \leq L \leq L_2 \text{ for every } (A, S) \in V$$

where

$$L_2 = \max \{\frac{C\omega}{\alpha k_0^2}, \frac{1}{\alpha} + \frac{C^2\omega}{\alpha^2 k_0^2}\}. \tag{23}$$

Moreover, remembering (14) and (20), we have

$$c^* = \frac{1}{\inf \{\|S^* x\| : \|x\| = 1\}} \leq \frac{1}{k_0}.$$

Then the first inequality of (18) is fulfilled by choosing

$$0 < a_1 \leq k_0 L_1. \tag{24}$$

By (11), the third is true if

$$0 < a_2 \leq \alpha L_1. \tag{25}$$

By lemmas 6 and 11, and by (20), the second inequality of (18) follows provided

$$a_3 \geq C L_2, \tag{26}$$

ending the proof.

As a direct consequence of the previous results, we obtain

COROLLARY 13. *Let V be defined as in theorem 12. Then there exist positive constants C_1, C_2 depending only on C, α, ω, k_0, such that*

$$\frac{C_1}{cond\ (P)} \leq dist\ (P, IC) \leq \frac{C_2}{cond\ (P)} \tag{27}$$

for every $P \in V$.

Proof. By theorems 10 and 12, the conclusion holds provided

$$0 < C_1 < \min \{\frac{k_0^2 \alpha}{\omega C^2}, \frac{k_0 \alpha^2}{\omega C^2}\},\ C_2 > C \max \{\frac{C\omega}{\alpha k_0^2}, \frac{1}{\alpha} + \frac{C^2 \omega}{\alpha^2 k_0^2}\}, \tag{28}$$

as we see by (19), (22), (23), (24), (25) and (26).

6. The finite - dimensional setting. Here we consider two positive integers N, K with $K \leq N$, and problems of the following form: to minimize

$$x \to \frac{1}{2} x^T A x,\ x \in R^N$$

subject to the constraint

$$Sx = 0$$

where A is a $N \times N$ real symmetric positive semidefinite matrix, and S is a $K \times N$ real matrix. By obvious identifications, the above finite - dimensional linear - quadratic problems are particular cases of those previously considered, with $E = R^N$ and $F = R^K$. Given positive constants α, ω, k_1, C, consider the set V of all pairs of matrices (A, S) as before, such that

$$\alpha \leq \lambda \leq \omega \text{ for every eigenvalue } \lambda \text{ of } A; \tag{29}$$

$$k_1 \leq \sigma \leq C \text{ for every singular value } \sigma \text{ of } S\ . \tag{30}$$

COROLLARY 14. *Let V be defined by (29), (30). Then there exist positive constants C_1, C_2 depending only on α, ω, k_1, C such that (27) holds for every $P \in V$.*

Proof. Due to corollary 13, we need only to check that (30) implies (20). As well known, $\|S\|$ is bounded above by a constant times the largest singular value of S. Moreover, if $u \in R^K$ minimizes $\|S^* x\|$ as $\|x\| = 1$ (Euclidean norms), then $\|S^* u\| = \sqrt{\lambda}$ where λ is some eigenvalue of SS^*. Then (20) is fulfilled and the proof is complete.

A significant example of uniformly well - conditioned finite - dimensional linear - quadratic problems (with variable domains) is obtained by applying Ritz - type numerical solution methods to a fixed infinite - dimensional problem, as follows. Let E, F be infinite - dimensional real Hilbert spaces, with E separable. Fix the problem $P = (A, S)$ and an orthonormal basis $\{\varphi_1, ..., \varphi_N, ...\}$ of E. Let

$$E_N = \text{ sp } \{\varphi_1, ..., \varphi_N\}, N = 1, 2, ...$$

and consider the sequence $(\bar{A}_N, \bar{S}_N)$ of the restrictions to E_N of A, S respectively. For each N, consider the basis $\{A\varphi_1, ..., A\varphi_N\}$ of $A(E_N)$, and fix an orthonormal basis of the linear subspace $S(E_N)$ of F. Denote by A_N, S_N the corresponding matrices associated to the linear operators

$$\bar{A}_N : E_N \to A(E_N),\ \bar{S}_N : E_N \to S(E_N)$$

respectively. For each N let $P_N = (A_N, S_N)$ be the optimization problem, to minimize

$$\frac{1}{2} x^T A_N x \text{ subject to } S_N x = 0, x \in R^N.$$

For each N denote by $(IC)_N$ the set of ill - conditioned linear - quadratic problems (B, T) where B is a $N \times N$ real symmetric positive semidefinite matrix, and T is a $K \times N$ real matrix, $K = K_N =$dim $S(E_N)$.

