On the monad of proper factorisation systems in categories (*)

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Abstract. It is known that factorisation systems in categories can be viewed as unitary pseudo algebras for the monad $\mathcal{P} = (-)^2$, in $\textbf{Cat}$. We show in this note that an analogous fact holds for proper (i.e., epi-mono) factorisation systems and a suitable quotient of the former monad, deriving from a construct introduced by P. Freyd for stable homotopy. Some similarities of $\mathcal{P}$ with the structure of the path endofunctor of topological spaces are considered.

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Introduction

For a category $X$, the category of morphisms $\mathcal{P}X = X^2$ has a natural factorisation system. So equipped, it is the free category with factorisation system, on $X$.

This system induces a proper, or epi-mono, factorisation system on a quotient $\mathcal{F}rX = X^2/R$ [G3], the free category with epi-mono factorisation system on $X$ (the epi-mono completion), that generalises the Freyd embedding of the stable homotopy category of spaces in an abelian category [Fr]. "Weak subobjects" in $X$, of interest for homotopy categories, correspond to ordinary subobjects in $\mathcal{F}rX$; other results in [G3] concern various properties of $\mathcal{F}rX$ that derive from weak (co)limits of $X$.

Now, the "path" endofunctor $\mathcal{P} = (-)^2$ of $\textbf{Cat}$ has an obvious 2-monad structure (with diagonal multiplication), linked to the universal property recalled above (a pseudo adjunction); it is known, since some hints in Coppey [Co] and a full proof in Korostenski - Tholen [KT], that its (unitary) pseudo algebras correspond to the factorisation systems of $X$. Similarly, as stated without proof in [G3], the pseudo algebras for the induced 2-monad on $\mathcal{F}rX$ correspond to proper factorisation systems of $X$; more precisely, we prove here, in Theorem 4 (ii), that there is a canonical bijection between proper factorisation systems in $X$ and pseudo isomorphism classes of pseudo $\mathcal{F}r$-algebras on $X$. Similar, simpler relations hold in the strict case: strict factorisation systems are monadic on categories, as well as the proper such. Structural similarities of $\mathcal{P}$ with the topological path functor $PX = X^{[0,1]}$ are discussed at the end (Section 5).

We shall use the same notation of [G3]. For factorisation systems, one can see Freyd - Kelly [FK], Carboni - Janelidze - Kelly - Paré [CJKP], and their references; the strict version is much less used: see [G3] and Rosebrugh-Wood [RW]. Lax $\mathcal{P}$-algebras are studied in [RT]. General lax and pseudo algebras can be found in Street [St].

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1. **The factorisation monad.** Let $X$ be any category and $X^2$ its category of morphisms. An object of the latter is an $X$-map $x: X' \to X''$, which we may write as $\hat{x}$ when it is viewed as an object of $X^2$: a morphism $f = (f', f'') : \hat{x} \to \hat{y}$ is a commutative square of $X$, as in the left diagram

\begin{align*}
x' &\xrightarrow{f} Y' \\
x &\downarrow \\
x'' &\xrightarrow{f''} Y''
\end{align*}

and the composition is obvious. The strict factorisation of $f$, shown in the right diagram, is $f = (f', 1)(1, f'')$; note that its middle object is the diagonal $\hat{1} = f'\hat{x} = yf''$ of the square $f$.

Thus, $X^2$ has a canonical factorisation system ($fs$ for short), where the map $f = (f', f'')$ is in $E$ (resp. in $M$) iff $f'$ (resp. $f''$) is an isomorphism. This system contains a canonical strict factorisation system, where $(f', f'')$ is in $E_0$ (resp. in $M_0$) iff $f'$ (resp. $f''$) is an identity. (As in [G3, 2.1], this means that: (i) $E_0, M_0$ are subcategories containing all the identities; (ii) every map $u$ has a strictly unique factorisation $u = me$ with $e \in E_0, m \in M_0$. A strict $fs$ $(E_0, M_0)$ is not a $fs$, of course; but, there is a unique $fs$ $(E, M)$ containing the former, where $u = me$ is in $E$ iff $m$ is iso, and dually. Two strict systems are said to be equivalent if they span the same $fs$.)

