Cohesive categories and manifolds (*)

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Sunto. Le strutture ottenibili per incollamento di "spazi elementari", come le varietà, i fibrati, le varietà fogliettate, possono essere definite da "atlanti di incollamento" e, formalmente, come categorie arricchite su opportune categorie ordinate.

0. Introduction

0.1. Glueing structures, for instance manifolds, fibre bundles or foliations, can be obtained by patching together a family \((U_i)\) of suitable "elementary spaces" by means of partial bijections \(u^i_j: U_i \rightarrow U_j\) expressing the glueing conditions and forming a sort of "glueing atlas", instead of the more usual atlas of charts.

The goal of this paper is to treat these structures as enriched categories over "totally cohesive" categories, that is ordered categories having binary meets and arbitrary joins of pairwise "compatible" morphisms. The morphisms of these "generalized manifolds" are obtained as compatible modules (or profunctors) between enriched categories, which can be composed precisely because of the existence of compatible joins. The condition of Cauchy-completeness corresponds to the maximality of the glueing atlas; however, since our morphisms are modules, the procedure of Cauchy-completion just produces an isomorphic object.

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This approach to glueing structures is clearly related to Ehresmann's one, based on pseudogroups of transformations (see [E1, E2] and the end of this Introduction).

On the other hand, our setting inscribes in Lawvere's remark that interesting mathematical structures not only organize in categories, but are themselves categories, enriched over some suitable base: a monoidal category as in Lawvere's original formulation [La], or more generally a bicategory as in Betti [Be]. The bases we actually use are suitable ordered categories (very particular bicategories).

Last, this work is closely related with the notion of "glueing data", considered by Kasangian and Walters [KW] in an involutive ordered category with arbitrary (instead of compatible) suprema of parallel maps.

A short version of some of these results, in a more particular setting, appeared in [G4].

0.2. A cohesive category $A$ is equipped with an order relation $f \leq g$ and a compatibility (or linking) relation $f ! g$, both concerning parallel morphisms (same domain and codomain), consistent with composition and satisfying some further axioms (2.1). In particular the relation $!$ is reflexive and symmetric, generally non transitive; binary linked meets $f \land g$ have to exist. A totally cohesive category, moreover, has arbitrary linked joins $\lor \psi$ (of sets $\psi$ of parallel, pairwise compatible maps).

The paradigmatic example is the category $S$ of sets and partial mappings, where $f \leq g$ means that $f$ is a restriction of $g$, while $f ! g$ means that $f$ and $g$ agree wherever they are both defined.

Analogously for the category $T$ of topological spaces and continuous partial mappings, defined on open subsets; or the category $\mathcal{C}^r$ of open euclidean spaces (i.e. open subspaces of some $\mathbb{R}^p$) and partial mappings of class $\mathcal{C}^r$, defined on open subsets.

This category $\mathcal{C}^r$ contains the elementary spaces we want to glue in order to get $\mathcal{C}^r$-manifolds, together with the morphisms for the glueing. More precisely, the glueing morphisms will live in the inverse subcategory $\text{Inv}\mathcal{C}^r$ of open euclidean spaces and partial $\mathcal{C}^r$-diffeomorphisms (between open subsets of domain and codomain), which in our setting replaces Ehresmann's pseudogroup of (everywhere defined) $\mathcal{C}^r$-diffeomorphisms between open euclidean sets; $\text{Inv}\mathcal{C}^r$ is an inverse category, meaning that each morphism $u$ has a unique generalized inverse $u^\#$, with $uu^\#u = u$ and $u^\#uu^\# = u^\#$. Notice, however, that we need the whole category $\mathcal{C}^r$ to construct the morphisms of manifolds.

0.3. It may be remarked that, in these examples, the linking relation is determined by the order: indeed $f ! g$ if and only if $f$ and $g$ have a common upper bound. Such cohesive categories are here called link-filtered.

However inverse categories, which form an important class of categories having a canonical cohesive structure, need not be so, and we think useful to keep our present definition of cohesive category, based on independent, if related, order and linking.

0.4. In the previous examples the cohesive structure is also determined by the endomorphisms $\leq 1$ (the partial identities), which we call projections

1. $f \leq g$ iff $f = ge$ (for some projection $e$),

2. $f ! g$ iff $f = fe$, $g = ge'$, $fe' = ge$ (for some projections $e, e'$).
Moreover every morphism $f$ has a support $e(f)$, namely the least projection $e$, of the domain of $f$, such that $f = fe$; and we have

\[ (3) \quad f \leq g \iff f = g e(f), \quad (4) \quad f[g \iff f e(g) = g e(f). \]

These facts suggest the more particular notions of prj-cohesive and e-cohesive category (prj-category and e-category, for short). Notice that these structures are determined by the order, but need not be link-filtered: e.g. consider the cohesive subcategory $S_0$ of $S$ consisting of those partial mappings whose definition-set has no more than (say) five elements.

Every prj-cohesive category $A$ has a canonical inverse subcategory, $\text{Inv}A$ (cf. 5.7). Every dominical category, in the sense of Di Paola-Heller [He, Di, DH] and more generally every p-category in the sense of Rosolini [Ro] is e-cohesive (see 3.8).

0.5. Cohesive categories present two interesting notions: the totally cohesive completion, concerning linked joins, and the glueing completion, concerning the "glueing of manifolds".

The first construction is achieved by considering equivalence classes of linked sets of parallel morphisms (cohesive completion theorem, 2.7-8).

As to the second, a manifold $(U_i, u^i_j)$ in the prj-category $A$ is here an enriched category over $A$ (i.e. $u^i_i = 1$ and $u^i_j u^j_k \leq u^i_k$, for all indices $i, j, k$) satisfying a symmetry condition: $u^i_j u^i_k = u^i_j$, which forces the glueing morphisms $u^i_j$ to belong to the inverse subcategory $\text{Inv}A$. Its glueing, if existing, is the lax colimit of the diagram of all $u^i_j$.

If $A$ is prj-cohesive, with linked joins, the category $\text{Mf}A$ of manifolds over $A$ and "linked" modules (or profunctors) between them, with the usual matrix composition, is the glueing completion of $A$: it is glueing-complete and every prj-functor $A \to B$ with values in a glueing-complete prj-category, that preserves linked joins, extends uniquely to $\text{Mf}A$ (glueing completion theorem, 7.7).

The e-categories $S$ and $T$ are already complete in both regards. Instead the e-category $\mathcal{C}^r$ is just totally cohesive: its glueing completion $\text{Mf}\mathcal{C}^r$ yields the category of $\mathcal{C}^r$-manifolds as "glueing atlases". The topological realization of a manifold is given by the functor $\text{Mf}\mathcal{C}^r \to T$ extending the natural embedding $\mathcal{C}^r \to T$ by the universal property of the completion itself; it transforms a manifold $(U_i, u^i_j)$ into its glueing in $T$, i.e. the quotient of the sum-space $\Sigma U_i$ modulo the obvious equivalence relation produced by the glueing morphisms.

0.6. Analogously, fibre bundles and vector bundles can be considered as manifolds over the e-cohesive categories $\mathcal{B}$ and $\mathcal{V}$ of trivial fibre or vector bundles, with suitable partial mappings. The topological realization can now be constructed in a (glueing) category whose objects are generalized fibrations, or also Serre fibrations.

A unified formal treatment of differentiable manifolds and fibre bundles clearly presents advantages. For instance, the trivial tangent bundle functor $T: \mathcal{C}^r \to \mathcal{V}$ $(r \geq 1)$, that transforms the open set $U$ of $\mathbb{R}^n$ into the trivial vector bundle $U \times \mathbb{R}^n$, automatically extends to the (complete) tangent bundle functor $\text{Mf}\mathcal{C}^r \to \text{Mf}\mathcal{V}$ for $\mathcal{C}^r$-manifolds, by the glueing completion theorem.
0.7. In a different context, the category $L^\infty(a, Ban)$ of Banach spaces with spectral measures (on a fixed Boolean $\sigma$-algebra $a$) and bounded measurable operators between the former, has a natural prj-cohesive structure which will be sketched here in 1.5 and studied in a subsequent work [G5]. It does not consist of partial mappings and its projections are idempotent operators.

0.8. Chapter 1 contains a more detailed exposition of the examples and motivations recalled above; it also treats compositive joins of morphisms in an ordered category (see 1.7) and the type of "cardinal bound" $\rho$ that we use to restrict completeness conditions (1.8).

Cohesive, prj-cohesive and e-cohesive categories are introduced and studied in Chapters 2-4, together with the $\rho$-cohesive completion (with regard to suprema of linked $\rho$-sets of parallel morphisms).

Chapter 5 is concerned with inverse categories $K$ and their canonical cohesive structure; the inverse $\rho$-glueing completion $\rho IMfK$ of $K$ is constructed in Chapter 6. The $\rho$-glueing completion $\rho IMfA$ of an e-cohesive category $A$ is derived from this result in Chapter 7.

Fibre bundles, vector bundles and foliations are briefly considered in Chapter 8. Finally, Chapter 9 contains the proof of some completion theorems.

Calligraphic letters, like $S$ or $T$, usually denote categories of partial mappings.

0.9. Last, some words on the connections of this setting with C. Ehresmann's one. I thank Mrs. Andrée Ehresmann for her suggestions on this point.

An ordered category $(C, \triangleleft)$ in Ehresmann's sense (let us say o-category, to avoid confusion) abstracts the usual category $Set$ of small sets and (total) mappings, provided with the following order on morphisms: $(f: X \rightarrow Y) \triangleleft (f': X' \rightarrow Y')$ if $X \subset X'$, $Y \subset Y'$ (weak inclusions) and $f$ is a restriction of $f'$. Thus, in an o-category, $f \triangleleft f'$ does not imply that $f$ and $f'$ are parallel; indeed, if $f, f'$ are parallel morphisms and $f \triangleleft f'$, it is assumed that $f = f'$.

These o-categories $C$, with suitable regularity conditions, should correspond to e-cohesive categories $A$ with splitting of projections (and possibly some further conditions). Given $C$, we construct $A = P(C)$ as the category of "partial maps" of $C$, obtained by spans $X \leftarrow \bullet \rightarrow Y$ whose first morphism $i$ is an "inclusion" ($i \triangleleft 1_X$). Given $A$, we let $C$ be the subcategory of "total maps" $u$ of $A$ (such that $e(u) = 1$).

Thus, the present glueing completion theorem, restricted to totally cohesive e-categories, probably reduces to Ehresmann's "théorème d'élargissement complet d'un foncteur local" [E2]. The connections at the level of cohesive or prj-cohesive categories should be more involved, if possible.

From our viewpoint, ordered categories in the present sense allow to treat manifolds as enriched categories over 2-categories, and their partial mappings as modules between enriched categories. Moreover, this setting seems to be more adapted to applications to measurable operators, using the prj-cohesive Banach categories $L^\infty(a, Ban)$, which are not e-cohesive (see 1.5, in [G5]).
1. Examples and preliminary notions

1.1. Cohesion in the category of partial mappings. Let $\mathcal{S}$ be the category of small sets and partial mappings (i.e. univocal correspondences), composed as correspondences. We write $\text{Def } f$ the subset of the domain of $f$ on which $f$ is defined.

$\mathcal{S}$ is an ordered category, via

1. $f \preceq g$ if $f$ and $g$ are parallel maps and $f$ coincides with $g$ on $\text{Def } f$,

if and only if $f$ and $g$ are parallel maps and the graph of $f$ is contained in the graph of $g$.

Moreover $\mathcal{S}$ is provided with a proximity relation $(1)$ which will be called linking (or compatibility) and written $f!g$

2. $f!g$ if $f$ and $g$ are parallel maps and coincide on $\text{Def } f \cap \text{Def } g$.

These two relations, order and linking, are closely related. For instance, if $\psi \subset S(X, Y)$ is a linked set of parallel maps (i.e. $f!f'$ for all $f, f' \in \psi$), the supremum $f_1 = \bigvee \psi$ and (for $\psi \neq \emptyset$) the infimum $f_0 = \bigwedge \psi$ exist; they are given, respectively, by the set-theoretical union and intersection of the graphs; moreover they are compositive, i.e. preserved by composition. It may be noticed that $\bigvee \psi$ exists if and only if $\psi$ is linked (every set of maps having an upper bound is linked), while $\bigwedge \psi$ always exists for $\psi \neq \emptyset$; however it is easy to check that the meet is compositive precisely when the set $\psi$ is linked.

1.2. Cohesion and projections. A projection of the set $X$ in $\mathcal{S}$ is any "partial identity" $e: X \to X$, i.e. any endomorphism $e \preceq 1_X$. The projections of $X$ form an ordered set $\text{Prj } X$ which is isomorphic to the Boolean algebra $\mathcal{P} X$ of the subsets of $X$, via $e \leftrightarrow \text{Def } e$.

The projections of $\mathcal{S}$ are determined by the order; conversely, they determine both the order and the linking relation

1. $f \preceq g$ iff there is some projection $e$ such that $f = ge$,

2. $f!g$ iff there are projections $e, e'$ such that $f = fe$, $g = ge'$, $fe' = ge$.

In the latter case the pair $(e, e')$ will be called a resolution of $f$ and $g$, and we have $f \wedge g = fe' = ge$.

Finally, every partial mapping $f: X \to Y$ has a least projection $e(f) \in \text{Prj } X$ such that $f = fe$, namely the partial identity on $\text{Def } f$; it will be written as $ef$ and called the support of $f$. Clearly

3. $f \preceq g$ iff $f = g e(f)$,  

4. $f!g$ iff $f e(g) = g e(f)$.

In Chapters 2 and 3 we shall introduce the notions of cohesive category $(A, \preceq, !)$, of prj-cohesive category $(A, \text{Prj})$ and of e-cohesive category $(A, e)$. Every prj-cohesive category $A$ has an

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$(1)$ We mean: a binary relation between parallel maps, reflexive, symmetric and consistent with composition.
associated cohesion structure defined as in (1)-(2), or more simply as in (3)-(4) if \( A \) is also e-cohesive.

1.3. Some categories of continuous partial mappings. Consider the category \( \mathcal{T} \) of small topological sets and continuous partial mappings, defined on open subsets. Consider also the subcategory \( \mathcal{C}^r \) of \( \mathcal{T} \) whose objects are the open subspaces of all \( \mathbb{R}^n \) \((n \in \mathbb{N})\), with partial mappings of class \( \mathcal{C}^r \) defined on open subsets; as usual, \( r \in \mathbb{N} \cup \{\infty, \omega\} \) and class \( \mathcal{C}^\omega \) means analytic.

If \( A \) is any of these categories, the (faithful) forgetful functor \( U: A \to \mathcal{S} \) creates an e-cohesive structure on \( A \), provided with arbitrary linked joins and binary linked meets (1.1), distributive with respect to the former. The projections of the object \( X \) form an ordered set \( \text{Prj} X \), isomorphic to the locale (2) \( \mathcal{O}(X) \) of the open sets of \( X \).

Other examples, related to fibre bundles and foliations, will be considered in Chapter 8.

1.4. Cohesion for measurable functions. Let \( X \) be a measurable space and \( Y \) a normed one. The following very simple cohesive structure on the set \( Y \)

\[
(1) \quad y \leq y' \iff (y = 0 \text{ or } y = y'), \quad y!y' \iff (y \leq y' \text{ or } y' \leq y),
\]

yields, by the usual "pointwise" argument, a cohesive structure on the normed space \( L^\infty(X, Y) \) of bounded measurable mappings from \( X \) to \( Y \)

\[
(2) \quad f \leq f' \iff (\forall x \in X: fx \leq gx) \iff (\forall x \in X: fx \neq 0 \Rightarrow fx = gx),
\]

\[
(3) \quad f!f' \iff (\forall x \in X: fx \not\leq gx) \iff (\forall x \in X: fx \neq 0 \not\Rightarrow fx = gx),
\]

which is finitely cohesive, i.e. provided with finite linked joins. It is easy to guess that the universal completion of \( L^\infty(X, Y) \) with respect to \( \sigma \)-joins of linked sets is the space \( M(X, Y) \) of all measurable mappings from \( X \) to \( Y \).

This may also be noticed that the category \( \mathcal{S} \) considered in 1.1 is equivalent to the category \( \mathcal{S}' \) of pointed sets and pointed (everywhere defined) mappings; writing 0 the base point, the cohesive structure of the hom-sets \( \mathcal{S}'(X, Y) \) can be described as above.

1.5. Cohesion for operators. The category \( L^\infty(a, \mathcal{Ban}) \) of bounded measurable operators in the category \( \mathcal{Ban} \) of Banach spaces, on the Boolean \( \sigma \)-algebra \( a \), has for objects all the pairs \((X, E)\) where \( X \) is a Banach space and \( E: a \to \mathcal{Ban}(X) \) is a (bounded) \( \sigma \)-additive spectral measure with values in \( X \) (see [DS], XV.2.3-4). A morphism \( S: (X, E) \to (Y, F) \) is a bounded linear mapping \( S: X \to Y \) that commutes with the measures \( E, F \), in the sense that \( S.E(a) = F(a).S \) for all \( a \in a \).

This category has a natural prj-cohesive structure, defined as in 1.1.3-4, the projections of the object \((X, E)\) being the endomorphisms \( E(a) \), for \( a \in a \). The structure is not complete with regard to linked

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(2) I.e. a complete lattice in which binary meets distribute over arbitrary joins.
joins: its $\sigma$-cohesive completion may be concretely described as the category $M(a, \text{Ban})$ of closed, densely defined, measurable operators, as it will be shown in [G5].