THEOREM 15. *If P is well - conditioned and A fulfills (9), then every P_N is well - conditioned, and there exist positive constants C_1, C_2 such that*

$$\frac{C_1}{\textit{cond } P} \leq \frac{C_1}{\textit{cond } P_N} \leq \ \textit{dist } [P_N, (IC)_N] \leq \frac{C_2}{\textit{cond } P_N}$$

for every $N = 1, 2, ...$

Proof. For each N, the eigenvalues of A_N are between α and ω of (9). If σ is a singular value of S_N, then there exists $y \in R^K, y \neq 0$, where $K =$dim $S(E_N)$, such that

$$\|S^* y\|^2 =< SS^* y, y >= \sigma^2 \|y\|^2.$$

By theorem 2 and lemma 6, S fulfills (20), hence $\sigma \geq k_0$ for some positive constant k_0 independent of N. Moreover $\sigma \leq \|S\|$, and cond $P_N \leq$ cond P for every N, due to theorem 5. Then every P_N is well - conditioned. From corollary (14), the conclusion follows by (27) and (28), due to the uniform bounds on the eigenvalues of A_N and the singular values of $S_N, N = 1, 2...$.

By theorem 15, computational complexity of each P_N as measured by its distance to ill - conditioning, is uniformly bounded, moreover a uniform version of the condition number theorem is available for the sequence P_N of finite - dimensional approximations of P. Convergence properties of the condition number of P_N as $N \to +\infty$, which are known for unconstrained problems, although in a different setting (as shown in [21]) will be considered elsewhere.

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REFERENCES

[1] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces,* Ed. Academiei (Bucuresti) - Reidel, 1986.

[2] L. Blum, F. Cucker, M. Shub and S. Smale, *Complexity and Real Computation,* Springer, New York, 1998.

[3] J.F. Bonnans and A. Shapiro, *Perturbations analysis of optimization problems,* Springer, New York, 2000.

[4] H. Brezis, *Analyse Fonctionelle,* Masson, Paris, 1983.

[5] D. Cheung and F. Cucker, *A new condition number for linear programming,* Math. Program., 91 (2001), pp. 163 - 174.

[6] D. Cheung, F. Cucker and J. Peña, *Unifying condition numbers for linear programming,* Math. Oper. Res., 28 (2003), pp. 609 - 624.

[7] A. Coulibaly and J.P. Crouzeix, *Condition numbers and error bounds in convex programming,* Math. Program. Ser. B, 116 (2009), pp. 79 - 113.

[8] J. P. Dedieu, *Approximate solutions of numerical problems, condition number analysis and condition number theorem,* Lectures in Appl. Math. 32 (1996), Amer. Math. Soc., Providence, pp. 263 - 283.

[9] J. Demmel, *The geometry of ill - conditioning,* J. Complexity, 3 (1987), pp. 201 - 229.

[10] J. Demmel, *On condition numbers and the distance to the nearest ill - posed problem,* Numer. Math., 51 (1987), pp. 251 - 289.

[11] N. Dunford and J. Schwartz, *Linear Operators, part I,* Interscience, New York, 1958.

[12] P. Frank, *Généralisation à un espace de Banach d' une propriètè sur la plus courte distance d' une matrice aux matrices singuliéres de même ordre,* C. R. Acad. Sci. Paris A - B, 262 (1966), pp. 1379 - 1380.

[13] P. Gill, W. Murray and M. Wright, *Numerical Linear Algebra and Optimization,* vol. 1, Addison - Wesley, Redwood City, CA, 1991.

[14] A. M. Gleason, *Finitely generated ideals in Banach algebras,* J. Math. Mech., 13 (1964), pp. 126-132.

[15] T. Kato, *Perturbation theory for linear operators,* Springer, Berlin - Heidelberg, 1966.

[16] J. Peña, *Conditioning of convex programs from a primal - dual perspective,* Math. Oper. Res., 26 (2001), pp. 206 - 220.

[17] J. Renegar, *Linear programming, complexity theory and elementary functional analysis,* Math. Program., 70 (1995), pp. 279 - 351.

[18] J. Renegar, *Incorporating condition measures into the complexity theory of linear programming,* SIAM J. Optim., 5 (1995), pp. 506 - 524.

[19] T. Zolezzi, *On the distance theorem in quadratic optimization,* J. Convex Anal., 9 (2002), pp. 693 - 700.

[20] T. Zolezzi, *Condition number theorems in optimization,* SIAM. J. Optim., 14 (2003), pp. 507 - 516.

[21] T. Zolezzi, *Condition numbers and Ritz type methods in unconstrained optimization,* Control Cybern., 36 (2007), pp. 811 - 822.

[22] T. Zolezzi, *Stability under perturbations of some condition numbers in optimization,* Math. Program. Ser. B, 116 (2009), pp. 579 - 593.