The full embedding that identifies the object $X$ of $X$, with $\hat{1}X$

\begin{align*}
x' &\xrightarrow{x} X' \\
x &\downarrow \\
x' &\xrightarrow{x} X'
\end{align*}

makes $X^2$ the free category with factorisation system on $X$, in the "ordinary" sense (as well as in a strict sense): for every functor $F: X \to A$ with values in a category with $fs$ (resp. strict $fs$), there is an extension $G: X^2 \to A$ that preserves factorisations (resp. strict factorisations), determined up to a unique functorial isomorphism $G(\hat{x}) = \text{Im}_A(Fx)$. The (obvious) proof is based on the canonical factorisation of $\eta X(x) = (x, 1), (1, x): \hat{1}X \to \hat{1}Y$ in $X^2$

\begin{align*}
x' &\xrightarrow{x} X' \\
x &\downarrow \\
x' &\xrightarrow{x} X'
\end{align*}

One might now expect that "factorisation systems be monadic on categories", but this is only true in a relaxed 2-dimensional sense.

First, by the strict universal property, the forgetful 2-functor $\mathcal{U}_0: \mathcal{F}_0\text{Cat} \to \text{Cat}$ (of categories with strict $fs$) has a left 2-adjoint $\mathcal{F}_0(X) = (X^2; E_0, M_0)$, and we shall see that $\mathcal{U}_0$ is indeed 2-monadic: the comparison 2-functor $\mathcal{K}_0: \mathcal{F}_0\text{Cat} \to \mathcal{P}_{-}\text{Alg}$ establishes an isomorphism of $\mathcal{F}_0\text{Cat}$ with the 2-category of algebras of the associated 2-monad, $\mathcal{P} = \mathcal{U}_0\mathcal{F}_0: \text{Cat} \to \text{Cat}$, $\mathcal{P}(X) = X^2$.

Secondly, by the "relaxed" universal property, the forgetful 2-functor $\mathcal{U}: \mathcal{F}s\text{Cat} \to \text{Cat}$ (of categories with $fs$) acquires a left pseudo adjoint 2-functor $\mathcal{F}(X) = (X^2; E, M)$: the unit $\eta: 1 \to \mathcal{U}\mathcal{F}$ is 2-natural, but the counit is pseudo natural and "ill-controlled", each component $\epsilon A: (|A|^2; E, M) \to A$ depending on a choice of images in $A$; the triangle conditions are – rather – invertible 2-cells. This
would give an ill-determined pseudo monad structure on $\mathcal{P} = \mathcal{U} \mathcal{F} = \mathcal{U}_0 \mathcal{F}_0$, isomorphic to the previous 2-monad; we will therefore settle on the latter and "by-pass" the pseudo adjunction.

In fact, the structure of the category $\mathbf{2} = \{0 \rightarrow 1\}$ as a diagonal comonoid (with $\epsilon: \mathbf{2} \rightarrow \mathbf{1}$, $\delta: \mathbf{2} \rightarrow \mathbf{2} \times \mathbf{2}$) produces a diagonal monad on the endofunctor $\mathcal{P} = (-)\times (-)$ of $\mathbf{Cat}$, precisely the one we are interested in. The unit $\eta_X = X^2$: $X \rightarrow X^2$ is the canonical embedding considered above, $\eta_X(X) = \hat{1}_X$. The multiplication $\mu_X = X^2$: $\mathcal{P}^2 \mathcal{X} \rightarrow \mathcal{P} \mathcal{X}$ is a "diagonal functor" defined on $\mathcal{P}^2 \mathcal{X} = X^{2 \times 2}$:

- an object of $\mathcal{P}^2 \mathcal{X}$ is a morphism $\xi_0 = (a_0, b_0): x_0 \rightarrow y_0$ of $\mathcal{P} \mathcal{X}$, and a commutative square in $\mathcal{X}$ (the front square of the diagram below); $\mu_X(\xi_0) = d_0 = b_0 x_0 = y_0 a_0$ is the diagonal of this square;
- a morphism of $\mathcal{P}^2 \mathcal{X}$ is a commutative square $\Xi$ of $\mathcal{P} \mathcal{X}$, and a commutative cube in $\mathcal{X}$; $\mu_X(\Xi)$ is a diagonal square of the cube

\[
\begin{array}{cccc}
X_1' & \xrightarrow{f_1} & X_2' \\
\downarrow{\xi_0} & & \downarrow{\xi_1} \\
X_0' & \xrightarrow{f} & X_0' \\
\downarrow{\alpha_0} & & \downarrow{\alpha_1} \\
Y_0' & \xrightarrow{g} & Y_0' \\
\end{array}
\]

$\mu$ coincides with the multiplication coming from the strict adjunction, $\mathcal{U}_0 \mathcal{F}_0: \mathcal{P}^2 \rightarrow \mathcal{P}$ (and would also coincide with the pseudo multiplication $\mathcal{U}\mathcal{F}$, if one might control the choice of images in $\mathcal{F} \mathcal{X}$ by its strict fs).