1.6. Cohesion for inverse categories. A category $K$ is inverse if every morphism $a: A \to A'$ has a unique generalized inverse $a': A' \to A$, with $aa' = a$ and $a'a = a'$. For example: the category $J = \text{Inv} S$ of sets and partial bijections, or the category $\text{Inv} T$ of topological spaces and partial homeomorphisms between open subspaces. In fact every prj-cohesive category $A$ has an associated inverse subcategory, $\text{Inv} A$, as shown below, in 5.7.

The inverse category $K$ has a canonical cohesive structure

1.7. Compositive joins and meets (3). Let $A$ be an ordered category; this means that $A$ is equipped with an order relation $\preceq$ on parallel maps, which is reflexive, transitive, antisymmetric and compositive.

We say that a set $\alpha \subset A(A, B)$ of parallel maps has compositive join (or union) $\dot{\alpha} = \bigvee \alpha$ if

- for all the maps $x: A' \to A$, $y: B \to B'$ we have $y\dot{\alpha}x = \bigvee a$ if $yax$, in the ordered set $A(A', B')$, so that, in particular, $\dot{\alpha}$ is the supremum of $\alpha$ in the ordered set $A(A, B)$.

Compositive joins have the following elementary properties:

(a) associativity: if $a = \bigvee a_i$ (i $\in I$), and for every i, $a_i = \bigvee a_{ij}$ (j $\in J_i$) are compositive joins, then $a = \bigvee a_{ij}$ (i $\in I$, j $\in J_i$) is also;

(b) composition: if $a = \bigvee a_i$ is a compositive join, $yax = \bigvee (yaix)$ is also;

(c) if $a = \bigvee a_i$ is compositive and for every i, $a_i \preceq a'_i \preceq a$, then $a = \bigvee a'_i$ is a compositive join.

Dually one defines compositive meets (or intersections), that enjoy dual properties. A category provided with binary compositive meets (of parallel pairs of maps)

(2) $y(\bigwedge a')x = yax \land ya'x$,

is "the same" as a category enriched over the closed category of $\land$-semilattices; it will be called a semilatticed category.

The stronger cartesian compositive property

(3) $(b \land b').(a \land a') = ba \land b'a'$,

(3) The two last sections of Chapter 1 contain some preliminary tools.
will appear in prj-cohesive categories, with respect to linked meets (3.3.3). A category provided with binary meets, compositive in this stronger sense, is the same as a category enriched over the category of $\land$-semilattices, provided with the monoidal structure of cartesian product (instead of the closed structure considered above).

1.8. Smallness and cardinal bounds. A universe $\mathcal{U}$ is fixed throughout; a small set is any set belonging to $\mathcal{U}$. A $\mathcal{U}$-category $\mathbf{A}$ is assumed to have each object and each hom set $\mathbf{A}(A, B)$ belonging to $\mathcal{U}$; e.g. the category of small sets, small groups and so on. It is small if also its object-set belongs to $\mathcal{U}$. All the categories we explicitly use are assumed to be $\mathcal{U}$-categories, except of course some "very large" 2-category of categories, like the 2-category $\rho\mathcal{C}\mathcal{H}$ of $\rho$-cohesive $\mathcal{U}$-categories mentioned in 2.6.

A section of cardinals will be a set $\rho$ of small cardinals such that:

(a) $1 \in \rho$, if $x, y \in \rho$ then $x \cdot y \in \rho$,
(b) if $x \in \rho$ and $0 \neq y \leq x$, then $y \in \rho$.

Thus $\rho$ is either $\{1\}$, or $\{0, 1\}$, or an interval of cardinals $[0, x]$ or $[1, x]$ (where $x$ is any small infinite cardinal), or the set $\Omega$ ($\Omega'$) of all small (non-null) cardinals. If $\rho$ is infinite, it is also closed with respect to binary sum. In particular, we write $f = [0, \aleph_0]$, the set of finite cardinals, and $\sigma = [0, \aleph_1] = [0, \aleph_0]$. We also write $\rho'$ the section of non-null cardinals of $\rho$.

A $\rho$-set is a small set whose cardinal belongs to $\rho$; the section $\{0, 1\}$ will be shortened to 0 in prefixes.

A $\rho$-lattice will be a (small) ordered set having join and meet of all its $\rho$-subsets; thus 0-lattices are ordered sets with supremum and infimum, $f$-lattices are lattices with least and greatest element, $\Omega$-lattices are complete lattices. Ordinary lattices coincide with $f'$-lattices and ordered sets with $\{1\}$-lattices.

Analogously one can consider $\rho$-distributive lattices, Boolean $\rho$-algebras, $\rho$-locales and so on. An ordinary locale is the same as an $\Omega$-locale.

A section $\rho$ is fixed throughout this paper.

2. Cohesive categories

2.1. Definition. A cohesive category will be a category $\mathbf{A}$ provided with two binary relations on parallel morphisms, the order $\preceq$ and the linking (or compatibility) relation $\triangleright$, that satisfy:

(CH.1) $\preceq$ is an order of categories (reflexive, transitive, antisymmetric and consistent with composition);
(CH.2) \( ! \) is reflexive, symmetric and consistent with composition in the strong sense \(^{(4)}\): if \( a!a' \) and \( b!b' \) are consecutive, then \( b!b'a' \);

(CH.3) if \( a \leq a', \ b \leq b' \) and \( a'!b' \) then \( a!b \);

(CH.4) if \( a!b \) the (linked) meet \( a \wedge b \) exists and is compositive in \( A \).

The notion of cohesive category is selfdual.

If \( a, b \leq c \) then \( a!b \), by (CH.2, 3). We say that the cohesive category \( A \) is *link-filtered* if the converse holds too

\[ \text{(1) } a!b \iff \text{a and b have a common upper bound,} \]

in which case the linking relation is determined by the order. A link-filtered cohesive category is clearly the same as an ordered category provided with binary filtered meets, consistent with composition. The cohesive category \( S_0 \) considered in 0.4 is not link-filtered.

Every category has a *discrete* cohesive structure, with \( a \leq b \iff a!b \iff a = b \). On the other hand, a cohesive category with *trivial* linking (\( a!a' \iff \text{a and a' are parallel} \)) is the same as a semilatticed category, i.e. a category enriched over the closed category of semilattices (1.7).

In this chapter, \( A \) will always be a cohesive category.

2.2. **Linked joins of morphisms.** A linked (or compatible) set \( \alpha \) of \( A \) is any set of parallel morphisms such that \( a!a' \) for all \( a, a' \in \alpha \); if also \( \beta \) is a linked set, \( a!\beta \) will mean that \( \alpha \) and \( \beta \) are parallel and \( a!b \) for all \( a \in \alpha, \ b \in \beta \); or equivalently, that \( \alpha \cup \beta \) is linked. Any subset which has an upper bound is linked.

We say that the set \( \alpha \) (of parallel morphisms) has a linked join if:

(a) \( \alpha \) has a compositive join \( \lor \alpha \) (in particular, it is a linked set),

(b) for each linked morphism \( b \) (i.e. \( b!a \), for all \( a \) in \( \alpha \)), \( (\lor \alpha)!b \) and \( (\lor \alpha) \wedge b \) is the compositive join of \( \{a \wedge b \mid a \in \alpha\} \) (which is linked, by (CH.3)).

It is easy to see that linked joins satisfy properties similar to those considered in 1.7 (a) - (c) for compositive joins.

2.3. **Definition.** A \( \rho \)-localic cohesive category (or \( \rho \)-cohesive category, for short) will be a cohesive category \( A \) such that every linked \( \rho \)-set of parallel morphisms has a linked join.

Equivalently, \( A \) has to satisfy:

(CH.5\( _\rho \)) every linked \( \rho \)-set \( \alpha \subseteq A(A, B) \) has a join \( \lor \alpha \), compositive in \( A \); linked binary meets distribute over joins of linked \( \rho \)-sets

\[ \text{(1) } \lor \alpha \wedge b = \lor_{a \in \alpha} (a \wedge b), \quad \text{if } \alpha!b. \]

\(^{(4)}\) As \( ! \) is not transitive, this consistency is stronger than "left and right consistency with composition with one map".
The necessity of (CH.5ρ) being obvious, let us assume that it holds and \( \alpha \not\!\! \beta \). Then \( (V \alpha) \not\!\! \beta \) is trivial for \( \rho \subset \{0, 1\} \); otherwise the set \( \beta = \alpha \cup \{b\} \) is a linked \( \rho \)-set and \( V \alpha, b \leq V \beta \), whence \( V \alpha \not\!\! b \). Moreover the meets \( a \land b \) (\( a \in \alpha \)) form a linked \( \rho \)-set (by (CH.3) or by (1) itself), so that their join has to be compositive.

In particular we have the notions of cohesive, 0-cohesive, f-cohesive (or finitely cohesive), \( \sigma \)-cohesive, totally cohesive categories when, respectively, \( \rho = \{1\}, \{0, 1\}, f, \sigma, \Omega \) (1.8). The categories \( S, T, \mathcal{C} \) are totally cohesive (1.1-3); \( L_\infty^\mathbb{K}(a, \text{Ban}) \) is just finitely cohesive (see 1.5).

2.4. Elementary properties. Let \( A \) be a \( \rho \)-cohesive category. A non-empty \( \rho \)-set \( \alpha \subset A(A, B) \) of parallel morphisms is linked if and only if it has some upper bound (e.g. \( V \alpha \)).

If \( \alpha \) and \( \beta \) are parallel linked \( \rho \)-sets of morphisms and \( \alpha \not\!\! \beta \), then \( V \alpha \not\!\! V \beta \) and

\[
(1) \quad (V \alpha) \land (V \beta) = V a \land b \quad \text{(for \( a \in \alpha, b \in \beta \)).}
\]

Furthermore, if \( \alpha \) and \( \gamma \) are consecutive linked \( \rho \)-sets of morphisms then

\[
\gamma \alpha = \{ca \mid a \in \alpha, c \in \gamma\}
\]

is again a linked \( \rho \)-set (CH.2) and

\[
(2) \quad V(\gamma \alpha) = V \gamma V \alpha.
\]

2.5. Characterizations. A cohesive category \( A \) is 0-cohesive (resp. f-cohesive, \( \sigma \)-cohesive) if and only if it satisfies the first (resp. the first two, the following three) conditions:

(CH.5a) for all objects \( A, B \) the ordered set \( A(A, B) \) has a least element \( 0_A^B \) (called the zero morphism from \( A \) to \( B \)), compositive in \( A \), in the sense that the composition of a zero morphism with any other is a zero morphism;

(CH.5b) every pair \( a, b \in A(A, B) \) of linked morphisms \( a \not\!\! b \) has join \( a \lor b \), compositive in \( A \); linked binary meets distribute over joins of linked pairs;

(CH.5c) every increasing sequence \( (a_n) \) in \( A(A, B) \), obviously linked, has join \( V a_n \), compositive in \( A \); linked binary meets distribute over increasing countable joins.

The proof reduces to calculate the join of a countable linked set \( \alpha = \{a_n \mid n \in \mathbb{N}\} \) in \( A(A, B) \) by means of an increasing sequence of finite suprema \( b_n = V \{a_k \mid k \leq n\} \).

Moreover, if \( 2 \in \rho \) (i.e. \( \Gamma \subset \rho \)), every \( \rho \)-cohesive category is link-filtered (2.1). Thus an ordered category \( X \) is \( \rho \)-cohesive, with linking relation expressed by 2.1.1, if and only if:

(C.1) \( X \) has compositive filtered binary meets,

(C.2ρ) \( \rho \)-sets of parallel maps, filtered in \( X \), have a compositive join; filtered binary meets distribute over these \( \rho \)-joins.

2.6. Cohesive functors and transformations. A \( \rho \)-cohesive functor \( F: A \to B \) will be a functor between \( \rho \)-cohesive categories which preserves order, linking, linked binary meets and linked \( \rho \)-joins.

For \( \rho \subset \sigma \) there are characterizations of such functors, similar to those in 2.5.
A ρ-cohesive transformation \( \psi: F \to G: A \to B \) will be a natural transformation between ρ-cohesive functors.

A ρ-cohesive subcategory of the ρ-cohesive category \( A \) is any subcategory \( A' \) which is closed under linked binary meets and linked ρ-joins; then \( A' \), equipped with the induced order and linking relation, is ρ-cohesive as well as the inclusion \( A' \to A \).

A ρ-cohesive embedding \( F: A \to B \) will be a ρ-cohesive functor that is injective on objects and reflects the order and linking relations. Then \( F \) is also faithful and \( F(A) \) is a ρ-cohesive subcategory of \( B \), isomorphic to \( A \).

The concrete 2-category \( \rho \text{CH} \) of ρ-cohesive \( \mathcal{U} \)-categories (cf. 1.8), ρ-cohesive functors and natural transformations is easily seen to be 2-complete (i.e. to have all small indexed 2-limits). The cohesive structure on a cartesian product \( \prod \mathcal{U}_i \) of ρ-cohesive categories is quite obvious.

2.7. Theorem (The ρ-cohesive completion). Let \( \rho \) be a fixed section of cardinals (cf. 1.8). Every cohesive category has a universal cohesive embedding \( \eta: A \to \rho c A \) in a ρ-cohesive category, that preserves the existing linked ρ-joins; it will be called the ρ-cohesive completion of \( A \).

The universality of \( \eta \) means that: for each cohesive functor \( F: A \to B \) preserving the existing linked ρ-joins, with values in a ρ-cohesive category, there exists precisely one ρ-cohesive functor \( G: \rho c A \to B \) that extends \( F \) (i.e. \( F = G \eta \)).

**Proof.** See 9.1-2.

2.8. A description of the ρ-cohesive completion. The ρ-cohesive completion \( \rho c A \) may be constructed in the following way.

First form the category \( \mathcal{P}_\rho A \) having the same objects of \( A \) and morphisms \( \alpha: A \to B \) given by the linked ρ-sets \( \alpha \subset A(A, B) \), with composition

\[
\beta \alpha = \{ ba \mid a \in \alpha, b \in \beta \}.
\]

The category \( \mathcal{P}_\rho A \) is equipped with the following preorder \( \prec \)

\[
\alpha \prec \beta \text{ iff } \alpha \beta \text{ and } \forall a \in \alpha^*, \ a = \bigvee_{b \in \beta} (a \wedge b) \text{ (linked join)},
\]

where \( \alpha^* \) denotes the down-closed subset of \( A(A, B) \) generated by \( \alpha \). This preorder determines the quotient category

\[
\rho c A = \mathcal{P}_\rho A / \sim,
\]

where \( \alpha \sim \beta \) is the congruence associated to \( \prec \), namely \( \alpha \prec \beta \) and \( \beta \prec \alpha \).

The order and the linking relation in \( \rho c A \) are given by

\[
[\alpha] \leq [\beta] \text{ iff } \alpha \beta, \quad [\alpha] [\beta] \text{ iff } \alpha \beta \text{ as linked sets of } A,
\]

independently of the choice of representatives.

Linked meets and linked ρ-joins are calculated in \( \rho c A \) by the following formulas

\[
\alpha \wedge [\beta] = \{ a \wedge b \mid a \in \alpha, b \in \beta \},
\]

\[
\forall \Sigma = [\bigcup \Sigma],
\]
where $[\alpha]![[\beta]]$, $\Sigma$ is any linked $\rho$-set of $\rho$-sets of $A$ ($\alpha!\alpha'$, for all $\alpha, \alpha' \in \Sigma$) and $\Sigma' = \{[\alpha] | \alpha \in \Sigma\}$. In particular

(7) $\forall \alpha = [\alpha]$ (in $\rho c A$, for any linked $\rho$-set $\alpha$ of $A$).

The universal embedding $\eta: A \to \rho c A$ takes the object $A$ to itself and the morphism $a$ to the equivalence class of $\{a\}$.

The $\sigma$-cohesive completion of a finitely cohesive category $A$ may be given a simpler description, since for each morphism $a$ in $\sigma c A$ there is an increasing sequence of parallel morphisms $(a_n)_{n \in \mathbb{N}}$ of $A$ such that $a = \{\{a_n\} | n \in \mathbb{N}\}$ (see 2.5). This case will be considered in [G5].

2.9. Density. If $A$ is a cohesive subcategory of a $\rho$-cohesive category $B$, with the same objects, the embedding $F: A \to B$ is the $\rho$-cohesive completion of $A$ if and only if:

(i) $F$ preserves the existing linked $\rho$-joins,

(ii) $A$ is $\rho$-dense in $B$: for every map $b$ in $B$ there is a linked $\rho$-set $\alpha$ in $A$ whose join in $B$ is $b$.

Indeed, the necessity of these conditions being obvious, let us assume that they hold: we must show that the $\rho$-cohesive functor $G: \rho c A \to B$ extending $F$ is an isomorphism of cohesive categories. Since it is surjective, by (ii), it suffices to show that it reflects the order (hence it is injective) and the linking relation.

Let $\alpha$ and $\beta$ be parallel, linked $\rho$-sets in $A$. If $G[\alpha] \leq G[\beta]$ in $B$, for every $a \in \alpha$ we have $a = Ga \leq G[\beta] = \lor b$, hence $a = \lor b, a \land b$ is a linked join in $B$. Since $a$ and all $a \land b$ are in $A$, the linked join holds in $A$, and therefore $[\alpha] \leq [\beta]$ in $\rho c A$.

