COLLARED COSPANS, COHOMOTOPY AND TQFT (COSPANS IN ALGEBRAIC TOPOLOGY, II)

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Abstract.

Topological cospans and their concatenation, by pushout, appear in the theories of tangles, ribbons, cobordisms, etc. Various algebraic invariants have been introduced for their study, which it would be interesting to link with the standard tools of Algebraic Topology, (co)homotopy and (co)homology functors.

Here we introduce *collarable* (and *collared*) *cospans* between topological spaces. They generalise the cospans which appear in the previous theories, as a consequence of a classical theorem on manifolds with boundary. Their interest lies in the fact that their concatenation is realised by means of *homotopy* pushouts. Therefore, cohomotopy functors induce 'functors' from collarable cospans to spans of sets, providing - by linearisation - topological quantum field theories (TQFT) on manifolds and their cobordisms. Similarly, (co)homology and homotopy functors take collarable cospans to relations of abelian groups or (co)spans of groups, yielding other 'algebraic' invariants.

This is the second paper in a series devoted to the study of cospans in Algebraic Topology. It is practically independent from the first, which deals with *higher* cubical cospans in abstract categories. The third article will proceed from both, studying cubical topological cospans and their *collared* version.

Introduction

A cospan of topological spaces is a pair of maps with the same codomain

$$u = (u^-\colon X^- \to X^0 \leftarrow X^+ : u^+), \tag{1}$$

viewed as a kind of morphism $u\colon X^- \rightarrow X^+$; they are composed by pushouts, and - of course - this composition is not strictly associative. They form a bicategory [1] and, also, the weak arrows of a *weak double category* $\mathbb{C}\mathrm{osp}(\mathbf{Top})$, whose strict arrows are the ordinary maps of topological spaces [11]. But it is well known that pushouts of spaces are not homotopy invariant, generally, and do not behave well under homology or homotopy functors.

Here, we are interested in particular cospans, the *collarable* ones (2.2), which give rise to a substructure $\mathbb{C}\mathrm{blc}(\mathbf{Top}) \subset \mathbb{C}\mathrm{osp}(\mathbf{Top})$. Typically, (non oriented) *tangles, ribbons* and *cobordisms between manifolds* (as studied in Topological Quantum Field Theories)

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are collarable cospans and are composed by means of pushouts (in the following figures, collars are suggested by dashes or dots)

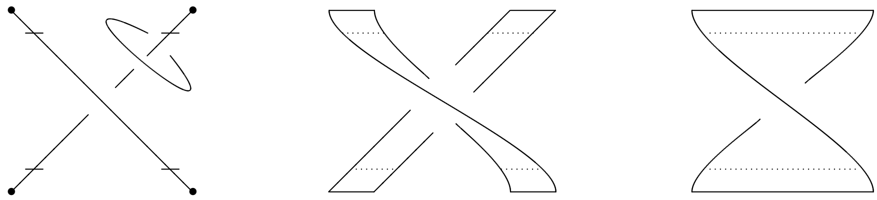

figure 1

The structural interest of restricting to collarable cospans abides in the fact that the 'collarable pushouts' appearing in their composition are always *homotopy pushouts* (Thm. 2.5), hence homotopy invariant and 'respected' by most homotopy and homology functors, in some sense specified below.

First, the Borsuk cohomotopy functor $\pi^S = [-, S] \colon \mathbf{Top} \to \mathbf{Set}^{\mathrm{op}}$ (where S is an arbitrary topological space) always turns a collarable pushout into a weak pullback of sets, and extends thus to a functor

$$\pi^S \colon \mathrm{Cblc}_{iso}(\mathbf{Top}) \to \mathrm{Rel}(\mathbf{Set}), \tag{2}$$

defined on the category of topological spaces and isomorphism-classes of collarable cospans, with values in the category of relations of sets (Thm. 3.3). Under suitable restrictions, we get an extension with values in $\mathrm{Sp}_{iso}(\mathrm{f}\mathbf{Set})$, the category of isomorphism-classes of spans of finite sets; this extension can be 'linearised' with values in finite-dimensional vector spaces (or free modules on a given ring), providing various topological quantum field theories.

Similar extensions can be obtained for relative homology or cohomology (4.2, 4.4). The same holds for the fundamental-groupoid functor, with values in *cospans of groupoids* (4.5), as a consequence of R. Brown's version of the Seifert-van Kampen theorem [5, 6]. (One can not use the fundamental group, as a pointed cospan can not admit collars.)