$\mathcal{P}$ will also be called the factorisation monad on $\mathbf{Cat}$, while a $\mathcal{P}$-algebra $(X, t)$ will also be called a factorisation algebra; it consists of a functor $t: X^2 \rightarrow X$ such that $t \eta_X = 1_X$, $t \mathcal{P} t = t \mu_X$.

2. The proper factorisation monad. Consider now the quotient $\mathcal{F} \mathcal{R} \mathcal{X} = X^2/R$, modulo the "Freyd congruence" $[Fr]$; two parallel $X^2$-morphisms $f = (f', f''): x \rightarrow y$ and $g = (g', g''): x \rightarrow y$ are R-equivalent whenever their diagonals $\bar{f}$, $\bar{g}$ coincide (cf. 1.1); the morphism of $\mathcal{F} \mathcal{R} \mathcal{X}$ represented by $f$ will be written as $[f]$ or $[f', f'']$. As a crucial effect of this congruence, if $f'$ is epi (resp. $f''$ is mono) in $\mathcal{X}$, so is $[f]$ in $\mathcal{F} \mathcal{R} \mathcal{X}$.

As in [G3], a canonical epi (resp. mono) of $\mathcal{F} \mathcal{R} \mathcal{X}$ will be a morphism which can be represented as $[1, f']$ (resp. $[f, 1]$). Every map $[f]$ has a precise canonical factorisation $[f] = [f, 1][1, f'']$, formed of a canonical epi and a canonical mono (both their diagonals being $\bar{f}$). $\mathcal{F} \mathcal{R} \mathcal{X}$ has thus a proper strict fs $(E_0, M_0)$, which spans a (proper) fs $(E, M)$: the map $[f]: x \rightarrow y$ belongs to $E$ iff there is some $u: Y' \rightarrow X'$ such that $y f' u = y$ (y sees $f'$ as a split epi).

The full embedding $\eta_X = p: \eta_X: X \rightarrow \mathcal{F} \mathcal{R} \mathcal{X}$ takes $f: X \rightarrow Y$ to $[f, f]$: $\hat{1}_X \rightarrow \hat{1}_Y$; $\mathcal{F} \mathcal{R} \mathcal{X}$ is thus the free category with proper factorisation system on $\mathcal{X}$ [G3, 2.3], called the Freyd completion, or epi-mono completion of $\mathcal{X}$. The 2-monad structure of $\mathcal{F} \mathcal{R}$, induced by the one of $\mathcal{P}$ (by-passing again a pseudo adjunction $\mathcal{F} \mathcal{R} \rightarrow \mathcal{U} \mathcal{F}$), will be called the proper factorisation monad on $\mathbf{Cat}$. The unit is $\eta'_X$. For the multiplication $\mu_X: \mathcal{F} \mathcal{R}^2 \mathcal{X} \rightarrow \mathcal{F} \mathcal{R} \mathcal{X}$, note that now

- an object of $\mathcal{F} \mathcal{R}^2 \mathcal{X}$ is a morphism of $\mathcal{F} \mathcal{R} \mathcal{X}$, $\xi_0 = (a_0, b_0): x_0 \rightarrow y_0$,
- a morphism of $\mathcal{F} \mathcal{R}^2 \mathcal{X}$ is an equivalence class $\Xi$ of commutative squares of $\mathcal{F} \mathcal{R} \mathcal{X}$

\[
\Xi = ((f', g'), (f'', g'')): \xi_0 \rightarrow \xi_1,
\]

\[
\Xi = (f', g''): d_0 \rightarrow d_1;
\]
(1) $\Xi = [f', g', [f'', g'']]$: $(\xi_0: x_0 \to y_0) \to (\xi_1: x_1 \to y_1)$, and we have