Last, if $G[\alpha] ! G[\beta]$ in $B$, then $\lor \alpha = \lor \beta$ in $B$, whence $a \land b$ in $B$ for every $a \in \alpha$ and $b \in \beta$, and the same holds in the cohesive subcategory $A$; in other words, $[\alpha] ! [\beta]$ in $\rho c A$.

3. Prj-cohesive and e-cohesive categories

As we have seen in Chapter 1, cohesive structures are often defined by assigning, to each object $A$, a set of commuting idempotent endomorphisms of $A$, which will be called "projections". This yields the notions of prj-cohesive and e-cohesive category, the latter being stronger than the former.

3.1. Definition. A prj-cohesive category (or prj-category for short) will be a category $A$ equipped, for every object $A$, with a set $Prj A \subset A(A, A)$ of endomorphisms of $A$ (the projections of $A$) so that:

(PCH.1) every identity is a projection; if $e$ is a projection, $ee = e$; if $e$ and $f$ are parallel projections, $ef = fe$ is a projection;

(PCH.2) if $a: A \to B$ is in $A$ and $f \in Prj B$, there exists some $e \in Prj A$ such that $fa = ae$.

Thus $Prj A$ is a commutative idempotent submonoid of $A(A, A)$ and a 1-semilattice in its own right, with $e\alpha f = ef = fe$, $e \leq f$ iff $e = ef (= fe)$ and greatest element $1_A$. 

A prj-cohesive functor \( F : A \rightarrow B \) is a functor between prj-cohesive categories which preserves projections.

3.2. The cohesive structure. The prj-category \( A \) has the following associated order and linking relations (which make \( A \) into a cohesive category, as proved below)

1. \( a \preceq b \) if there is a projection \( e \) such that \( a = be \) (note that \( ae = a \)),
2. \( a \!\! \mid b \) if there are projections \( e, f \) such that \( a = ae, \ b = bf, \ af = be \); in this case we say that the pair \( (e, f) \) is a resolution of the linked pair \( (a, b) \).

This order extends the canonical order of projections: indeed, if \( e = fg \) in \( \text{Prj} A \), then \( ef = fgf = fg = e \). An endomorphism \( a \in A(A, A) \) is a projection if and only if \( a \preceq 1_A \); thus all (the existing) joins and non-empty meets \(^5\) of projections are the same in \( \text{Prj} A \) or in \( A(A, A) \). The identity \( 1_A \) is maximal in \( A(A, A) \): if \( a \succeq 1_A \) then \( 1 = ae \), hence \( e = aee = ae = 1 \) and \( a = 1 \).

It will also be useful to remark that the projection \( e \) in (1) and (2) may be replaced with each projection \( e_0 \) such that \( e_0 \preceq e \) and \( ae_0 = a \).

3.3. Proposition. The prj-category \( A \) with the associated order and linking relations is a cohesive category (2.1). If \( (e, e') \) is a resolution of the linked pair \( (a, b) \), the meet of the latter is

1. \( a \wedge b = ae' = be \).

Moreover, if \( a \!\! \mid b \) and \( c \!\! \mid d \) in the left diagram below

\[
\begin{array}{ccc}
a & & c \\
\downarrow & & \downarrow \\
b & \Rightarrow & C, \\
\end{array}
\]

then the cartesian compositive property (3) holds (see 1.7).

Every set \( \varepsilon \) of parallel projections is linked; the linked meet of two parallel projections is their meet in \( \text{Prj} A \), \( ef = ef = fe \), which is therefore compositive in \( A \).

A functor between prj-cohesive categories is cohesive if and only if it is prj-cohesive, if and only if it preserves the order.

Proof. The letters \( e, f, e', f' \ldots \) always denote projections.

For the first two axioms (CH.1-2) the only non-trivial checkings concern the composition. Let the diagram (2) be given.

If \( a \preceq b \) and \( c \preceq d \), let \( a = be, \ c = df \); by (PCH.2) there is a projection \( e' \) such that \( fa = ae' \), and \( db e'e' = d be' = dae' = dfa = ca \).

\(^5\) Warning: the empty set has infimum 1 in \( \text{Prj} A \), but generally (e.g. in the examples of Chapter 1) no infimum in \( A(A, A) \): the latter has no greatest element.
Instead, if \( a!b \) and \( c!d \), let \( a = ae, \ b = be', \ ae' = be, \ c = cf, \ d = df' \), \( cf' = df \). By (PCH.2) there are projections \( \hat{e}, \hat{e}' \) such that \( fa = a\hat{e}, \ f'b = b\hat{e}' \); we want to show that \((e\hat{e}, e'\hat{e}')\) is a resolution of the pair \((ca, db)\). Indeed \( ca\hat{e} = c a\hat{e} = cfa = ca \), and analogously \( db e'\hat{e}' = db \); last

\[
(4) \quad ca.e'\hat{e}' = cb\hat{e}e = cf'be = dfae' = dae'e = db.e\hat{e}.
\]

As to (CH.3), if \( a \leq a' \), \( b \leq b' \) and \( a'!b' \), let \( a = a'e, \ b = b'f \) and \((e', f')\) be a resolution of \((a', b')\). It is sufficient to check that \((ee', ff')\) is a resolution of \((a, b)\); then \(a.ee' = a, \ b.ff' = b\) and \(a.ff' = a'eff' = b'efe' = b.ee'\).

Now we prove (CH.4) and the properties (1), (3). Let \( a!b \), with resolution \((e, e')\): \( a = ae, \ b = be', \ ae' = be \); we must show that \( h = ae' = be\) is the meet of \( a \) and \( b \); clearly \( h \leq a, b \), while if \( k \leq a, b \) then \( k = af = bf' \) and \( k = ae f = bf' e \leq h \). It is now easy to deduce (3), hence the compositive property of meets: with the hypothesis \( a!b, \ c!d \) and the notation above (in the proof of (CH.2)), we have

\[
(5) \quad ca\land db = ca.e'\hat{e}' = df.be = (c\land d).a\land b).
\]

The last remarks are now trivial; in particular a functor between prj-cohesive categories preserves the order if and only if it preserves the projections, in which case it also preserves resolutions, hence the linking relation and also binary linked meets, because of (1).

### 3.4. Remark.

A cohesive category \((A, \leq, !)\) may be determined by at most one prj-cohesive structure on \( A \), given by

\[
(1) \quad \text{Prj } A = \{a \in A(A, A) \mid a \leq 1_A\},
\]

which happens iff \( a \leq 1 \) implies \( a = a \) and moreover the characterizations 3.2.1-2, concerning the ordering and linking relation, hold.

Indeed, if this is the case, we define the projections by (1). (PCH.1): if \( e, f \in \text{Prj } A \) then \( ef \leq 1 \) is again a projection, hence an idempotent: it follows that \( ef = ef.ef \leq f.e \); analogously \( fe \leq ef \). (PCH.2): from \( fa \leq a \) and the condition 3.2.1 it follows the existence of a projection \( e \) such that \( fa = a.e \).

Thus a cohesive category will be said to be prj-cohesive when these facts hold.

Analogously, an ordered category \((A, \leq)\) is prj-cohesive, with projections defined by (1), iff \( a \leq 1 \) implies \( a = a \) and property 3.2.1 holds true.

### 3.5. Definition.

An \( e \)-cohesive category (or \( e \)-category for short) will be a category \( A \) equipped, for every object \( A \), with a projection-set \( \text{Prj } A \subset A(A, A) \) satisfying (PCH.1) and the axioms:

(ECH.1) for each \( a: A \to B \) in \( A \), the set of projections \( e \) of \( A \) such that \( ae = a \) has a least element \( e(a) \), called the support of \( a \),

(ECH.2) for every \( a: A \to B \) and \( b: B \to C \) in \( A \), we have \( e(b).a = a.e(ba) \).

Elementary properties, for \( a, a': A \to B, \ b: B \to C, \ e \in \text{Prj } A, \ f \in \text{Prj } B \):

\[
\begin{align*}
(1) & \quad e(e) = e, & (2) & \quad e(ba) \leq e(a), \\
(3) & \quad fa = a.e(fa), & (4) & \quad e(a.e) = e.e(ae) = e(a).e, \\
(5) & \quad a \leq a' \Rightarrow e(a) \leq e(a'), & (6) & \quad \text{if } a \text{ is monic, then } e(a) = 1.
\end{align*}
\]
In particular, (3) shows that the axiom (PCH.2) is satisfied: \( \mathbf{A} \) is prj-cohesive, hence cohesive.

An \( e \)-cohesive functor will be a functor between \( e \)-cohesive categories which preserves supports; by (1) it also preserves projections, hence it is prj-cohesive and cohesive (cf. 3.3).

### 3.6. The cohesive structure.

By the last remark in 3.2, if the category \( \mathbf{A} \) is \( e \)-cohesive the associated order and linking relations are characterized by

1. \( a \leq b \) iff \( a = b.e(a) \),
2. \( a!b \) iff \( a.e(b) = b.e(a) \), iff \((e(a), e(b))\) is a resolution of \((a, b)\),
3. \( a.e(b) \leq b \) and \( b.e(a) \leq a \).

Furthermore, if \( a!b \), by 3.3 and 3.5.4

\[
(3) \quad a \wedge b = a.e(b) = b.e(a),
\]

\[
(4) \quad e(a \wedge a') = e(a)e(a').
\]

Similarly, if \( \mathbf{A} \) is \( \rho \)-cohesive, it is easy to check that

\[
(5) \quad e(\lor a) = \lor a.e(a),
\]

(for every linked \( \rho \)-set \( \alpha \)).

### 3.7. Inverse images of projections.

If \( \mathbf{A} \) is \( e \)-cohesive, every morphism \( a : \mathbf{A} \rightarrow \mathbf{B} \) in \( \mathbf{A} \) determines a mapping

1. \( a^P : \text{Prj} \mathbf{B} \rightarrow \text{Prj} \mathbf{A} \), \( a^P(f) = e(fa) \).

Thus \( \text{Prj} \) becomes a contravariant functor from \( \mathbf{A} \) to the category of semilattices

\[
(2) \quad 1^P = 1, \quad (ba)^P = a^Pb^P, \quad a^P(f \land g) = a^P(f) \lor a^P(g).
\]

Indeed

\[
a^Pb^P(g) = a^P(e(gb)) = e(e(gb)a) = e(a.e(gba)) = e(a).e(gba) = e(g.ba) = (ba)^P(g).
\]

\[
a^P(fg) = a^P(f^P(g)) = (fa)^P(g) = (a.a^P(f))^P(g) = (a^P(f))^P(a^P(g)) = a^P(f).a^P(g) = a^P(f) \land a^P(g).
\]

Other properties, for \( a, a' : \mathbf{A} \rightarrow \mathbf{B}, b : \mathbf{B} \rightarrow \mathbf{C}, e, e' \in \text{Prj} \mathbf{A} \) and \( f \in \text{Prj} \mathbf{B} \)

\[
(3) \quad e^P(1) = e, \quad (4) \quad fa = a.a^P(f),
\]

\[
(5) \quad e^P(e') = ee', \quad (6) \quad e(ba) = e(e(b).a) \leq e(a).
\]

To verify (6), let \( f = b^P(1) \); then

\[
e(ba) = (ba)^P(1) = a^Pb^P(1) = a^P(f) = a^Pb^P(1) = (fa)^P(1) = e(fa) = e(e(b).a).
\]

Conversely, if \( \mathbf{A} \) satisfies (ECH.1) and the mappings (1) are given, satisfying (2)-(4), hence (5), then \( \mathbf{A} \) is \( e \)-cohesive, with \( e(a) = a^P(1) \). Indeed \( a.a^P(1) = a \); if \( ae = a \) then \( a^P(1) = (ae)^P(1) = e^P(a^P(1)) = e.a^P(1) \), i.e. \( a^P(1) \leq e \); furthermore \( a.(ba)^P(1) = a.a^Pb^P(1) = a.a^P(f) = fa = b^P(1).a \), where \( f = b^P(1) \).

### 3.8. Examples.

(a) The cohesive categories \( \mathbf{S}, \mathbf{T}, \mathbf{C}' \) described in Chapter 1 are \( e \)-cohesive, with projections given by the partial identities.

(b) An \( e \)-cohesive category need not be link-filtered; e.g. the subcategory of \( \mathbf{S} \) considered in 2.1.
(c) Every *dominical* category ([He, Di, DH]), and more generally every *p-category* [Ro] \( A \), is e-cohesive, with:

(1) \( \text{Prj} \ A = \{ \text{dom} \ x \mid x \in A(A, A) \} = \{ e \in A(A, A) \mid \exists \ a \ \text{in} \ A \ \text{such that} \ e = \text{dom} \ a \} = \{ e \in A(A, A) \mid e = \text{dom} \ e \} \),

(2) \( e(a) = \text{dom} \ a \).

This follows from the following properties of *domains*, proved in [Ro] 2.1.4-5, for morphisms \( a: A \to B \), \( b: B \to C \), \( a': A \to B' \):

(i) \( \text{dom} \ 1_A = 1_A \),

(ii) \( \text{dom}(ba) = \text{dom} \left( (\text{dom} \ b) \ a \right) \),

(iii) \( \text{dom} \ b, a = a.\text{dom}(ba) \),

(iv) \( (\text{dom} \ a).\text{dom}(a') = (\text{dom} a').(\text{dom} a) \),

(v) \( a.\text{dom} \ a = a \),

(vi) \( \text{dom}(\text{dom} \ a) = \text{dom} \ a \),

(vii) \( \text{dom} \ a.\text{dom} a = (\text{dom} a) \),

(viii) \( \text{dom}((\text{dom} a).(\text{dom} a')) = (\text{dom} a).(\text{dom} a'). \)

Indeed the second and third equalities in (1) come from the property (vi). The axiom (PCH.1) follows from (i), (vii), (iv) and (viii), while (ECH.2) coincides with (iii). As to (ECH.1): if \( a: A \to B \), then \( a = a.\text{dom} \ a \), by (v); on the other hand, if \( a = ae \) and \( e \in \text{Prj} \ A \), then \( e \leq \text{dom} \ a \), since (by (ii) and (viii))

(3) \( \text{dom} \ a = \text{dom} (ae) = \text{dom}((\text{dom} a) e) = \text{dom}((\text{dom} a)(\text{dom} e)) = (\text{dom} a).(\text{dom} e) = (\text{dom} a).e \).

(d) The category \( L^\infty(a, \text{Ban}) \) described in 1.5 is prj-cohesive (see [G5]).

3.9. Cartesian products and duality. The cartesian product \( A = \Pi A_i \) of a family of prj-cohesive categories \( (A_i)_{i \in I} \) is prj-cohesive, with \( \text{Prj} A = \Pi(\text{Prj} A_i) \). If the factors \( A_i \) are e-cohesive, so is the product \( A \), with \( e((a_i)_{i \in I}) = (e(a_i))_{i \in I} \).

A *prj*-cohesive category will be a pair \( A = (A, \text{Prj}) \) satisfying (PCH.1) and (PCH.2*): for all \( a \) and \( e \) there is some \( f \) such that \( f a = ae \); the associated cohesion structure has \( a \leq b \) iff there is some projection \( f \) such that \( a = fb \), and analogously for the linking (determined by *coresolutions* of pairs of morphisms). Then \( A \) is an *e*-cohesive category if it is provided with *cosupports* \( e^*(a) \) that satisfy (ECH.1*, 2*).

4. Adequate prj-cohesive categories

\( A \) is always a prj-category; we examine conditions ensuring that the \( \rho \)-cohesive completion of \( A \) is again a prj-category.

4.1. Resolution of sets. It is easy to show that a set \( \alpha \subset A(A, B) \) of parallel morphisms is linked if and only if there is a family of projections \( e_{ab} \in \text{Prj} A \) (for \( a, b \in \alpha \)) such that

(1) \( a = a.e_{ab} \), \( a.e_{ba} = b.e_{ab} \) for all \( a, b \in \alpha \).

More particularly, a *resolution* of \( \alpha \) will be a family \( (e_{a})_{a \in \alpha} \) of projections of \( A \) such that
The second condition can also be written as \( a.e_b \leq b \). A set admitting a resolution is clearly linked, but these two facts are indeed equivalent in most cases we are interested in, as we shall soon see (cf. 4.3-4).

Every prj-cohesive functor preserves resolutions of sets.

4.2. Transfer of resolutions. A resolution \((e_a)\) of the set \( \alpha \) may be transferred by composition in the following way. Given the morphisms \( x, y \)

1. \[ A' \xrightarrow{x} A \xrightarrow{a} B \xrightarrow{y} B' \] (\( a \in \alpha \)),

let us choose, for each \( a \in \alpha \), a projection \( e_a' \in \Prj A' \) such that \( e_a.x = x.e_a' \). Then, a trivial checking shows that

2. \((e_a')\) is a resolution of \( y \alpha x = \{ yax \mid a \in \alpha \} \).

4.3. Existence of resolutions. Let \( A \) be a prj-cohesive category.

(a) Every set \( \epsilon \) of parallel projections has a canonical resolution, namely \((e)e \in \epsilon \).

(b) More generally, every set \( \alpha \) that has an upper bound \( \hat{\alpha} \) has a resolution. Indeed, if we let \( a = \hat{\alpha}.e_a \) (for \( a \in \alpha \)), we have

1. \( a.e_a = \hat{\alpha}.e_a.e_a = a \),
   \( a.e_b = \hat{\alpha}.e_a.e_b = \hat{\alpha}.e_b.e_a = b.e_a \).