In Section 1 we recall the construction of the *weak double category* $\mathbb{C}\mathrm{osp}(\mathbf{Top})$ of topological cospans. It contains the corresponding bicategory, which could also be used here; but we want to have general 'transversal maps' in the structure, which allows for limits (cf. 1.3) and adjunctions, as studied in [11, 12]. Moreover, this setting is adequate for the cubical extension which will be given in Part III, on the basis of the weak cubical category of higher cospans of Part I [10].

In Section 2 we define *collarable cospans*; they form a weak double subcategory

$$\mathbb{C}\mathrm{blc}(\mathbf{Top}) \subset \mathbb{C}\mathrm{osp}(\mathbf{Top}).$$

Notice that, as motivated in 1.6, the definition is more general than one might expect, and also the degenerate cospans $e_1(X) = (\mathrm{id}X, \mathrm{id}X)$ are accepted: *a collarable cospan decomposes into a sum of a trivially collarable part* (a pair of homeomorphisms) *and a*

1-collarable part; only the second admits 'real' collars. We end with a notion of *weak equivalence* of topological cospans, which is sufficiently general to make the cylindrical cospan on a space equivalent to the degenerate one (2.8); *homotopy invariance* of functors on topological cospans is defined with respect to this notion. *Collared* cospans, which are *equipped* with collars, are only hinted at (see 2.2); they will be studied in Part III of this series, together with their higher cubical versions.

Sections 3 and 4 show that cohomotopy functors, (co)homology theories and the fundamental-groupoid functor can be extended to collarable cospans, as outlined above, obtaining homotopy-invariant (co)lax or pseudo double functors. Some Frobenius algebras derived from cohomotopy functors are computed in 3.4, 3.5. A few computations on (absolute or relative) homology relations can be found in 4.6, 4.7. Finally, the Appendix in Section 5 deals with extending functors to (co)spans and relations, by classifying the squares of 'ordinary maps' which become 'bicommutative' in these involutive categories.

As to literature, a clear, elementary exposition of low-dimensional TQFT can be found in [16]. Other invariants for links, tangles and ribbons can be seen, for instance, in [15, 2] and their references.

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1. Topological cospans and homotopy pushouts

We review here the basic notions of this paper: the weak double category of topological cospans, with its limits and colimits, and the classical notion of homotopy pushout. The index α always takes the values ± 1, which are written $-, +$ in superscripts. $IX = X \times [0,1]$ denotes the cylinder on a space X.

1.1. Topological cospans and their transversal maps. The weak double category $\mathbb{C}\mathrm{osp}(\mathbf{Top})$ of topological spaces, maps and cospans, has been introduced and studied in [11]. It is also the *1-truncated* structure of the weak cubical category of higher cospans $\mathbb{C}\mathrm{osp}_*(\mathbf{Top})$ introduced in Part I of this series [10]: truncation acts on the weak directions, leaving only one of them (together with the strict direction); here we will use a terminology and notation similar to the one of [10], which is more suited for a future study of higher topological cospans.

Therefore, a *0-cube*, or *object*, is a topological space, viewed as a functor $X\colon \mathbf{1} \to \mathbf{Top}$ defined on the singleton category $\mathbf{1} = \{*\}$. A *1-cube* is a (topological) cospan, i.e. a pair of continuous mappings between topological spaces

$$u = (u^-\colon X^- \to X^0 \leftarrow X^+ : u^+), \tag{3}$$

and is viewed as a functor $u\colon \wedge \to \mathbf{Top}$, defined on the category

$$\wedge : \qquad -1 \to 0 \leftarrow 1 \qquad \text{(the } \textit{formal cospan}\text{)}. \tag{4}$$

(An *n-cube* in the higher cubical structure $\mathbb{C}\mathrm{osp}_*(\mathbf{Top})$ is a functor $\wedge^n \to \mathbf{Top}$ [10]). Ordinary cospans form a 1-truncated cubical set, or reflexive graph, with (cubical) *faces* and *degeneracies*

$$\partial_1^\alpha u = X^\alpha, \qquad\qquad e_1(X) = (\mathrm{id}\colon X \to X \leftarrow X : \mathrm{id}). \tag{5}$$

We often write $u = (u^-, u^+)\colon X^- \nrightarrow X^+$ to specify the cubical faces of u.