(2) $\mu'(\Xi_0) = d_0$, $\mu'(\Xi) = [f', g'']$: $d_0 \to d_1$;

in fact, the class $[f', g'']$: $d_0 \to d_1$ is well defined, since its diagonal $g''d_0 = g''b_0x_0$ only depends on the class $[f'', g'']$ and the object $x_0$. The projection $p$ is thus a strict morphism of monads $(P, \eta, \mu) \to (F_r, \eta', \mu')$, as shown in the left diagram below (with $p_2 = Fr(p).p_{2F} = pFr.F(p)$)

$$
\begin{align*}
\xymatrix{X \ar[r]^\eta & PX \ar[l]_{\mu} & P^2X \\
X \ar[r]^\eta' & FrX \ar[l]_{\mu'} & Fr^2X}
\end{align*}
$$

Moreover, any $Fr$-algebra $t: FrX \to X$ determines a $P$-algebra $t = t'p: P^2X \to X$, while a $P$-algebra $t: PX \to X$ induces a $Fr$-algebra $t: FrX \to X$ (with $t = t'p$) iff $t$ is compatible with $R$.

3. Pseudo algebras. Actually, we want to compare the 2-category $FsCat$ (of categories with fs, functors which preserve them, and natural transformations of such functors) with the 2-category $Ps-P-Alg$ of pseudo $P$-algebras, always understood to be unitary (or normalised).

According to a general definition (cf. [St], §2), a (unitary) pseudo $P$-algebra $(X, t, \theta)$, or factorisation pseudo algebra, consists of a category $X$, a functor $t$ (the structure) and a functorial isomorphism $\theta$ (pseudo associativity), so that

(1) $t: X^2 \to X$, $t.\eta X = 1$,

(2) $\theta: tPt \cong t.\mu X: P^2X \to X$,

(3) $\theta(\eta PX) = 1_t = \theta(\eta P^2X): t \to t: PX \to X$,

(4) $\theta(\mu PX).t(\theta) = \theta(\mu P^2X).\theta(P^2t): tPt.P^2t \to t.\mu X,\mu PX: P^3X \to X$.

$$
\begin{align*}
\xymatrix{P^2X & P^2X \ar[l]_{\mu} & P^2X \\
\ar[r]_{\mu'} & P^2X \ar[u]_{\mu} & P^2X \ar[l]_{\mu} \\
\ar[r]_{\mu'} & P^2X \ar[u]_{\mu} & P^2X \ar[l]_{\mu} \ar[u]_{\mu}
\end{align*}
$$

but here (i.e., for $P$) the conditions (3), (4) follow from the rest (as proved below, 4 (A), (B)).

A morphism $(F, \varphi): (X, t, \theta) \to (Y, t', \theta')$ of pseudo $P$-algebras is a functor $F: X \to Y$ with a functorial isomorphism $\varphi: Ft \to t'.PF$: $PX \to Y$ satisfying the following coherence conditions (again, the second is redundant for $P$, cf. 4 (A), (B))

(5) $\varphi.\eta Y = 1_Y, \varphi X, F\theta = \theta^tP^2F.t'.\varphi.pPt: FtPt \to t'.\mu Y, P^2F: P^2X \to Y$. 
Finally, a 2-cell \( \alpha: (F, \psi) \to (G, \psi) \) is just a natural transformation \( \alpha: F \to G \); it is automatically coherent (cf. 4 (B))

\[
(7) \quad \psi \cdot \alpha = t' \rho_{\alpha, \psi}: \quad F \cdot t \to t'. \rho_G: \quad \mathcal{P} \cdot X \to Y.
\]

Similarly, we have the 2-category \( \text{Ps-Fr-Alg} \) of pseudo Fr-algebras, or proper-factorisation pseudo algebras; these amount to pseudo \( \mathcal{P} \)-algebras \((X, t, \emptyset)\) where both \( t \) and \( \emptyset \) are consistent with \( R \) (the consistency of \( \emptyset \) being redundant, cf. 4 (D)). Again, (3), (4), (6), (7) are redundant.

4. Theorem (The comparison of factorisation algebras). (i) (Coppey-Korostenski-Tholen) With respect to the diagonal 2-monad for the endofunctor \( \mathcal{P} = (-)^2 \) of \( \text{Cat} \), there is a canonical equivalence of categories – described below – between \( \text{FsCat} \) and \( \text{Ps-\mathcal{P}-Alg} \), which induces a bijection between fs on a category \( X \) and pseudo isomorphism classes of pseudo \( \mathcal{P} \)-algebras on \( X \). In the strict situation, the canonical comparison functor \( \mathcal{K}_0: \text{Fs}_0 \text{Cat} \to \mathcal{P} \text{-Alg} \), between strict fs and \( \mathcal{P} \)-algebras, is an isomorphism.