Therefore, if \( A \) is \( \rho \)-cohesive, each linked \( \rho \)-set has a resolution.

(c) If \( A \) has \( \rho' \)-meets of projections (in \( \Prj A \) or equivalently in \( A(A, A) \), by 3.2), that are compositive in \( A \), we show now that each linked \( \rho \)-set \( \alpha \) has a resolution \((e_a)\), and also (if \( \alpha \neq \emptyset \)) a compositive meet

2. \( \wedge \alpha = b.\Lambda_a e_a \), for every \( b \in \alpha \).

Indeed, with the notation of 4.1.1, the family \( e_a = \Lambda_b e_{ab} \) (for \( a \in \alpha \)) is a resolution of \( \alpha \)

3. \( a.e_a = a \Lambda_b e_{ab} = \Lambda_b a.e_{ab} = \Lambda_b a = a \),
4. \( a.e_b = a.e_a.e_b = (a.e_{ba} e_b)e_a = b.e_b.e_{ab} e_a = b.e_{ab} e_a = b.e_a \).

As to (2), \( b.\wedge a \leq b.e_a = a.e_b \leq a \) for all \( a \in \alpha \); if \( x \leq a \) for all \( a \in \alpha \), then \( x \leq a \wedge b = b.e_a \), hence \( x \leq b.\wedge e_a = b.a.e_a \). Last, the compositive property of the meet is a straightforward consequence of the transfer of resolutions (4.2).

(d) In particular, as every prj-category has linked \( f' \)-meets, it follows that each finite linked set has a resolution.

(e) If \( A \) is \( e \)-cohesive, a set \( \alpha \) of parallel morphisms is linked if and only if (cf. 3.6.2)

5. \( a.e(b) \leq b \), for all \( a, b \in \alpha \),

if and only if the family \( e_a = e(a) \) of their supports is a resolution of \( \alpha \) (the \( e \)-resolution).
4.4. *Adequate prj-cohesive categories*. We shall say that the prj-coherent category $\mathbf{A}$ is $\rho$-adequate if it satisfies:

(PCH.3) each linked $\rho$-set of $\mathbf{A}$ has a resolution,

(PCH.4) $\mathbf{A}$ has $\rho$-joins of projections, compositive in $\mathbf{A}$.

A prj-category which is $\rho$-cohesive is also $\rho$-adequate (by 4.3(b)); trivially, it is 0-cohesive if and only if it is 0-adequate. The category $\mathcal{L}^\infty(a, \mathbf{Ban})$ (cf. 1.5) is $\sigma$-adequate, because of 4.3 (c), whereas it is not $\sigma$-cohesive.

A $\rho$-adequate functor will be a prj-cohesive functor between $\rho$-adequate (prj-cohesive) categories, which preserves $\rho$-joins of projections.

4.5. **Proposition.** If $\mathbf{A}$ is a $\rho$-adequate prj-category, a linked $\rho$-set of parallel morphisms has linked join (2.2) if and only if it has an upper bound. Every existing $\rho$-join of morphisms is linked.

A $\rho$-adequate functor preserves all the existing $\rho$-joins.

**Proof.** The thesis being trivial for $\rho \in \{0, 1\}$, let us assume that $\rho$ is infinite.

First, every $\rho$-set $\alpha$ with an upper bound has compositive join: if $a = c.e_a$, for $a \in \alpha$, set $\hat{e} = V e_a$ (compositive join) and $\hat{a} = c.\hat{e}$; then $\hat{a} = c.V e_a = V_a c e_a = V_{\alpha} a = V \alpha$ is a compositive join.

Take now a parallel map $b$ linked with $\alpha$; we have to show that $\hat{a}! b$ and $\hat{a} \land b = V_a (a \land b)$. By (PCH.3), the linked $\rho$-set $\gamma = \alpha \cup \{b\}$ has a resolution $(e_c)c \in \gamma$; the compositive join $\hat{e} = V e_a (a \in \alpha)$ yields a resolution $(\hat{e}, e_b)$ of the pair $(\hat{a}, b)$, proving that it is linked

1. $\hat{a}.\hat{e} = V_{a, a'} e_{a'} = V_a a = \hat{a}, \quad \hat{a}.e_b = V_a (a.e_b) = V_a (b.e_a) = b, V_a e_a = b.e.$

The distributivity follows, calculating the meets by the resolutions (3.3.1)

2. $(V a) \land b = \hat{a}.e_b = (V_a a).e_b = V_a (a.e_b) = V_a (a \land b).$

Thus, every existing $\rho$-join in $\mathbf{A}$ is linked. Moreover, if $F : \mathbf{A} \to \mathbf{B}$ is a $\rho$-adequate functor, $\hat{a} = V \alpha$ is a $\rho$-join with $a = \hat{a}.e_a$ ($a \in \alpha$) and $\hat{e} = V e_a$ as above

3. $F(V a) = F(\hat{a}) = F(\hat{a}.\hat{e}) = F(\hat{a}).F(\hat{e}) = F(\hat{a}).(V e_a) = V (F \hat{a} . F e_a) = V (F a).F e_a = F(V a).F e_a = V F a.$

4.6. **Corollary.** A prj-category is $\rho$-cohesive if and only if it is $\rho$-adequate and every linked $\rho$-set of parallel morphisms has an upper bound. A functor between $\rho$-cohesive prj-categories is $\rho$-cohesive if and only if it is $\rho$-adequate.

4.7. **Theorem** (the $\rho$-completion for $\rho$-adequate prj-categories). If $\mathbf{A}$ is a $\rho$-adequate prj-category, the $\rho$-cohesive completion $\rho \mathbf{c A}$ (cf. 2.8) is prj-cohesive, with the same projections. The embedding $\mathbf{A} \to \rho \mathbf{c A}$ is $\rho$-adequate, and may also be considered as the universal $\rho$-adequate functor from $\mathbf{A}$ into a $\rho$-cohesive prj-category.

The linking and order relations in $\rho \mathbf{c A}$ can also be described as follows, for two parallel linked $\rho$-sets of $\mathbf{A}$-morphisms $\alpha$ and $\beta$

1. $[\alpha] ! [\beta]$ iff there is a resolution $(e_x)_{x \in \alpha \cup \beta}$ of $\alpha \cup \beta$ in $\mathbf{A}$,
(2) \([\alpha] \preceq [\beta]\) iff there is such a resolution \((e_x)_{x \in \alpha \cup \beta}\) with \(a = a^*(V_{b \in \beta} e_b)\) for all \(a \in \alpha\),

iff there is such a resolution \((e_x)_{x \in \alpha \cup \beta}\) with \(\forall_{a \in \alpha} e_a \preceq V_{b \in \beta} e_b\),

iff for every \(a \in \alpha\), \(a = V_{b \in \beta} (a \wedge b)\).

**Proof.** If \(\varepsilon\) is a \(\rho\)-set of parallel projections of \(A\), \(e = V\varepsilon\) is a linked join in \(A\), by the previous proposition (4.5); as linked joins are preserved by the embedding in \(\rho c A\), we have \(e = [\varepsilon]\) in \(\rho c A\). It follows that the projections of \(A\) coincide with the endomaps \([\alpha] \preceq 1\) of \(\rho c A\); indeed the relation \(e \preceq 1\) in \(A\) is preserved by the embedding, while if \([\alpha] \preceq 1\) in \(\rho c A\), each morphism \(a \in \alpha\) is a projection \((a \preceq [\alpha] \preceq 1\) in \(\rho c A\), hence \(a \preceq 1\) in \(A\) and \([\alpha] = \vee \alpha \preceq 1\) in \(A\).

We have to prove that the \(\rho\)-cohesive category \(\rho c A\) is prj-cohesive, with the same projections as \(A\). Because of 3.4, this reduces to check the characterizations 3.2.1-2 for the order and the linking relation in \(\rho c A\); in the same time, we shall also verify the characterizations (1) and (2) of these relations.

First, consider the linking. If \(\alpha \cup \beta\) in \(P_\rho A\), then \(\alpha \cup \beta\) is a linked \(\rho\)-set, with resolution \((e_x)_{x \in \alpha \cup \beta}\). Given such a resolution, the subsets \(\varepsilon = \{e_a | a \in \alpha\}\), \(\eta = \{e_b | b \in \beta\}\) and their joins \(e = V\varepsilon = [\varepsilon]\), \(f = V\eta = [\eta]\) yield a resolution of \([\alpha]\) and \([\beta]\) in \(\rho c A\), in agreement with 3.2.2

(3) \(\alpha . e = \{a.e | a \in \alpha\} = \alpha\), \(\beta . f = \beta\),

(4) \([\alpha]. f = [\alpha], [\eta] = \{\{a.e_b | a \in \alpha, b \in \beta\}\} = \{\{b.e_a | a \in \alpha, b \in \beta\}\} = [\beta], \varepsilon = [\beta]. e\).

Last, let us assume that we have such a resolution: \([\alpha] = [\alpha]. e\), \([\beta] = [\beta]. f\), \([\alpha]. f = [\beta]. e\). By (PCH.3p) there are resolutions \((e_a)\) of \(\alpha\) and \((f_b)\) of \(\beta\); thus \(a = [\alpha]. e_a = [\alpha]. e_a = a.e\), and we may assume that \(e_a \preceq e\), for all \(a \in \alpha\); similarly \(f_b \preceq f\) for \(b \in \beta\). Then, in \(\rho c A\)

(5) \(a.f_b = [\alpha]. e_a.f_b = [\alpha]. f . e_a.f_b = [\beta]. e . e_a.f_b = [\beta]. e_a.f_b = b.e_a\),

whence \(a.f_b\) in \(A\) (for all \(a\) and \(b\)) and \([\alpha]\) in \(\rho c A\).

Then, let us consider the order. If \([\alpha] \preceq [\beta]\) in \(\rho c A\), every resolution \((e_x)\) of \(\alpha \cup \beta\) (with \(e\) and \(f\) as above) yields: \(a = V_b(a \wedge b) = V_b a e b = af\). Given such a resolution, we replace each \(e_a\) with \(e_a.f\); this gives a new resolution of \(\alpha \cup \beta\) such that \(e \preceq f\). If this property holds we have, by (3) and (4), \([\alpha] = [\alpha]. e = [\alpha]. f = [\beta]. e\), as required by 3.2.1. Now, if \([\alpha] = [\beta]. h\) for some projection \(h\), the relation \(h \preceq 1\) in \(\rho c A\) implies \([\alpha] \preceq [\beta]\). The last characterization of \([\alpha] \preceq [\beta]\) in (2) is equivalent to 2.8.2, by 4.5.

Finally the embedding \(A \to \rho c A\) preserves the projections by the above remarks, and their \(\rho\)-joins (as all the existing linked \(\rho\)-joins) by definition; the new universal property is a particular case of the known one (cf. 2.7).

4.8. **Theorem (The \(\rho\)-completion for \(\rho\)-adequate e-categories).** If \(A\) is a \(\rho\)-adequate e-category, then \(\rho c A\) is e-cohesive, with supports

(1) \(e[a] = \{e(a) | a \in \alpha\}\) = \(V_{a \in \alpha} e(a)\) (for every linked \(\rho\)-set \(\alpha\) of \(A\)).

The embedding \(\eta: A \to \rho c A\) is a \(\rho\)-adequate e-functor; it is the universal \(\rho\)-adequate e-functor from \(A\) to a \(\rho\)-cohesive prj-category.
Proof. Let us consider the $\rho$-set of projections $\varepsilon = \{ e(a) \mid a \in \alpha \}$; it is an endomap in $\mathcal{P}_\rho A$. Clearly $\alpha.\varepsilon = \{ a.e(a') \mid a, a' \in \alpha \} \sim \alpha$; on the other hand, if $[\alpha].e = [\alpha]$ then (as in the proof of 4.7) $ae = a$ for all $a \in \alpha$, i.e. $e(a) \leq e$, for all $a$, and $[\varepsilon] \leq e$. Hence, in $\rho c A$, $[\alpha]$ has support $[\varepsilon] = \lor e(a)$; this proves also that the embedding $\eta$ preserves supports.

As to (ECH.2), if $\beta$ is a linked $\rho$-set of $A$, composable with $\alpha$, for all $a, a' \in \alpha$ and $b \in \beta$,

\begin{align*}
(2) & \hspace{1cm} a.e(ba') \leq a.e(ba').e(a) = a.e(ba.e(a)) = a.e(ba.e(a')) \leq a.e(ba), \\
(3) & \hspace{1cm} e[\beta], [\alpha] = \{ \{ e(b).a \mid a \in \alpha, b \in \beta \} \} = \{ \{ a.e(ba) \mid a \in \alpha, b \in \beta \} \} = \alpha.e[\beta].
\end{align*}

4.9. Remark. Let $A$ be a prj-category. It can be shown that its $\rho$-cohesive completion $\rho c A$ is prj-cohesive provided that $A$ satisfies (PCH.3$\rho$) and the following condition, weaker than (PCH.4$\rho$):

(a) for every morphism $a: A \rightarrow B$, every $e_0 \in \text{Prj } A$ and every $\rho$-set $\varepsilon$ of projections of $A$, if $ae_0 = a = \lor e \in \varepsilon$ then there exists a projection $e_1$ of $A$ such that $e_1 \leq e_0$, $ae_1 = a$ and $e_1 = \lor e \in \varepsilon e_1 e$ is a linked join (i.e. $e_1 \leq e \in \mathcal{P}_\rho A$).

In such a case the projections of $\rho c A$ are the equivalence classes $[\varepsilon]$, where $\varepsilon$ is any $\rho$-set of parallel projections of $A$. However, the stronger but simpler condition (PCH.4$\rho$) is sufficient for our purposes.

5. Inverse categories and cohesion

Inverse categories are the obvious generalization of inverse semigroups. They are used here to supply "glueing morphisms" for generalized manifolds; for instance, the usual $C^r$-manifolds will be constructed in Chapters 6, 7 by means of open euclidean spaces and partial $C^r$-diffeomorphisms between open subsets, forming the inverse category $\text{Inv } C^r$ associated to $C^r$.

After a review of elementary properties of inverse categories from [G1, 2], we introduce here their canonical cohesive structure and study their $\rho$-cohesive completion. Other references on inverse categories can be found in [G3].

5.1. A review of inverse categories. A category $K$ is said to be inverse if every morphism $a: A \rightarrow A'$ has precisely one generalized inverse $a^\#: A' \rightarrow A$

\begin{align*}
(1) & \hspace{1cm} aa^\#a = a, \quad a^\#aa^\# = a^\#.
\end{align*}

Then (see [G1], Theorem 1.25) the mapping $a \mapsto a^\#$ defines an involution of $K$ (i.e. a contravariant functor, identical on the objects and involutive)

\begin{align*}
(2) & \hspace{1cm} 1^\# = 1, \quad (ba)^\# = a^\#b^\#, \quad (a^\#)^\# = a, \quad \text{and } K \text{ is selfdual.}
\end{align*}

A projection of the object $A$ is any idempotent endomorphism $e: A \rightarrow A$; clearly $e^\# = e$. The projections of $A$ are closed with respect to composition (ef = ef.(ef)$^\#$, ef = ef.fe.ef = ef.ef) and commute (ef = (ef)$^\#$ = fe), so that they form a unitary semilattice $\text{Prj } A$. 

Every morphism \( a: A \to B \) defines two mappings, the covariant and contravariant transfer of projections

\[(3) \quad a^P: \text{Prj} \ A =: \text{Prj} \ B, \quad a^P(e) = aea^\# \]
\[(4) \quad a^P: \text{Prj} \ B =: \text{Prj} \ A, \quad a^P(f) = a^\#fa = a^\#P(f). \]

These mappings are easily seen to be homomorphisms of semilattices and to behave functorially \(((ba)_P = b_P a_P, (ba)^P = a^P b^P)\). Furthermore

\[(5) \quad a \text{ is mono} \iff a^P(1) = a^\#a = 1 \iff a \text{ has a left inverse}, \]
\[(6) \quad a \text{ is epi} \iff a_P(1) = aa^\# = 1 \iff a \text{ has a right inverse}, \]
\[(7) \quad a \text{ is mono and epi} \iff (a^\#a = 1, \ aa^\# = 1) \iff a \text{ is an isomorphism}. \]

The category \( K \) is provided with a canonical order (that generalizes the canonical order of inverse semigroups) \( a \preceq b \), characterized by the following seven equivalent conditions (for \( a, b: A \to B \)):

\[(i) \quad a = ab^\#a; \quad (ii) \quad a = b.a^\#a; \quad (iii) \quad a = aa^\#.b; \quad (iv) \quad a = aa^\#.b.a^\#a; \]
\[(v) \exists e \in \text{Prj} \ A: a = b.e; \quad (vi) \exists f \in \text{Prj} \ B: a = f.b; \]
\[(vii) \exists e \in \text{Prj} \ A, \exists f \in \text{Prj} \ B: a = fbe. \]

Notice that the endomorphisms \( x \preceq 1 \) are precisely the projections and that \( x \succeq 1 \) implies \( x = 1 \).

Since for each morphism \( a \)

\[(8) \quad aa^\#a = a, \quad a^\#aa^\# = a^\#, \quad a^\#a \preceq 1, \quad aa^\# \preceq 1, \]

it follows that \( a \) is a monomorphism if and only if it has a right adjoint \( b \) (characterized by \( ba \succeq 1, \ ab \succeq 1 \)); then \( b = a^\# \) is also left inverse to \( a \).

A functor between inverse categories preserves all the notions considered above.