(ii) With respect to the 2-monad of the endofunctor \( \text{Fr} \), the previous equivalence induces an equivalence between categories with proper factorisation systems and pseudo \( \text{Fr} \)-algebras, as well as a bijection between proper fs on a category \( X \) and pseudo isomorphism classes of pseudo \( \text{Fr} \)-algebras on \( X \). The comparison functor \( \mathcal{K}_0': \text{PFs}_0 \text{Cat} \to \text{Fr-Alg} \), of proper strict fs, is an isomorphism.

Proof. Part (i) is mostly proved in [KT], and we only need to complete a few points. 

(A) First, there is a canonical 2-functor \( \mathcal{L}: \text{Ps-\mathcal{P}-Alg} \to \text{FsCat} \). Given a (unitary) pseudo \( \mathcal{P} \)-algebra \((X, t, \emptyset)\), every map \( x: X' \to X^* \) in \( X \) inherits a precise \( t \)-factorisation through the object \( t(\hat{x}) \), by letting the functor \( t \) act on the canonical factorisation of \( \#X(x) = (x, 1). (1, x) \) in \( X^2 \) (1.3)

\[
\begin{align*}
X' & \xrightarrow{\tau^*(\hat{x})} X^* \\
1 \downarrow_x & \xrightarrow{1} X' & \xrightarrow{x} & \xrightarrow{1} & \xrightarrow{\tau(x)} & \xrightarrow{\tau^*(\hat{x})} & X^* \\
X' & \xrightarrow{\tau^*(\hat{x})} X^* & \xrightarrow{\tau(\hat{x})} & \xrightarrow{\tau^*(\hat{x})} & X^* \\
\end{align*}
\]

Define \( E \) as the class of \( X \)-maps \( x \) such that \( \tau^*(\hat{x}) \) is iso; dually for \( M \). This is indeed a fs, as proved in [KT], thm. 4.4, without assuming the coherence condition 3.3 (cf. the Note at the end of the paper) nor 3.4; the fact that these properties will be obtained in (B), from the backward procedure, shows that they are redundant. (In the strict case, a strict \( \mathcal{P} \)-algebra \( t \) gives a strict fs, where \( E_0 \) contains the maps \( x \) such that \( \tau^*(\hat{x}) \) is an identity, and dually for \( M_0 \).)

Given a morphism \( (F, \psi): (X, t, \emptyset) \to (Y, t', \emptyset') \) of pseudo \( \mathcal{P} \)-algebras, the fact that the functor \( F: X \to Y \) preserves the associated fs follows from the following diagram, commutative by the naturality of \( \psi: F \cdot t \to t'. \rho_F \) on \((1, x): X' \to \hat{x}, (x, 1): \hat{x} \to X^*, (1, y) \) and \((y, 1) \)
Again, we do not need the condition 3.6: any natural iso \( \varphi \) such that \( \varphi \eta X = 1_F \) has this effect.

(B) Conversely, one can construct a 2-functor \( \mathcal{K}: \mathbf{FsCat} \to \mathbf{Ps-P-Alg} \) depending on choice. Let \((X, E, M)\) be a category with fs; for every map \( x: X' \to X^* \), let us choose one structural factorisation \( x = \tau(x), \tau'(x): X' \to t(x) \to X^* \), respecting all identities: \( 1 = 1.1 \) (We are not saying that this choice comes from a strict fs contained in \((E, M)\)). By orthogonality, this choice determines one functor \( t: X \to X^* \) with this action on the objects and such that \( \tau^{-} \cdot \delta^{-} \to t, \tau ': \cdot \delta^{+} \) are natural transformations (\( \delta^{-}, \delta^{+}: X^2 \to X \) being the domain and codomain functors)

\[
\begin{align*}
X' \xrightarrow{\tau x} & \quad t(x) \quad \xrightarrow{\tau' x} \quad X^* \\
Y' \xrightarrow{\tau y} & \quad \xrightarrow{t(f)} \quad Y^*
\end{align*}
\]