The paradigmatic inverse category is the category \( I \) of small sets and partial bijections: any small inverse category may be embedded in this ([Ks, G3]). Other examples of interest for our context are given in 5.8 and in Chapter 9.

### 5.2. Inverse categories and regularity

Let the category \( A \) be regular in the sense of von Neumann (or \( \nu N \)-regular), in the sense that each morphism \( a: A \to A' \) has \( \text{some} \) generalized inverse \( a': A' \to A \) (satisfying \( aa'a = a, \ a'aa' = a' \)). Then \( A \) is inverse (i.e. the generalized inverses are uniquely determined) if and only if the idempotents of \( A \) commute ([G1], 1.25).

More particularly, let the category \( A \) be equipped with a regular involution \( a \mapsto a^\# \), where 'regular' means that \( aa^\#a = a, \) for all \( a \). Let us call projection of \( A \) any symmetric idempotent, i.e. any endomorphism \( e: A \to A \) such that \( e = ee = e^\# \) (or equivalently \( e = ee^\#, \) or also \( e = e^\#e \)). Then each idempotent \( a \) is the product of two projections (since \( a = aa^\#a = aa^\#, a^\#a \)), so that \( A \) is inverse iff its idempotents commute, iff its projections commute, iff every idempotent is symmetric; in this case the involution of \( A \) yields the (unique) generalized inverse of every morphism.

### 5.3. The canonical cohesive structure

From now on, \( K \) is an inverse category.
It is easy to see that the projections of \( K \) satisfy the axioms (PCH.1) and (ECH.1, 2), and define an \( e \)-cohesive structure with \( e(a) = a^e = a^p(1) \). Indeed

- \( a.e(a) = a.a^e = a \);
- if \( a = ae \) then \( a^e = a^e.a \) and \( a^e = e \);
- \( a.e(ba) = a.a^b b a = b^a b.a.a^e = e(b).a \).

Now, the involution of \( K \) determines also an \( e^* \)-cohesive structure (cf. 3.9), with cosupports given by \( e^*(a) = e(a^e) = aa^e = ap(1) \).

Thus \( K \) is provided with a first cohesive structure (determined by supports) and with a second one (determined by cosupports)

(1) \( a \leq^1 b \iff a = ba^e, \quad a \!
\!
\! \leq^1 b \iff a = a b a^e \iff ba^e \in \text{Prj} B \quad (6), \)

(2) \( a \leq^2 b \iff a = aa^e b, \quad a \!
\!
\! \leq^2 b \iff b a^e a = aa^e b \iff b^e a \in \text{Prj} A.\)

These orders coincide with the canonical order \( \leq \) of \( K \) (cf. 5.1), while the two linking relations are generally different (7), and related by the involution

(3) \( a \!
\!
\! \leq^1 b \iff a^e \!
\!
\! \leq^2 b^e. \)

The canonical cohesive structure of \( K \) will be given by the canonical order \( \leq \) together with the following linking relation (preserved by the involution of \( K \))

(4) \( a \!
\!
\! \leq^1 b \iff (a \!
\!
\! \leq^1 b \text{ and } a \!
\!
\! \leq^2 b), \)

\[ \text{iff } (a.b^e = b.a^e \text{ and } bb^e.a = aa^e b), \]

\[ \text{iff } (ba^e \in \text{Prj} B \text{ and } b^e a \in \text{Prj} A). \]

\( K \) need not be link-filtered: e.g. consider the inverse subcategory of \( I \) formed by those partial bijections whose definition-set has no more than five elements.

Every functor between inverse categories preserves the canonical cohesive structure.

5.4. Proposition. This is indeed a cohesive structure on \( K \) (but not a prj-cohesive structure in the sense of Chapter 3). If \( a!b \)

(1) \( a \land b = ab^e = ba^e = bb^e a = aa^e b = ab^e a = ba^e b, \)

(2) \( (a \land b)^p(e) = a^p(e) \land b^p(e), \quad (a \land b)^p(f) = a^p(f) \land b^p(f), \)

(6) If \( a.b^e = b.a^e \), then \( ba^e , ba^e = b.b^e a.a^e = ba^e , ba^e = b.b^e a.a^e = b.a^e; \) conversely, if \( ba^e \) is a projection, then \( b.a^e = ba^e , a = ab^e , a = a^b b.a^e = a.a^e b = a.b^e b. \)

(7) For instance, take the inverse category \( I \) of partial bijections: the projections of \( I \) coincide with those of \( S \), thus \( a \leq^1 b \iff a \text{ and } b \text{ are compatible functions}, \) while \( a \leq^2 b \iff a^e \text{ and } b^e \text{ are compatible functions}. \) Thus, any pair \( a, b \) with \( \text{Def} a \cap \text{Def} b = \emptyset \) and \( \text{Val} a \cap \text{Val} b = \emptyset \) yields a counterexample.
(3) \( e(a \land b) = e(a) \land e(b) = b^a a = a^b b \), \( e^*(a \land b) = e^*(a) \land e^*(b) = b a^b = a b^b \).

A set \( \alpha \) of parallel morphisms in \( K \) is linked if and only if it has a double resolution \((e_a), (f_a)\) of projections, satisfying

\[
(4) \quad a = a.e_a = f_a.a, \quad a.e_b = f_a.b, \quad f_b.a = f_a.b, \quad (a, b \in \alpha),
\]

the smallest double resolution being given by \( e_a = e(a), \ f_a = e^*(a) \).

The cartesian compositive property of meets \((1.7.3)\) holds.

**Proof.** The axioms (CH.1-4) are a straightforward consequence of the definition: the first and second structure are both cohesive structures, with the same order relation. Linked meets may be calculated according to the first structure, \( e \)-cohesive \((a \land b = a \cdot b^# = b \cdot a^#)\) or according to the second one, \( e^* \)-cohesive \((a \land b = b \cdot b^# \cdot a = a \cdot a^# \cdot b)\); the last two expressions in \((1)\) follow at once from \( b a^# \in \text{Prj} B \) and \( b^# \cdot a \in \text{Prj} A \).

The cartesian compositive property of meets follows from \((3.3)\) (applied to the first cohesive structure of \( K \)). For \((2)\):

\[
(a \land b)_{(e)} = (a \land b).e.(a \land b)^# = (a e^# a \cdot b^#).a = a e^# \land b e^# = a p(e) \land b p(e).
\]

The last assertions are obvious.

5.5. **Remark.** It may be noticed that \( a \not\!\! \land b \) (or \( a \not\!\! \land b \)) is a sufficient condition in order that \( a \) and \( b \) have compositive meet with respect to the canonical order (use the associated \( e \)-cohesive or \( e^* \)-cohesive structure). However, in an inverse category, compositive intersections are not 'satisfactory': e.g. they do not satisfy \((5.4.2)\), nor \((5.4.3)\). The good notion seems to be linked meets, in the present sense.

5.6. **Theorem (The \( \rho \)-completion of an inverse category).** The \( \rho \)-cohesive completion of the inverse category \( K \) (with respect to its canonical cohesive structure) is an inverse category, provided with the canonical cohesive structure. The involution of \( \rho c K \) is given by \( \alpha^# = \{ a^# \mid a \in \alpha \} \), while its projections are the classes \([\varepsilon]\), where \( \varepsilon \) is any \( \rho \)-set of parallel projections of \( K \).

**Proof.** The mapping \( \alpha \mapsto \alpha^# = \{ a^# \mid a \in \alpha \} \) plainly defines an involution on \( \mathcal{P}_{\rho K} \), and further in \( \rho c K \); the latter is regular (cf. \((5.2)\)), as

\[
(1) \quad \alpha \alpha^# = \{ab^# \mid a, b \in \alpha \} \sim \{ aa^# \mid a \in \alpha \},
\]

\[
(2) \quad \alpha^# \alpha = \{ aa^# b \mid a, b \in \alpha \} \sim \{ a a^# a \mid a \in \alpha \} = \alpha.
\]

An endomorphism \([\alpha]\) is a projection (with regard to the regular involution, see \((5.2)\)) iff \( \alpha \sim \alpha^# \alpha \sim \{a^# a \mid a \in \alpha\} \), iff \( \alpha \) is a \( \rho \)-set of projections of \( K \). Therefore the projections of \( \rho c K \) commute and the latter is an inverse category (by \((5.2)\)); we only need to prove that the cohesive structure of \( \rho c K \) coincides with the canonical one, determined by supports and cosupports.

If \( [\alpha] \preceq [\beta] \) in the "completion" structure of \( \rho c K \), the projection \([\varepsilon]\) = \( \alpha^# \alpha \sim \{ a^# a \mid a \in \alpha \} \) yields

\[
(3) \quad [\beta],[\varepsilon] = [(b.a^# a \mid a \in \alpha, b \in \beta)] = [(a a b \mid a \in \alpha, b \in \beta)] = [\alpha] \wedge [\beta] = [\alpha],
\]

hence \( [\alpha] \preceq [\beta] \) in the "inverse" structure. Conversely, if \([\alpha] = [\beta].e\) for some projection \( e \) of \( \rho c K \), the relation \( e \preceq 1 \) in \( \rho c K \) implies \( [\alpha] \preceq [\beta] \) in the completion structure.
Last $[\alpha] \not\in [\beta]$ in the completion structure iff $\alpha ! \beta$ in $K$, iff $a ! b$ for all $a \in \alpha$ and $b \in \beta$, iff all the endomorphisms $ba^\rho$ and $b^\rho a$ are projections, iff $[\beta^\rho a]$ and $[\beta^\rho \alpha]$ are projections of $\rho c K$, iff $[\alpha] ! [\beta]$ in the inverse structure.

5.7. The inverse subcategory of a prj-category. Now, let $A$ be a prj-category. Define $K = Inv A$ as the subcategory of $A$ having the same objects and those morphisms $u: A \to B$ having a Morita inverse $u': B \to A$ in $A$, satisfying

(a) $u = uu'u$, \quad $u' = u'u$, \quad $u'u \leq 1_A$, \quad $uu' \leq 1_B$.

We prove now that $K$ is an inverse category whose projections (i.e. idempotent endomorphisms) coincide with the ones of $A$, the generalized inverse in $K$ being given by the Morita inverse in $A$.

First, $K$ is a subcategory of $A$; indeed, if $u: A \to B$ and $v: B \to C$ have Morita inverses $u'$ and $v'$, respectively, then $u'v'$ is a Morita inverse for $vu$:

1. $vu.u'v'.vu = v.uu'.v'v.u = vv'.vu'u = vu$,
2. $vu.u'v' = v(uu')v' \leq vv' \leq 1$.

Thus, $K$ is vN-regular (5.2). Every idempotent endomorphism $e$ of $K$ is a projection of $A$; indeed, if $v$ is a Morita inverse of $e$ then $v = vev = ve.ev$ is a projection, hence an idempotent, and $e = eve = ev.ve$ is a projection.

As the converse is trivial, the idempotent endomorphisms of $K$ coincide with the projections of $A$, hence commute; by 5.2, $K$ is inverse, and the generalized inverse of a morphism $u$ in $K$ is unique - it will be written as $u^\#$.

The embedding $Inv A \to A$ preserves the cohesive structure; generally, it does not reflect it. The Inv-construction is clearly functorial on prj-cohesive functors.

If $A$ is $\rho$-cohesive, so is $K$ with respect to its canonical cohesive structure: if $q_\rho$ is a linked $\rho$-set in $K$, so is $q^\# = \{u^\# \mid u \in q\}$; both $q$ and $q^\#$ are also linked in $A$, with resolutions $e_u = u^\# u$, $eu^\# = uu^\#$ ($u \in q$), and

3. $(Vq^\#)(Vq) = V u^\# v = V e_v e_u = V e_u \in \Prj A$, \quad $(u, v \in q)$
4. $(Vq)(Vq^\#)(Vq) = (Vq).e = Vu.e_v = Vq$.

It may also be noticed that an adjunction $u \dashv v$ in $A$ (characterized by $vu \geq 1, uv \leq 1$) implies $vu = 1$, hence is "the same" as a monomorphism $u$ of $K$ (with $v = u^\#$).

5.8. Examples. If $A = S$, the prj-category of small sets and partial mappings (see 1.1), then $J = Inv S$ is the subcategory of small sets and partial bijections.

Analogously $Inv T$ (resp. $Inv C^\rho$) is the category of topological spaces (resp. open euclidean sets) and partial homeomorphisms (resp. partial $C^\rho$-diffeomorphisms) between open subsets of the domain and codomain. All these inverse categories are totally cohesive (see 5.7).
6. Manifolds and glueing completion for inverse categories

In this chapter \( K \) is always an inverse category and \( \rho \) is an infinite section of cardinals (in the sense of 1.8). Manifolds over \( K \) are introduced as symmetrical enriched categories over \( K \). If \( K \) is \( \rho \)-cohesive, bilinked modules between \( \rho \)-manifolds produce the \( \rho \)-glueing completion \( \rho \text{IMf}K \) of \( K \).

6.1. Manifolds. A diagram \( U = (U_i, u^i_j)_I \), consisting of objects \( U_i \) (the charts) and morphisms \( u^i_j: U_i \rightarrow U_j \) of \( K \) (the glueing maps), indexed over a small set \( I \), will be said to be a manifold in \( K \) if

1. \( u^i_i = 1_{U_i} \) (identity law),
2. \( u^j_k u^k_i \leq u^j_i \) (composition law, or triangle inequality),
3. \( u^j_i = (u^j_i)^\# \) (symmetry law).

In other words, \( U \) is a small symmetric category enriched over the involutive (ordered) 2-category \( K \) \([Be, Wa, BC]\): notice that the first condition is equivalent to the usual one, \( u^i_i \geq 1 \) (by 5.1). We say that \( U \) is a \( \rho \)-manifold if its object-set \( I \) is a \( \rho \)-set.

The glueing of the manifold \( U \) in \( K \) (if existing) will be an object \( X = \text{gl} U \) provided with a family of morphisms \( u^i: U_i \rightarrow X \) (\( i \in I \)), such that

4. \( u^j u^i_j \leq u^i \),
5. \( u^j u^i_j = u^j_i \),

for all \( i, j \in I \), and universal in the obvious sense. According to definition 6.3, below, the family \( (u^i) \) is a "bilinked" module from \( U \) to the trivial manifold \( (X, 1_X) \).

\( K \) will be said to be \( \rho \)-glueing (as an inverse category) if it is \( \rho \)-cohesive and every \( \rho \)-manifold has a glueing; totally glueing inverse category, or just glueing inverse category, will mean \( \Omega \)-glueing.

From now on, we assume that the inverse category \( K \) is \( \rho \)-cohesive.

6.2. Proposition. With the previous notation, a family of morphisms \( u^i: U_i \rightarrow X \) (\( i \in I \)) is the glueing of the manifold \( U \) if and only if, for all \( i, j \in I \)

1. \( u^j u^i_j \leq u^i \),
2. \( u^j u^i_j = u^j_i \),
3. \( \bigvee_i u^i u^i_\# = 1_X \) (8); the condition (2) can be replaced with
4. \( u^j u^i_j \leq u^j_i \),
5. \( u^j u^i_j = u^j_i \),

Moreover, if \( y^i: U_i \rightarrow Y \) (\( i \in I \)) is any family of morphisms satisfying 6.1.4-5, the unique morphism \( y: X \rightarrow Y \) such that \( y^i = y u^i \) is given by

4. \( y = \bigvee_i y^i u^i_\# \) (linked join).

Every \( \rho \)-cohesive functor between \( \rho \)-cohesive inverse categories preserves the existing glueings of \( \rho \)-manifolds.

---

(8) These conditions mean that \( u = (u^i): U \rightarrow X \) is an isomorphism, in the category of manifolds over \( K \) (cf. 6.3, 6.4), from \( U \) to the one-index manifold \( X = (X, 1_X) \).
Proof. First, assume that $X$ is the glueing of $U$; then (1) and (2') hold by definition. To prove (3), we consider the projection $e = V_i u^i_\# j; X \rightarrow X$; clearly $eu^i = u^i$, for all $i$; by the uniqueness in the universal property of the glueing, it follows that $e = 1$.

Now, for (2"), let us fix some $h \in I$ and consider the family of morphisms $z^i: U_i \rightarrow U_h$, $z^i = u^i_h$ (for $i \in I$); since it satisfies the conditions 6.1.4-5

$$z^i_\# z^i = u^i_\# u^i \leq u^i,$$
there is exactly one morphism $z: X \rightarrow U_h$ such that $z^i = z u^i_i$ for all $i$; in particular $z u^h = z^h = u^h_\# = 1$, whence $u^h$ is monic and $u^h_\# u^h = 1$.

Secondly, (1, 2', 2") imply (2): $u^j_\# z^i = z u^j_\# u^j \leq u^j_\#$.

Last, if (1-3) hold, it is easy to check the universal property for $(X, u^i)$ by means of the formula (4), which concerns the join of a linked $\rho$-set, since

$$u^j_\# y^j u^j_\# \leq u^j_\# u^j_\# \leq u^j_\#$,

there is exactly one morphism $z: X \rightarrow U_h$ such that $z^i = z u^i_i$ for all $i$; in particular $z u^h = z^h = u^h_\# = 1$, whence $u^h$ is monic and $u^h_\# u^h = 1$.

The final assertion on $\rho$-cohesive functors is now trivial.

6.3. Bilinked modules. We now form the category $\rho \text{IMfK}$ of $\rho$-manifolds over $K$ and "bilinked modules" between them; we show below that this category is the inverse $\rho$-glueing completion of $K$.

A bilinked module $a = (a^h)_{i, h, i}: (U_i, u^i) \rightarrow (V_h, v^h)$ between the $\rho$-manifolds $U$ and $V$ will be a family of $K$-morphisms $a^h_i: U_i \rightarrow V_h$, satisfying (for $i, j \in I$ and $h, k \in H$)

$$a^h_\# a^i_k \leq u^i_j,$$
$$a^h_\# a^i_k \leq a^h_\# a^h_\# \leq 1,$$
where (1) is the usual condition for a module $a: U \rightarrow V$ between categories enriched on an ordered category [Be, Wa], while (2) expresses the linking property on domains and codomains.

Once that the category of bilinked modules is constructed (below), and provided with its canonical order as an inverse category (cf. 6.4), the condition (2) may be thought to mean that the modules $a = (a^h)_{i, h}$ and $a^\# = (a^h_\#)_{i, h}$ form a Morita context [Bi], i.e. $a^\# a = 1_U$ and $a a^\# = 1_V$.

On the other hand, notice that arbitrary modules cannot be composed, because $K$ lacks arbitrary joins.

The (matrix) composition of a with $(b^h)_{i, h, i}: (V_h, v^h) \rightarrow (W_m, w^m)$ is given by

$$c^h_i = V_h (b^h_i a^i)$,$$
where the join is legitimate and gives a bilinked module, because

$$a^h_\# a^h_\# a^h_\# a^h_\# \leq a^h_\# a^h_\# a^h_\# a^h_\# \leq 1,$$
$$a^h_\# a^h_\# a^h_\# a^h_\# \leq a^h_\# a^h_\# a^h_\# a^h_\# \leq 1.$$

It is easy to see that this is indeed a category, with identity of $U = (U_i, u^i)$ given by the bilinked endomodule $1_U = (u^i)$. The category $K$ has an obvious embedding in $\rho \text{IMfK}$ that identifies the object $U$ with the $\rho$-manifold $(U, 1_U)$. 

6.4. Theorem (the inverse structure). The category $\mathbf{M} = \rho \text{ImfK}$ is inverse.

The following conditions are equivalent

(i) $e = (a_{ij})_{I,I} : (U_i, u_{ij})_I \rightarrow (U_j, u_{ij})_I$ is a projection of $\mathbf{M}$,

(ii) $e = (a_{ij})_{I,I}$ is an endomorphism and $a_{ij} \leq u_{ij}$ for all indices $i, j$,

(iii) $e = (a_{ij})_{I,I}$ is an endomorphism, $e_i = a_{ii} \in \text{Prj } U_i$ and $a_{ij} = u_{ij} e_i = e_j u_{ij}$,

(iv) $e = (u_{ij} e_i)_{I,I}$ where $e_i \in \text{Prj } U_i$ and $u_{ij} e_j u_{ij} \leq e_i$ for all $i, j$.

Let $a = (a_{ih})_{I,H}$ and $b = (b_{ih})_{I,H}$ be maps from $U = (U_i, u_{ij})_I$ to $V = (V_h, v_{hk})_H$; let $e = (u_{ij} e_i)_{I,I}$ and $f = (v_{hk} f_{ih})_{H,H}$ be projections of $U$ and $V$, respectively. Then

1. $(fae)_h = f_h a_{ih} e_i$,
2. $a \leq b \iff a_{ih} \leq b_{ih}$ in $\mathbf{K}$, for all $i, h$;
3. $a \triangleright b \iff b_{ih} a_{ih} \leq u_{ij}$ and $b_{ih} a_{ih} \leq v_{hk}$ for all $i, j$ and $h, k$,
   \[ \iff a_{ih} \triangleright b_{ih} \text{ for all } i, h \text{ and } (a_{ih} v b_{ih})_{I,H} \text{ is a linked module}; \]
4. $a \wedge b = (a_{ih} \wedge b_{ih})_{I,H}$, $a \vee b = (a_{ih} \vee b_{ih})_{I,H}$ (for $a \triangleright b$).

Last, if $e_i \in \text{Prj } U_i$ ($i \in I$) is an arbitrary family of projections of our charts, the least projection $\hat{e} = (a_{ij})_{I,I}$ of the manifold $U$, with $\hat{e}_i \geq e_i$ for all $i$, is given by

5. $a_{ij} = \bigvee_h u_{ij} \hat{e}_i u_{ij}$.

Proof. See 9.3.

6.5. Inverse glueing completion theorem. The category $\mathbf{M} = \rho \text{ImfK}$ is the inverse $\rho$-glueing completion of $\mathbf{K}$.

Proof. See 9.4.

6.6. Examples. The inverse category $\mathcal{J} = \text{InvS}$ of small sets and partial bijections (see 5.8) is totally glueing: the glueing of the manifold $(U_i, u_{ij})_I$ is the set $X = \text{glU} = (\Sigma U_i)/R$, where $R$ is the equivalence relation (in the disjoint union $\Sigma U_i$) that identifies every $x \in \text{Def } u_{ij} \subset U_i$ with $u_{ij}(x) \in U_j$. The partial bijections $u^i : U_i \rightarrow X$ are obvious (and everywhere defined).

Analogously for $\text{InvT}$: we take on the glueing-set $X$ the finest topology that makes all the mappings $u^i$ continuous.

On the other hand, $\text{InvC}$ is totally cohesive and not glueing, even finitely: its (total) glueing completion "is" (i.e. can be interpreted as) the category of $\mathcal{C}$-manifolds and partial $\mathcal{C}$-diffeomorphisms.

Indeed, the inclusion $\text{InvC} \rightarrow \mathcal{J}$ extends, by the universal property of the glueing completion, to a unique glueing functor $\text{Imf(InvC)} \rightarrow \mathcal{J}$ (the topological realization of a manifold), that transforms the manifold $U = (U_i, u_{ij})_I$ into the space $X = \text{glU}$, the glueing of $U$ in $\mathcal{J}$.

This space $X$ is locally euclidean (with locally constant dimension), because of the partial homeomorphisms $u^i : U_i \rightarrow X$ (everywhere defined), whose images cover $X$; it is not necessarily
paracompact nor Hausdorff. It allows to reconstruct the manifold in the usual setting: a topological space \( X \) provided with an open covering \( (V_i) \) and a \( C^r \)-atlas of charts (onto open euclidean sets) \( v^i : V_i \to U_i \); namely, we take \( V_i = u^i(U_i) \) and \( v^i \) as the restriction of \( (u^i)^\# \) to its definition-set \( V_i \); the partial \( C^r \)-diffeomorphisms \( u^i_j \) are thus the coordinate changes.

6.7. Cauchy-completion and maximal manifolds. The notion of Cauchy-complete enriched category was introduced by Lawvere \([La]\) for a monoidal base and extended by Betti \([Be]\) to enrichment over a bicategory.

This notion has a straightforward adaptation to our case: symmetrical categories over a \( \rho \)-cohesive inverse category \( K \). However, the interest of such a notion in the present case is small: since the natural morphisms for manifolds are modules, the Cauchy-completion of a manifold would just produce an isomorphic object: the associated maximal glueing atlas; moreover these completions are still small manifolds provided that \( K \) is small, which in our examples may be true (e.g. for \( \text{Inv} C^r \)) or not (see \( \text{Inv} F \) and \( \text{Inv} B \) in Chapter 8).

Let us recall that, in the \emph{inverse} category \( \rho \text{IMf} \, K \), the data of an adjoint pair \( a \to b \) (i.e. a pair of bilinked modules satisfying \( ba \leq 1 \) and \( ab \leq 1 \)) are equivalent to giving a monomorphism \( a \) (cf. 5.1, with \( b = a^\# \)).

Now, a \emph{linked functor} \( f : (U_i, u^i_j)_{i} \to (V_h, v^h_j)_H \) between manifolds over \( K \) will be a mapping \( f : I \to H \) between their index-sets, such that \( U_i = V_{fi}, \ u^i_j = v^fi_j \, (9) \) for \( i, j \in I \). It produces a bilinked module

\[
  f = (f^i_h) : (U_i, u^i_j)_I \to (V_h, v^h_j)_H, \quad f^i_h = v^fi_h,
\]

which is monic \((f \to f^\#)\)

\[
(1) \quad (f^\# f_j^i) = v^f_j \ v^f_i = v^fi = u^i_j.
\]

Actually, the only case we are interested in is a (trivially linked) functor \( f : W \to M = (U_i, u^i_j)_I \) defined on a one-index manifold \( W = (W, 1) \); this is just the same as selecting an index \( h \in I \) such that \( W = U_h \), and produces the monic bilinked module \((f_i) : W \to M, \ f_i = u^i_h \).

The manifold \( M = (U_i, u^i_j)_I \) over \( K \) is said to be \emph{Cauchy-complete} if, for every \( W \) in \( K \), every monic bilinked module \((u_i) : W \to M \) is produced by such a functor \( f \), i.e. there is some \( h \in I \) such that \( W = U_h \), \( u_i = u^i_h \).

Now, it is easy to see that the data of a monic bilinked module \( u = (u_i) : W \to M \) \((u^\# u = 1)\) is equivalent to "adding to \( M \) a redundant chart": in other words, giving a larger glueing atlas \( M' = (U_i, u^i_j)_I \) with \( J = I \cup \{k\} \, (k \notin I) \) and requiring that the bilinked module \((u^i_j)_{I,J} : M \to M' \) be an isomorphism. The correspondence between these notions is established by the equations \( U_k = W, \ u^k_i = u_i, \ u^k_i = u^i_k \).

Thus the manifold \( M \) is Cauchy-complete if and only if it is a \emph{maximal} glueing atlas, that is if "every compatible chart is already in \( M \).

If \( K \) is small every manifold is contained in a maximal isomorphic one, its Cauchy-completion.

\( (9) \) For a functor, one would here require \( \leq \) instead of equality.
7. Manifolds and glueing completion for prj-categories

A is always a ρ-cohesive prj-category and K = InvA the associated ρ-cohesive inverse category (cf. 5.7). The simpler, more particular case of a ρ-cohesive e-category is treated in 7.8.

7.1. Manifolds and glueing. A manifold over A will be a diagram U = (Ui, ui j)I in A, with ui j: Ui → U j (i, j ∈ I), such that
1. ui i = 1Ui (identity law),
2. ui j.ui j ≤ ui k (composition law, or triangle inequality),
3. ui j = ui jui jui j (symmetry law).

Since ui jui j ≤ ui i = 1Ui and because of (3), all the morphisms ui j are actually in K = InvA, and satisfy (ui j)♯ = ui j. In other words, the manifolds of A are precisely those of K.

The glueing X = gl U of the manifold U in A (if existing) will be, by definition, its lax colimit, that is an object X provided with a universal lax cocone ui: Ui = X (i ∈ I) in A. This means that
(a) ui jui j ≤ u i, for all i, j,
(b) for every lax cocone yi: Ui → Y (satisfying yi jui j ≤ yi), there exists a unique map y: X → Y in A such that yi = y.ui (for i ∈ I),
(c) if y', y": X → Y and y'.ui ≤ y".ui (for i ∈ I), then y' ≤ y".

We show below that this problem is equivalent to the glueing of U in K (cf. 6.1-2).

Also here, a prj-category will be said to be ρ-glueing if it is ρ-cohesive and every ρ-manifold has a glueing.

7.2. Theorem. Let U = (Ui, ui j)I be a manifold over A (and K), and ui: Ui → X (i ∈ I) a family of morphisms in A.

Then (X, u i) is the glueing of U in A if and only if it is so in K. In such a case the morphisms u i are monomorphisms of K and for every lax cocone yi: Ui → Y in A, the corresponding morphism y: X → Y is given by
1. y = V i yi u i (linked join in A).

A is glueing if and only if InvA is. Every ρ-cohesive functor between ρ-cohesive prj-categories preserves the existing glueings of ρ-manifolds.

Proof. If (X, u i) is the glueing of U in K, the formula (1) concerns the join of a linked ρ-set in A(X, Y), with resolution ei = u i u i♯ ∈ Prj X (i ∈ I)
2. yi u i♯.ei = yi,
y i u i♯.ei = yi u i u i u i♯ = yi u i u i♯ ≤ yi u i♯.

It is now easy to check, as in 6.2, the universal properties 7.1 (b), (c) in A.

Conversely, assume that (X, u i) is the glueing of U in A. Fix an index h ∈ I and consider, as in the proof of 6.2, the family zi = u i h: Ui → U h (i ∈ I) of morphisms of A; they form a lax cocone
from \( U \) (as in 6.2.5), hence there is one morphism \( z: X \to U_h \) of \( A \) such that \( z^i = zu^i \) (\( i \in I \)). In particular, \( zu^h = 1 \); moreover \( (u^h z)u^i = u^h z u^i u^h = u^i \), for all \( i \), so that \( u^h z \leq 1 \) (7.1 (c)); therefore \( u^h \) is in \( \text{Inv}A \), with generalized inverse \( (u^h)^\# = z \).

It suffices now to verify the conditions 6.2.2-3; the relation

\[
(3) \quad u^h\#u^i = zu^i = z^i = u^i_h,
\]
gives the first, by the arbitrariness of \( h \in I \). The second follows from

\[
(4) \quad (V_i u^i u^i\#)u^j = u^j,
\]
by means of the uniqueness property in 7.1 (a): \( (V_i u^i u^i\#) = 1 \).

The last statement follows now from the last assertion in 6.2.

7.3. Linked modules. We form the category \( \rho \text{Mf}A \) of \( \rho \)-manifolds over \( A \) and linked modules between them.

A module \( (a^i_h)_{i,H}: (U_i, u^i_j) = (V^h_k, v^h_k) \) is a family of \( A \)-morphisms \( a^i_h: U_i \to V^h_k \) such that, for all \( i, j \in I \) and \( h, k \in H \)

\[
(1) \quad v^h_k a^i_h \leq a^i_k, \quad a^i_h u^i_j \leq a^j_k \quad \text{(module laws)}.\]

It will be said to be linked (or compatible) if it has a resolution \( e^i_h \in \text{Prj} U_i \) (\( i \in I, \ h \in H \))

\[
(2) \quad a^i_h e^i_k = v^h_k a^i_k \quad \text{(linking law),}
\]
or equivalently

\[
(2^\prime) \quad a^i_h e^i_h = a^i_h, \quad (2^\prime^\prime) \quad a^i_h e^i_h \leq v^h_k a^i_k,
\]
since, from (2') and (2''), we have \( v^h_k a^i_k = v^h_k a^i_k e^i_h \leq a^i_h e^i_h \).

In a resolution, each \( e^i_h \) can be clearly replaced with any \( e^i_h \) with \( e^i_h \leq e^i_h \) and \( a^i_h e^i_h = a^i_h \).

Thus, in the e-cohesive case, the linking condition (2) may be more simply expressed by means of supports: \( e^i_h = e^i(a^i_h) \) (cf. 7.8).

Clearly \( \rho \text{Mf}K \subset \rho \text{Mf}A \). But note that a linked module over \( A \) whose components \( a^i_h \) are in \( K \) need not belong to \( \rho \text{Mf}K \); this happens if and only if the "reverse" module \( (a^i_h)^\# \) is also linked. It is easy to give counterexamples in the categories \( S \) and \( T \), where the linking condition (2) forces the module \( (a^i_h) \) (more precisely, its glueing) to be "single-valued" but not "injective", even if all the components are so. We shall prove in 7.6 that \( \rho \text{Mf}K \) coincides with \( \text{Inv}(\rho \text{Mf}A) \).

Again, the composition is matrix-like: if \( (b^j^h_m)_{H,M}: (V^h_k, v^h_k) \to (W^m_n, w^m_n) \) is a linked module

\[
(3) \quad (b^j^h_m)_{H,M} (a^i_h)_{I,H} = (c^i_j)_{I,M}, \quad c^i_j = V^h_k (b^j^h_m a^i_h),
\]

We prove that \( ba \) is well defined. Let \( (f^i_m) \) be a resolution of \( b = (b^i_h) \) and let us choose projections a family of \( e^i_h \in \text{Prj} U_i \) such that

\[
(4) \quad f^i_m a^i_h = a^i_h e^i_h, \quad e^i_h \leq e^i_h, \quad (i \in I, \ h \in H, \ m \in M).
\]

Then each family \( (b^j^h_m a^i_h)_{h \in H} \) is linked, with resolution \( (e^i_h m)_{h \in H} \)

\[
(5) \quad (b^j^h_m a^i_h) e^i_h m = b^j^h_m f^i_m a^i_h = b^j^h_m a^i_h,
\]

\[
(6) \quad (b^j^h_m a^i_h) e^i_k m = b^j^h_m a^i_h e^i_k m \leq b^j^h_m v^h_k a^i_k \leq b^j^h_m a^i_h.
\]
More generally, for \( n \in M \)
\[
(7) \quad (b_m^h a_i^h) e_{ikn} = b_m^h v_k^i a_i^h e_{ikn} = b_k^h a_i^h e_{ikn} = b_m^h f_k a_i^h = w_m^b b_n^k a_i^h,
\]
and \((c_i^m)\) is a linked module, with resolution \( e_{im} = V_h e_{ihm} \)
\[
(8) \quad c_m^i u_j^i = V_h (b_m^h a_i^h u_j^i) \leq V_h (b_m^h a_i^h) = e_m^i
\]
(9) \( c_m^i e_{im} = V_h (b_m^h a_i^h e_{ikm}) \leq V_h b_m^h a_i^h = c_m^i \quad \text{(by (5))},
\]
(10) \( c_m^i e_{im} = V_h (b_m^h a_i^h e_{ikm}) \leq V_k w_m^b b_n^k a_i^k = w_m^b. V_k b_n^k a_i^k = w_m^b c_n^i \quad \text{(by (7)).}
\]
This is indeed a category and \( A \) embeds in \( \rho MfA \) as in the inverse case (6.3): \( U \mapsto (U, 1_U) \).