Now \( t \eta(X) = t(1_X) = X \). Moreover, let \( t \cdot \mathcal{P}_t \) and \( t \cdot \mu X: \mathcal{P}_2X \to X \) operate on the object \((f', f^\gamma): x \to y \) of \( \mathcal{P}_2X \), producing \( t \cdot \mathcal{P}_t(f', f^\gamma) = Z' \) and \( t \cdot \mu X(f', f^\gamma) = t(\tilde{f}) = Z'' \)

\[
\begin{align*}
X' \xrightarrow{x} & \quad t(x) \quad \xrightarrow{x'} \quad X^* \\
Y' \xrightarrow{y} & \quad \xrightarrow{t(y)} \quad Y^*
\end{align*}
\]

so that there is precisely one isomorphism \( \vartheta(f): Z' \to Z'' \) linking the two EM-factorisations we have obtained for the diagonal, \( \tilde{f} = (y''z^*).(z'x^*) = d''.d' \) (a strict fs would give an identity, for \( \vartheta(f) \))

\[
\begin{align*}
X' \xrightarrow{x} & \quad X^* \\
Y' \xrightarrow{y} & \quad Y^*
\end{align*}
\]

The coherence relations for \( \vartheta \) do hold: the first (3.3) is obvious; the second (3.4) is concerned with two natural transformations, \( \vartheta(\mu X), t(\vartheta) \) and \( \vartheta(\mu X), \vartheta(\mathcal{P}_2 t) \), that take a commutative cube \( \Xi \in \text{Ob}(\mathcal{P}_3 X) \) to the unique isomorphism linking two precise EM-factorisations of the diagonal arrow of \( \Xi \), through \( t \cdot \mathcal{P}_t \mathcal{P}_2 t(\Xi) \) and \( t \cdot \mu X, \mu X(\mathcal{P}_X(\Xi)) \), respectively.

By similar arguments, a functor \( F: (X, E, M) \to (Y, E', M') \) that preserves fs is easily seen to produce a morphism \( (F, \varphi): (X, t, \vartheta) \to (Y, t', \vartheta') \) of the associated pseudo \( \mathcal{P} \)-algebras. Note that \( \varphi: F \cdot t \to t' \cdot \mathcal{P} F: \mathcal{P} X \to Y \) is determined by the choices which give \( t \) and \( t' \), and does satisfy the coherence condition 3.6, \( \varphi \mu X, F \theta = \vartheta'^{-} \cdot \mathcal{P} F, t \cdot \varphi, \vartheta \cdot \mathcal{P} t; \) these two natural transformations take a commutative square \( \xi \in \text{Ob}(\mathcal{P}_3 X) \) to the unique isomorphism linking two precise EM-factorisations of the diagonal arrow of the square, through \( F \cdot t \cdot \mathcal{P}_t(\xi) \) and \( t' \cdot \mu Y, \mathcal{P}_2 F(\xi) \). Similarly, a natural transformation \( \alpha: F \to G \) satisfies automatically the condition 3.7.
(C) The composite \( \text{FsCat} \to \text{Ps-P-Alg} \to \text{FsCat} \) is the identity. Let \((E, M)\) be a fs on a category \(X\), \((t, \theta)\) the associated pseudo \(\mathcal{P}\)-algebra and \((E', M')\) the fs corresponding to the latter. Then \(E' = \{x \mid \tau^+(x) \text{ is iso}\}\) plainly coincides with \(E\), and \(M' = M\).

The other composite, \(\text{Ps-P-Alg} \to \text{FsCat} \to \text{Ps-P-Alg}\), is just isomorphic to the identity. It is now sufficient to consider two pseudo \(\mathcal{P}\)-algebras \((t, \theta), (t', \theta')\) on \(X\), giving the same factorisation system \((E, M)\), and prove that they are pseudo isomorphic, in a unique coherent way. Actually, for each \(x: X' \to X''\) in \(X\) there is one iso \(\psi(x)\) linking the \(t\)- and \(t'\)-factorisation (both in \((E, M)\))

\[
\begin{align*}
X' & \xrightarrow{\tau x} t^x X' & \xrightarrow{\tau x} & X^* \\
X' & \xrightarrow{\tau x} t^x X' & \xrightarrow{\psi x} & X^* \\
\end{align*}
\]

this gives a functorial isomorphism \(\psi: t \to t': \mathcal{P}X \to X\) such that \((1_X, \psi): (X, t, \theta) \to (X, t', \theta')\) is a pseudo isomorphism of algebras.