7.4. The prj-structure. We define the projections of \( \rho MfA \) as those of \( \rho MfK \), described in 6.4. Note that, as in 6.4.1 and with the same proof, if \( a: U \rightarrow V \) is a morphism in \( \rho MfA \), \( e \in \text{Prj} U \) and \( f \in \text{Prj} V \), then
\[
(1) \quad (fae)_i^h = f_h a_i^h e_i.
\]
The axiom (PCH.1) holds, because \( \rho MfK \) is inverse. As to (PCH.2), given the linked module \( a: U \rightarrow V \) in \( \rho MfA \), with resolution \((e_i^h)\), and \( f \in \text{Prj} V \), let us choose a family of projections \( e'_ih \in \text{Prj} U_i \) such that
\[
(2) \quad f_h a_i^h = a_i^h e'_ih, \quad e'_ih \leq e_{ih} \quad (i \in I, \ h \in H).
\]
Further, we let
\[
(3) \quad e_i^h = V_h e_{ih}, \quad \hat{e}_i = V_j (u_j^i e'_j u_j^i),
\]
so that, by 6.4.5, \( \hat{e} = (u_j^i \hat{e}_j) \) is the projection of the manifold \( U \) spanned by the family \((e_i^h)\). We prove that \( fa = a\hat{e} \)
\[
(4) \quad a_i^h V_k e_{ih} = V_k (a_i^h e_{ih}) = V_k (v_k^i a_i^h e_{ih}) = V_k (v_k^i f_k a_i^h) =
\]
\[
= V_k (f_h v_k^i a_i^h) = f_h a_i^h,
\]
(5) \( (fa)_i^h = f_h a_i^h = a_i^h e_{ih} \leq a_i^h e_i^h \leq a_i^h \hat{e}_i = (a\hat{e})_i^h, \)
(6) \( (a\hat{e})_i^h = a_i^h \hat{e}_i = V_j (a_i^h u_j^i e'_j u_j^i) \leq V_j (a_i^h e'_j u_j^i) = V_j (a_i^h u_j^i u_j^i) = f_h a_i^h (fa)_i^h \quad \text{(by (4))}. \)

7.5. Lemma. If \( a = (a_i^h) \) and \( b = (b_i^h) \) are parallel morphisms in \( \rho MfA \)
\[
(1) \quad a \leq b \iff \text{ the modules } a, b \text{ have resolutions } (e_{ih}), (f_{ih}) \text{ such that:}
\]
\[
e_{ih} \leq f_{ih} \quad \text{and} \quad a_i^h = b_i^h e_{ih} \quad \text{(for all } i, h),
\]
\[
\iff a_i^h \leq b_i^h \quad \text{(for all } i, h); \]
(2) \( a \not\leq b \iff \text{ the modules } a, b \text{ have resolutions } (e_{ih}), (f_{ih}) \text{ such that:}
\]
\[
a_i^h f_{ik} \leq b_i^h \quad \text{and} \quad b_i^h e_{ik} \leq a_i^h \quad \text{(for all } i, h, k),
\]
\[
\iff a_i^h \not\leq b_i^h \quad \text{(for all } i, h) \quad \text{and} \quad (a_i^h \lor b_i^h) \text{ is a linked module};
\]
(3) \( a \land b = (a_i^h \land b_i^h)_{i,H}, \quad a \lor b = (a_i^h \lor b_i^h)_{i,H} \quad \text{(for } a \lor b). \)
Proof. If $a \leq b$, then $a = b.e$ and $a_l^i = b_l^i e_l^i \leq b_l^i$. Assume now that $a_l^i \leq b_l^i$ for all $i$ and $h$; let $(e_l^i h)$, $(f_l^i h)$ be resolutions of $a$ and $b$, respectively, and let us choose projections $e_l^i h$ such that $a_l^i = b_l^i e_l^i h$; then the family $e_l^i h = e_l^i h, e_l^i f_l^i h$ is a resolution of $a$ (by 7.3) that satisfies, with $(f_l^i h)$, our conditions. Last, if the resolutions $(e_l^i h)$ and $(f_l^i h)$ satisfy these conditions, we write $e_l^i = \bigvee h e_l^i h$ and $\hat{e}$ the projection of $U$ spanned by the family $(e_l^i)$, as in 7.4.3, so that $a = b\hat{e}$.

(4) $a_l^i = b_l^i e_l^i h \leq b_l^i, \hat{e}_l^i = (b\hat{e})_l^i$.

(5) $(b\hat{e})_l^i h = b_l^i, \hat{e}_l^i = b_l^i . V_j (u_j^i e_j^i u_j^i) = V_j (b_l^i, u_j^i e_j^i u_j^i) \leq V_j k b_l^i, e_k^i j u_j^i \leq V_j k b_l^i, f_k^i e_k^i j u_j^i = V_j k v_h^k b_l^i, e_k^i j u_j^i = V_j k v_h^k a_l^i e_h^i = a_l^i$.

The proof of (2) and (3) is similar (see also 6.4).

7.6. We prove now that, for the $\rho$-cohesive prj-category $A$, the inverse $\rho$-glueing completion of $K = Inv A$ coincides with the inverse subcategory of the $\rho$-glueing completion of $A$.

(1) $\rho IMf K = Inv (\rho Mf A)$.

Trivially, a bilinked module $a = (a_l^i h)_{l, h}$ over $K = Inv A$ is a linked module over $A$, provided with a Morita inverse $(a_l^i h#)_{h, l}$ (cf. 5.7) in $\rho Mf A$.

Conversely, let $a = (a_l^i h)_{l, h}: (U_l, u_l^i j)_{l, j} \rightarrow (V_l, v_l^i k)_{k, l}$ be a linked module, with resolution $(e_l^i h)$, that has a Morita inverse $b = (b_l^i h)_{l, h}$. Then $ba$ and $ab$ are projections of $\rho Mf A$, and all the composites $b_l^i a_l^i$ and $a_l^i b_l^i$ are also

(2) $b_l^i a_l^i \leq (ba)_l^i \leq 1$.

Moreover $(ba)_l^i e_l^i h = b_l^i a_l^i$ since

(3) $(ba)_l^i e_l^i h = V_k (b_l^i k a_l^i k e_l^i h) \leq V_k (b_l^i k v_l^i k a_l^i h) \leq b_l^i a_l^i \leq (ba)_l^i e_l^i h$.

Finally $a_l^i = a_l^i b_l^i a_l^i$ because

(4) $a_l^i = (aba)_l^i e_l^i h = V_j a_l^i j (ba)_l^i e_l^i h = V_j a_l^i j (ba)_l^i e_l^i h \leq a_l^i (ba)_l^i e_l^i h = a_l^i b_l^i a_l^i \leq (aba)_l^i \leq a_l^i$.

7.7. Glueing completion theorem. The prj-category $\rho Mf A$ is the $\rho$-glueing completion of $A$.

Proof. It is an easy consequence of the inverse glueing completion theorem 6.5 and the previous arguments. A direct proof, in the simpler e-cohesive case, can be found in [G4].

By 7.6 and the inverse glueing completion theorem, $Inv(\rho Mf A) = \rho IMf K$ is $\rho$-glueing (as an inverse category); hence the prj-category $\rho Mf A$ is $\rho$-glueing (by 7.2). Now, if $F: A \rightarrow B$ is a totally cohesive prj-functor with values in a glueing prj-category, $F$ transforms manifolds and linked modules over $A$ into manifolds and linked modules over $B$, which can be glued in $B$.

7.8. The e-cohesive case. Let $A$ be a $\rho$-cohesive e-category: then the previous results take a simpler form. Notice that, for every $u$ in $K = Inv A$, the support of $u$ in $A$ is $e(u) = u^e u$.

A module $a = (a_l^i h)_{l, h}: (U_l, u_l^i j)_{l, j} \rightarrow (V_l, v_l^i k)_{k, l}$ between $\rho$-manifolds over $A$ (satisfying the module laws 7.3.1) is linked if and only if it satisfies the following equivalent conditions
(1) \( a^i_h e(a^i_h) = v^k_h a^i_k \) (linking law),

(1') \( a^i_h e(a^i_k) \leq v^k_h a^i_k \), if and only if the family \((e(a^i_h))\) is a resolution of \(a\) (the least one).

The prj-category \(\rho MfA\) of \(\rho\)-manifolds and linked modules over \(A\) is now e-cohesive, with

(2) \((e(a))_i \cdot = V_h e(a^i_h), \quad (e(a))_i^j = u^j_i (e(a))_i = (e(a))_j u^j_i.\)

Indeed, \((e(a))_i\) is a projection of the manifold \((U_i, u^j_i)\), according to 6.4 (iv)

(3) \(a^i_h (u^j_i e(a^i_h)) u^j_i \leq a^i_h e(a^i_h) u^j_i = a^i_h u^j_i \leq a^i_h,\)

(4) \((u^j_i e(a^i_h)) u^j_i \leq e(a^i_h),\)

(5) \((u^j_i (e(a))_i^j) \leq V_h e(a^i_h) = (e(a))_i.\)

We verify now the axioms of the e-structure. For (ECH.1): \((a.e(a))_i^j = a^i_h,\) as follows from the argument below (for \(b = 1\)). On the other hand, \(a = ae\) in \(\rho MfA\) implies \(a^i_h = a^i_h e_i\) (by (4)), hence \(e_i \geq e(a^i_h)\) for every \(h\), and \(a \geq e(a)\). Last, for (ECH.2), given a second module \(b = (h^b)_j^m\): \((V_h, v^b_h) \to (W_m, w^m_n),\) we have

(6) \((a.e(ba))_i^j = a^i_h e(ba)_i = a^i_h V_k (e(ba)^i_k) = a^i_h V_{k,m} e(b^m_k a^i_k) = a^i_h V_k (e(a^i_k) e(b^m_k a^i_k)) = V_{k,m} (v^k_h e(a^i_k) e(b^m_k a^i_k)) = V_k (v^k_h (V_m e(b^m_k a^i_k))) = V_k ((e(b))_h v^k_h a^i_k) = (e(b))_h a^i_h = (e(b) a)_h.\)

Finally, from 7.5, for two parallel linked modules \(a, b\)

(7) \(a \leq b \iff a^i_h \leq b^i_h e(a^i_h)\) (for all \(i, h),\)

(8) \(a ! b \iff a^i_h e(b^i_k) \leq b^i_h \) and \(b^i_h e(a^i_k) \leq a^i_h\) (for all \(i, h, k).\)

7.9. Differentiable manifolds. The e-categories \(S\) and \(T\) are glueing. The e-category \(C^r\) is totally cohesive and not glueing (even finitely): its glueing completion is the category of \(C^r\)-manifolds (as in 6.6) with partial \(C^r\)-mappings (defined on open subsets). Also here, the inclusion \(C^r \to T\) extends to a glueing functor \(MFc^r \to T\), the topological realization of \(C^r\)-manifolds.

Manifolds with boundary can be obtained in a similar way, by glueing the open subspaces of the spaces \(H^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_n \geq 0\}\).

The category \(MFc^r\) can also be obtained (up to equivalence of categories) by glueing completion of the full subcategory of \(C^r\) whose objects are the euclidean spaces \(\mathbb{R}^n\), since each open euclidean space is a union (and a glueing in \(C^r\)) of open balls. It can be noticed that our totally cohesive e-subcategory of \(C^r\) yields back \(C^r\) by a projection-completion procedure (analogous to the well-known idempotent completion).

8. Fibre bundles, vector bundles and foliations

We sketch here a definition of fibre bundles, vector bundles and foliations as "manifolds" over the e-cohesive categories of the corresponding trivial structures. For fibre and vector bundles, the
topological realization takes place in a (glueing) category $\mathcal{F}$ of generalized "fibrations" $p: X \to B$, playing the role of $\mathcal{T}$ for differentiable manifolds.

8.1. A glueing category. Here a fibration will be just a continuous, surjective (everywhere defined) mapping $p: X \to B$ between topological spaces.

Let us form the category $\mathcal{F}$ of fibrations and partial maps $(f, \overline{f}): p \to p'$, provided by commutative diagrams in $\mathcal{T}$

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow p & & \downarrow p' \\
B & \xrightarrow{\overline{f}} & B'
\end{array}
\]

Thus $f$ and $\overline{f}$ are partial continuous mappings, defined on open subsets of $X$ and $B$ respectively, with

(2) $\text{Def } f = p^{-1}(\text{Def } \overline{f})$, $\text{Def } \overline{f} = p(\text{Def } f)$,

and $\overline{f}$ is determined by $f$.

A projection $(e, \overline{e}) : p \to p$ of $\mathcal{F}$ will be any pair of partial identities on a distinguished pair $(p^{-1}(W), W)$ of the fibration $p$, determined by any open subset $W$ of the base $B$. $\mathcal{F}$ becomes thus an e-category.

The inverse category $\text{Inv}\mathcal{F}$ has the same objects and for morphisms the pairs $(u, \overline{u}) : p \to p'$ composed by partial homeomorphisms between distinguished pairs of $p$ and $p'$, making (1) to commute.

$\mathcal{F}$ is totally cohesive and glueing: if $M = (p_i: X_i \to B_i, (u_{ij}, \overline{u}_{ij}))_i$ is a manifold over $\mathcal{F}$, its glueing $p: X \to B$ in $\mathcal{F}$ can be obtained by glueing in $\mathcal{T}$ the spaces $X_i$, the bases $B_i$ and the modules determined by the fibrations $p_i$

(3) $X = \text{gl}(X_i, u_{ij})_i$, $B = \text{gl}(B_i, \overline{u}_{ij})_i$, $p = \text{gl}(u_{ij}, p_i: X_i \to B_j)$.

The full subcategory $\mathcal{F}_0$ determined by Serre fibrations has similar properties; it can be substituted to $\mathcal{F}$ in the following, yielding straightforwardly the homotopy-lifting property for fibre bundles.

8.2. Fibre bundles. The "elementary spaces" that we want to patch together are the trivial fibre bundles, i.e. the cartesian projections $(10) p: B \times F \to B$, where $B$ and $F$ are topological spaces and $B \times F$ has the product topology.

Let $\mathcal{B}$ be the full subcategory of $\mathcal{F}$ determined by such objects, with the induced e-cohesive structure; this is totally cohesive but not glueing. For a morphism $(f, \overline{f}): p \to p'$ in $\mathcal{B}$

(1) $\text{Def } f = p^{-1}(\text{Def } \overline{f}) = (\text{Def } \overline{f}) \times F$,

(2) $f(b, y) = (\overline{f}(b), f_2(b, y))$,

(10) Not to be confused, of course, with the selected endomaps which we call projections.
so that a morphism can also be given by two morphisms in $\mathcal{J}$, $\tilde{f}: B \to B'$ and $f_2: B \times F \to F'$, with $\text{Def } f_2 = (\text{Def } \tilde{f}) \times F$.

The trivial fibre bundle $p: B \times F \to B$ will also be written $B \times F$; the morphism $(f, \tilde{f})$ will then be denoted by its component $f$ (that determines $\tilde{f}$).

The inverse category $\text{Inv}B$ has the same objects and for morphisms the pairs $(u, \tilde{u}): p \to p'$ composed by partial homeomorphisms between distinguished pairs of $p$ and $p'$, that make 8.2.1 to commute. As in (2), this is equivalent to giving two mappings of $\text{Inv} \mathcal{T}$, $u: B \to B'$ and $u_2: B \times F \to F'$ (partial homeomorphism between open subsets), such that $\text{Def } u_2 = (\text{Def } \tilde{u}) \times F$ and for every $b \in \text{Def } \tilde{u}$, the map $u_2(b, -): F \to F'$ is a homeomorphism. Thus, provided that the morphism $u$ is not empty, the fibres $F$ and $F'$ are homeomorphic.

The glueing completion $MfB$ has for objects the "manifolds" $M = (B_i \times F_i, u^i_j)_{i,j}$ over $B$, for morphisms their bilinked modules: it is the category of fibre bundles and partial maps. The inclusion $B \to \mathcal{J}$ (or, more tightly, $B \to \mathcal{J}_0$) extends to the topological realization functor $MfB \to \mathcal{J}$ (or $MfB \to \mathcal{J}_0$), that takes the above object $M$ to its glueing (8.1.3).

By the above characterization of the morphisms of $\text{Inv}B$, the topological type of the fibre $F_b = p^{-1}(\{b\})$ at the point $b$ of the base $B = gl(B_i, u^i_j)_I$ is locally constant, hence constant on every connected component of $B$.

8.3. Vector bundles. A trivial vector bundle is a trivial fibre bundle $p: B \times F \to B$, where $B$ is a topological space and $F$ is a finite-dimensional, real vector space (provided with the linear topology).

Let $\mathcal{V}$ be the subcategory of $\mathcal{B}$ (and $\mathcal{J}$) having such objects, with "fiberwise linear" morphisms $f: B \times F \to B' \times F'$; this means that, for every $b \in \text{Def } f_0$, the (everywhere defined) mapping $f_2(b, -): F \to F'$ (8.2.2) is $\mathbb{R}$-linear. A morphism $u: B \times F \to B' \times F'$ of $\text{Inv}\mathcal{V}$ is in $\text{Inv} \mathcal{B}$ (8.2); moreover, for every $b \in \text{Def } \tilde{u}$, $u_2(b, -): F \to F'$ is a linear isomorphism.

The glueing completion $Mf\mathcal{V}$ yields vector bundles and their usual morphisms (partially defined, on distinguished pairs). Also here we have the topological realization in $\mathcal{J}$, or in $\mathcal{J}_0$.

8.4. Differentiable manifolds and tangent bundles. Consider again the category $\mathcal{C}^r$ (of trivial $\mathcal{C}^r$-manifolds), with $r \geq 1$. The (trivial) tangent bundle functor, with the abuse of notation described in 8.2, is:

1. $T: \mathcal{C}^r \to \mathcal{V}$, $U \mapsto U \times \mathbb{R}^{\dim U}$, $f \mapsto Tf$,
2. $Tf(x, h) = (fx, D_hf(x))$, for $x \in \text{Def } f$ and $h \in \mathbb{R}^{\dim U}$,

where $D_hf(x)$ is the derivative of $f$ at $x$, along the vector $h$.

Since $\mathcal{J}$ is totally cohesive, $T$ extends to a glueing functor, namely the tangent bundle functor $Mf\mathcal{C}^r \to Mf\mathcal{V}$ of $\mathcal{C}^r$-manifolds.

8.5. Foliations. A trivial foliation is a cartesian product $U \times V$, where $U$ and $V$ are open euclidean spaces; the subsets $V_x = \{x\} \times V$ are its leaves (for $x \in U$). A partial $\mathcal{C}^r$-map $f: U \times V \to U' \times V'$ (of
trivial foliations) is a partial $C^r$-mapping, defined on an open subset of $U \times V$, which preserves leaves: if $(x, y_1)$ and $(x, y_2)$ are in $V$, their $f$-images are in the same leaf of $U' \times V'$ (11).

All this forms the category $C^r F$ of trivial $C^r$-foliations and partial $C^r$-maps, ordered by restriction. It is a totally cohesive e-category, whose glueing completion $MF C^r F$ yields the category of $C^r$-foliations, with partial $C^r$-maps of foliations.

9. Proof of the completion theorems

We now prove the $\rho$-cohesive completion theorem (2.7) and the two theorems on the $\rho$-glueing completion of an inverse category (6.4, 6.5).

9.1. The category of linked $\rho$-sets. Let $A$ be a category provided with a proximity relation $!$ (and no order): we embed $A$ in a category $P_\rho A$ with order and proximity satisfying (CH.1-3) and that part of (CH.5) which deals with joins.

The objects are the same. A morphism $\alpha \in P_\rho A (A, B)$ is given by any $\rho$-set $\alpha \subset A(A, B)$, linked in $A$ (including the empty subset $0_A^B$, if $0 \in \rho$). The composition of $\alpha: A \rightarrow B$ with $\beta: B \rightarrow C$ is obviously:

$$\beta \alpha = \{ ba | a \in \alpha, b \in \beta \},$$

that is again a linked $\rho$-set of $A$-morphisms from $A$ to $C$.

$P_\rho A$ is obviously a category, with the identity of $A$ given by the subset $\{1_A\}$; we equip $P_\rho A$ with the inclusion relation $\alpha \subset \alpha'$ (for parallel maps) and the linking relation:

$$\alpha ! \alpha' \text{ if } a!a' \text{ in } A, \text{ for all } a \in \alpha \text{ and } a' \in \alpha'.$$

Now (the axioms CH.1-3) are trivially satisfied. Let $\Sigma \subset P_\rho A (A, B)$ be a linked $\rho$-set of $P_\rho A$ and let $\beta = \bigcup \Sigma \subset A(A, B)$: this is again a $\rho$-set (1.8) of parallel morphisms of $A$, clearly linked. The morphism $\beta$ is the join of the set $\Sigma$ with respect to the order of $P_\rho A$. Moreover, the join is compositive: if $\gamma: A' \rightarrow A$ and $\delta: B \rightarrow B'$ are in $P_\rho A$, we have

$$\delta \beta \gamma = \{ dbc | c \in \gamma, b \in \bigcup \Sigma, d \in \delta \} = \bigcup_{\alpha \in \Sigma} \{ dbc | c \in \gamma, b \in \alpha, d \in \delta \} = V_{\alpha \in \Sigma} \delta \alpha \gamma.$$.

It may be noticed that $P_\rho A$ has arbitrary non-empty meets; however these are not compositive, even in the binary case, and will play no role in the following steps.

9.2. Proof of the $\rho$-cohesive completion theorem (2.7). Now $A$ is a cohesive category and $P_\rho A$ is the category of its linked $\rho$-sets, constructed on $(A, !)$.

\[\text{(11)}\] In other words: there exists a partial map $f: U \rightarrow U'$ (also of class $C^r$) defined on $p(\text{Def } f)$, such that $p'f \leq \tilde{f}p$, where $p: U \times V \rightarrow V$ and analogously $p'$. Compare this with the stronger condition of commutativity in 8.1: a partial map of foliations need not be defined on a union of leaves.
Consider the binary relation \( \alpha \prec \beta \), on parallel morphisms of \( \mathcal{P}_\rho A \), defined by \( \alpha \prec \beta \) and the following equivalent conditions

1. \( \forall a \in \alpha^*, \ a = V_{b \in \beta} (a \land b) \) (linked join),
2. \( \forall a \in \alpha^*, \ a = V_{b \in \beta^*} (a \land b) \) (linked join),
3. \( \forall a \in \alpha^*, \ a = V_{\{b \mid b \in \beta^*, \ b \leq a\}} \) (linked join),

where \( \alpha^* \) denotes the down-closed subset generated by \( \alpha \), and the equivalence of the three properties comes from 1.7 (c) (see the last remark of 2.2).

It is a preorder of categories: let \( \alpha \prec \beta \prec \gamma \). If \( a \in \alpha \) and \( c \in \gamma \), it follows that \( (a \land b) \downarrow c \) (for each \( b \in \beta \)) and \( a = V_{b \in \beta^*} (a \land b) \downarrow c \). Moreover, by 1.7 (a), (c), we have linked joins:

\[ a = V_{\{c \mid c \in \gamma^*, \ c \leq b \leq a \} \text{ for some } b \in \beta^*} \]

The pre-order is consistent with composition because linked joins and meets are.

Let \( \sim \) be the congruence associated to the preorder \( \prec \), and let us consider the quotient category

\( \rho \mathcal{A} = \mathcal{P}_\rho A / \sim \),

provided with the order \( \leq \) induced by the preorder \( \prec \), namely \([\alpha] \leq [\beta] \) iff \( \alpha \prec \beta \) (independently of the choice of representatives). The linking relation is defined by \([\alpha] \downarrow [\beta] \) as linked sets of \( A \) (again, independently of the choice).

As to (CH.4, 5\( \rho \)), linked meets and linked \( \rho \)-joins are calculated in \( \rho \mathcal{A} \) by the following formulas:

\[ [\alpha] \land [\beta] = \{a \land b \mid a \in \alpha, b \in \beta\} = [\alpha^* \cap \beta^*], \quad \text{for } [\alpha] \downarrow [\beta], \]

\[ \lor \Sigma' = \bigcup \Sigma, \]

where \( \Sigma \) is any linked \( \rho \)-set of \( \rho \)-sets of \( A \) (\( \alpha, \alpha' \) for all \( \alpha, \alpha' \in \Sigma \)) and \( \Sigma' = \{[\alpha] \mid \alpha \in \Sigma\} \).

Last, we define the functor \( \eta : A \to \rho \mathcal{A} \) taking the object \( A \) to itself and the morphism \( a \) to the equivalence class of \( \{a\} \). Clearly, \( \eta \) reflects the order and linking relations, is cohesive and preserves the existing linked \( \rho \)-joins of \( A \). To verify the universal property, we set \( G([\alpha]) = \lor Fa \ (a \in \alpha) \) and check that \( G \) is a \( \rho \)-cohesive functor; its uniqueness is trivial.

(a) \( \mathcal{M} \) has a natural regular involution :

1. \((a_{h})_{1,H} = (a_{h})_{H,1},
2. \((a_{h})_{1,h} (a_{h})_{h,1} = (a_{h})_{1,H,H},

where the last equality follows from \( a_{k} \land a_{k} \leq a_{k} \leq a_{k} \) for all \( k \) and \( j \), with equality for \( j = i \) and \( k = h \).

(b) We now prove the equivalence of (i) - (iv) (see 6.4), where a projection is any idempotent endomap, symmetric with respect to the previous involution.

(i) \( \Rightarrow \) (ii). \( a_{i} = c = c \downarrow c = (V_{h} a_{h} \downarrow a_{h}) \) and \( a_{j} = V_{h} a_{h} \downarrow a_{h} \leq u_{j} \).

(ii) \( \Rightarrow \) (iv). \( e_{i} = a_{i} \leq u_{i} = 1 \) is a projection of \( U_{i} \) and:

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(3) \[ a^i_j = a^i_j a^g_j \preceq a^i_j c_i \preceq u^i_j c_i = u^i_j a^i_j \preceq a^j_i, \]
so that \( a^i_j = u^i_j c_i \) and \( u^i_j c_i u^j_i = a^i_j \preceq a^i_j = e_i \) for all \( i, j \).

(iv) \( \Rightarrow \) (iii). It is easy to show that \( u^i_j c_i = e_j u^j_i \). The family \( e = (e^i_j) = (u^i_j e_i) \) is an endomorphism of \( U \), as (for \( i, j, h \in I \))

(4) \[ u^i_j c_i u^j_i = u^i_j, \]

(5) \[ e^h_j e^i_j = (e^h_j u^i_j)(u^j_i e^i_j) \preceq u^i_j, \]

(iii) \( \Rightarrow \) (i).

(6) \( (e^g e)^i_j = V_h a^i_h a^h_i = V_h (e^i_j u^i_h u^h_i c_i) \preceq e^j u^j_i c_i = a^i_j, \)

(7) \( (e^g e)^i_j = V_h a^i_h a^h_i \preceq a^i_j a^j_i = e^i_j u^j_i c_i = a^i_j. \)

(c) \( M \) is inverse. We just need to show that the product of two parallel projections \( e = (e^i_j), \ f = (f^i_j) \) is a projection

(8) \( (ef)^i_j = V_h e^j_i f^i_h = V_h (e^j_i u^j_i u^i_h f^i_h) = e^j_i f^i_h, \)

(9) \( (ef)^i_j = e^i_j f^i_i \in Pr_j A_i, \quad (ef)^i_j = e^j_i u^j_i (ef)^i_j = (ef)^i_j u^i_j. \)

Moreover, the property 6.4.1 is an easy consequence of the following inequality

\[ f^i_h a^i_j e^j_i = f^i_h v^k_i a^i_k u^j_i c_i \preceq f^i_h a^i_h e^i_i \]
(with equality for \( j = i \) and \( k = h \)).

(d) We check now the characterization 6.4.2 of the order of \( M \). If \( a^i_h \preceq b^i_h \), for all \( i \in I, \ h \in H \)

(10) \( (ab^g a)^h_i = V_{k,j} a^i_j b^g_j a^i_k \preceq V_{k,j} a^i_h a^i_k a^j_k = (ab^g a)^h_i = a^i_h, \)

(11) \( (ab^g a)^h_i = V_{k,j} a^i_j b^g_j a^i_k \preceq V_{k,j} a^i_h b^j_k b^i_k \preceq V_{k,j} a^i_h, \)

hence \( ab^g a = a \) and \( a \preceq b \) in \( M \). Conversely, if the last property holds, \( a = be \) for some projection \( e \) of \( M \) and

(12) \( a^i_h = (be)^h_i = V_j b^i_h c_j = V_j b^i_j c_i \preceq V_j b^i_h \preceq b^i_h. \)

(e) Finally we prove the characterization 6.4.3 of the linking relation in \( M \). First, we assume that \( a^i_h \preceq b^i_h \) in the inverse category \( M \); then \( b^g a \) and \( ba^g \) are projections, and \( b^g b^i_h a^i_h = (b^g a)^i_j \preceq u^i_j \) for all \( i, j \in I \) and \( h \in H \) (property (ii)). Analogously for \( ba^g \).

Now, if the previous conditions hold, \( b^g b^i_h a^i_h \preceq u^i_j = 1 \) and \( b^g b^i_h a^i_h \preceq v^i_h = 1, \) i.e. \( a^i_h \preceq b^i_h \) in \( K \) (for all \( i, h \)); moreover \( (a^i_h \lor b^i_h)_{I,H} \) is a linked module

(13) \( v^k_i (a^i_j \lor b^i_j) = (v^k_i a^i_j) \lor (v^k_i b^i_j) \preceq a^i_j \lor b^i_j, \)

(14) \( a^i_j \lor b^i_j \preceq (a^i_j \lor b^i_i) \lor (a^j_i \lor b^j_i) \lor (b^j_i \lor b^j_i) \preceq u^j_i. \)

Last, if \( x = (a^i_h \lor b^i_i)_{I,H} \) is a linked module, then \( a, b \preceq x \) (by (2)), and \( a \lor b \).

As to 6.4.4, if \( a \lor b \) one shows as before that \( y = (a^i_h \land b^i_i)_{I,H} \) is a linked module; by 6.4.2, \( x = a \lor b \) and \( y = a \land b \). The last remark follows now easily from our previous characterization of projections.
9.4. Proof of the inverse glueing completion theorem (6.5).

(a) \( M \) is \( \rho \)-cohesive. Assume that \( \alpha \) is a linked \( \rho \)-set of parallel maps \((a_{i,h})_{I,H}: (U_i, u_{ij}^h) \to (V_h, v_{ik}^j)_{H}\)
and write \( \alpha_i^h: U_i \to V_h \) the \( \rho \)-set of its \( i,h \)-components, which is linked by the characterization 6.4.3;
let \( b_i^h: U_i \to V_h \) be the join of the former set in \( K \). It is now easy to check that \( b = (b_i^h) \) is the
linked join of \( \alpha \).

(b) \( M \) is \( \rho \)-glueing. We have to show that each \( \rho \)-manifold \( U = (U^r, Z^{rs})_{R} \) of \( M \) has a glueing in \( M \).

The manifold \( U \) is given by a family of objects

\[ U^r = (U_i^r, u_{ij}^r)_{I} \quad (12) \]

with glueing morphisms

\[ Z^{rs}: U^r = (U_i^r, u_{ij}^r)_{I} \to U^s = (U_i^s, u_{ij}^s)_{I}, \]

\[ Z^{rs} = (z_{ij}^{rs}: U_i^r \to U_j^s)_{i,j} \quad (for \ r, s \in R). \]

These morphisms satisfy the following conditions (for \( r, s, t \in R \) and \( i, j, h \in I \)):

\[ Z^{rr} = 1, \quad \text{i.e.} \ z_{ij}^{rr} = u_{ij}^r, \]

\[ Z^{st}.Z^{rs} \leq Z^{rt}, \quad \text{i.e.} \ z_{ih}^{ts} z_{ij}^{rs} \leq z_{ih}^{rt}, \]

\[ (Z^{rs})^\# = Z^{sr}, \quad \text{i.e.} \ (z_{ij}^{rs})^\# = z_{ji}^{sr}. \]

Now the \( \rho \)-diagram (over \( K \)) \( X = (U_i^r, z_{ij}^{rs})_{R \times I} \) is in \( M \), by (4)-(6). It is provided with natural maps

\[ Z^r: U^r = (U_i^r, u_{ij}^r)_{I} \to X = (U_i^r, z_{ij}^{rs})_{R \times I}, \]

\[ Z^r = (z_{ij}^r: U_i^r \to U_j^s)_{i \in I, (s,j) \in R \times I}, \]

that satisfy the characterization 6.2.1-3 for the glueing:

\[ (Z^r.Z^{rs})_{i,1,h} = V_j (z_{ih}^{st} z_{ij}^{rs}) \leq z_{ih}^{rt} = (Z^r)_{i,1,h}, \]

\[ (Z^{rs}Z^{rt})_{i,j} = V_{t,h} (z_{ih}^{ts} z_{ij}^{rt}) = V_{t,h} (z_{ih}^{ts} z_{ih}^{rt}) = z_{ij}^{rt} = (Z^{rs})_{i,j}, \]

\[ (V_tZ^{rs}Z^{rt})_{i,j} = V_{t,i} (z_{ij}^{rs} z_{ij}^{rt}) = V_{t,i} (z_{ij}^{rs} z_{ij}^{rt}) = z_{ij}^{ss} = (1_X)_{s,j}. \]

(c) Finally the embedding \( K \to M \) satisfies this universal property: if \( F: K \to A \) is a \( \rho \)-cohesive
functor with values into a \( \rho \)-glueing inverse category, there is exactly one \( \rho \)-cohesive functor \( G: M \to A \)
that extends \( F \).

Obviously one defines \( G(U_i, u_{ij}^r)_{I} \) as the glueing of the manifold \( (FU_i, Fu_{ij}^r)_{I} \) in \( A \).

References

Différentielle 23 (1982), 243-256.

\[ (12) \] Clearly it is possible to index all the manifolds \( U^r \) on the same \( \rho \)-set \( I \).


(13) Talks on this subject where given by the first author at the University of Fribourg (1982) and by the second author at the Sussex Category Meeting (1982).