(D) For Part (ii), we only need now to prove that, in the previous transformations, pseudo \(\mathcal{F}_r\)-algebras (i.e., pseudo \(\mathcal{P}\)-algebras consistent with the Freyd congruence \(R\)) correspond to proper fs.

First, the consistency of \(t: X^2 \to X\) with \(R\) is sufficient to give an epi-mono factorisation system. Take, for instance, \(\mu = \tau(m)\) (so that \(\mu = \tau(m)\) is iso) and \(\mu f_1 = \mu = \mu f_2\) in the left-hand diagram below; then, the naturality of the transformation \(\tau^+: \partial^- \to t\) on the \(R\)-equivalent maps \((f, h): X' \to m\) of \(X^2\) gives \(uf_1 = t(f_1, h) = t(f_2, h) = uf_2\) and \(f_1 = f_2\)

\[
\begin{align*}
X' & \xrightarrow{f_1} X' & \xrightarrow{f} X' \\
Y & \xrightarrow{h} X & \xrightarrow{t} t(m) \\
\end{align*}
\]

Finally, if \((E, M)\) is epi-mono, then \(t(f)\) in (3) only depends on the diagonal \(\bar{f}\) of \(f = (f', f'')\): \(x \to y\) in \(X^2\), and similarly for \(\partial(f)\) in (5). Therefore they induce a functor \(t': \mathcal{F}_rX \to X\) and a functorial iso \(\theta': t': \mathcal{F}_r(t') \to t', \mu X\), which form a pseudo \(\mathcal{F}_r\)-algebra.

5. Remarks. A crucial tool for the proof of point (A), above, is the structure of \(\mathcal{P}X = X^2\) as a "path functor" (representing natural transformations): it forms a cubical comonad \([G1, G2]\), well linked to the previous monad structure. This interplay already arises in the exponent category \(2\) – a comonoid and a lattice (more precisely, a cubical monoid \([G1]\)) – and was exploited in this form in \([KT]\), Section 1.

The cubical comonad structure, relevant for formal homotopy theory \([G2]\), has one degeneracy \(\eta: 1 \to \mathcal{P}\) (the previous unit), two faces or co-units \(\bar{\partial}^x: \mathcal{P} \to 1\) (domain and codomain) and two connections or co-operations \(g^\pm: \mathcal{P} \to \mathcal{P}^2\)

\[
\begin{align*}
X' & \xrightarrow{x} X'' & X' & \xrightarrow{x} X' \\
X'' & \xrightarrow{x} X'' & X' & \xrightarrow{x} X'' \\
\end{align*}
\]
A cubical comonad satisfies axioms [G1, G2] essentially saying that $\partial^\varepsilon$ ($\varepsilon = \pm$) is a co-unit for the corresponding connection $g^\varepsilon$ and co-absorbat for the other, while $\eta$ makes everything degenerate; moreover, the connections are co-associative. Here the two structures, monad and cubical comonad, are linked by some equations (after the coincidence of the monad-unit with the degeneracy; the last formula is actually a consequence of the co-associativity of connections):

\[
\begin{align*}
\partial^\varepsilon \mu &= \partial^\varepsilon P \partial^\varepsilon = \partial^\varepsilon \partial^\varepsilon P, \\
\eta P &= P \mu g^\varepsilon P g^\varepsilon = P \mu P g^\varepsilon g^\varepsilon = g^\varepsilon \quad (\varepsilon \neq \varepsilon').
\end{align*}
\]

A natural question arises – if the previous arguments have a non-trivial rebound in the usual range of homotopy, the category $\mathbf{Top}$ of topological spaces. Replace the categorical interval $\mathbf{2}$ with the topological one, $I = [0, 1]$, which is, again, a diagonal comonoid and a lattice (and an exponentiable object); thus, the path functor $PX = X^I$ is a monad and a cubical comonad, consistently as above. But here, the interest of (pseudo?) $P$-algebras is not clear (once we have excluded the trivial, "universal" ones: for a fixed $a \in I$, every space $X$ has an obvious strict structure, $ev_a \colon PX \to X$; in the same way as each category $X$ has two trivial $P$-algebras, $\delta^\varepsilon \colon PX \to X$, and two trivial $\eta$). On the other hand, one can readily note that the Kleisli category of $P$ has for morphisms the homotopies, with "diagonal" horizontal composition: $(\beta \circ \alpha)(x; t) = \beta(\alpha(x; t); t)$, for $t \in I$.

References