A (transversal) *0-map* $f\colon X \to Y$ is a continuous mapping between topological spaces, also viewed as a natural transformation $f\colon X \to Y\colon \mathbf{1} \to \mathbf{Top}$. A (transversal) *1-map* $f\colon u \to v$, or *double cell*, is a natural transformation $f\colon u \to v\colon \wedge \to \mathbf{Top}$, which amounts to two commutative squares in $\mathbf{Top}$

$$\begin{tikzcd}
X^- \arrow[r, "f^-"] \arrow[d, "u^-"'] & Y^- \arrow[d, "v^-"] \\
X^0 \arrow[r, "f^0"] & Y^0 \\
X^+ \arrow[r, "f^+"'] \arrow[u, "u^+"] & Y^+ \arrow[u, "v^+"']
\end{tikzcd} \qquad\qquad f = (f^-, f^0, f^+)\colon u \to v. \tag{6}$$

Also transversal maps (of cubical degree 0 or 1) form a 1-truncated cubical set, with:

$$\partial_1^\alpha f = f^\alpha, \qquad\qquad e_1(f) = (f, f, f) \qquad\qquad (\alpha = \pm 1). \tag{7}$$

The 1-map f is said to be *special* if its cubical faces f^-, f^+ are identities.

Transversal maps compose (as natural transformations), forming the category of diagrams $\mathbf{Top}^\wedge$, with identities $\mathrm{id}(u) = (\mathrm{id}X^-, \mathrm{id}X^0, \mathrm{id}X^+)$. Cospans (and their transversal maps) are *concatenated* using pushouts: given u and $v = (v^-\colon Y^- \to Y^0 \leftarrow Y^+ : v^+)$, with $X^+ = A = Y^-$, the concatenation $w = u +_1 v$ is computed as:

$$\begin{tikzcd}
 & & Z^0 & & \\
 & X^0 \arrow[ur] & & Y^0 \arrow[ul] & \\
X^- \arrow[ur] & & A \arrow[ul] \arrow[ur] & & Y^+ \arrow[ul]
\end{tikzcd} \qquad\qquad w = u +_1 v = (X^- \to Z^0 \leftarrow Y^+). \tag{8}$$

This 'cubical' composition is categorical up to *isomorphisms* of cospans, i.e. invertible *special* transversal maps.

The weak double category structure is described more in detail in [11] (and [10]). In [11], the transversal (strict) direction is called *horizontal* and the cubical (weak) direction is called *vertical.* A 0-map is called a *horizontal arrow*, a 1-cube is a *vertical arrow*, a 1-map is always called a *double cell.* (In other articles on weak double categories, 'horizontal' and 'vertical' are interchanged, which agrees with the usual terminology of bicategories. Here, such terms will be avoided.)

We will also use the associated *category* $\mathrm{Cosp}_{iso}(\mathbf{Top})$, of topological spaces and *isomorphism* classes of cospans, *up to invertible special 1-maps.*

1.2. Remarks. The *bicategory* of topological cospans [1] can be identified to the weak double subcategory of $\mathbb{C}\mathrm{osp}(\mathbf{Top})$ obtained by restricting 0-maps to identities of topological spaces and 1-maps to the special ones.

We prefer to use the larger structure of a weak double category, where (double) limits and colimits exist and essentially amount to (co)limits in the category of diagrams $\mathbf{Top}^{\wedge}$, *together with the* (co)*tabulator*, i.e. the double (co)limit of a 1-cube (see [11]: the cotabulator of a cospan is its central space, while the tabulator is its pullback). The (co)limits of interest for the sequel are briefly described below (1.3).

Finally, let us remark that concatenation in $\mathbb{C}\mathrm{osp}(\mathbf{Top})$ is well-defined once we assume that we have in **Top** (or any other setting for cospans) a (symmetric) choice of a distinguished pushout for any span. As discussed in [10], 3.1, it is convenient to assume the following restriction on the choice (which is formulated there in a more general context)

(a) the pair $(f, 1)$ has distinguished pushout $(1, f)$, and symmetrically (*unitarity constraint*).

1.3. Limits and colimits. The product $\Pi\, u_j$ of a family of cospans $u_j\colon \wedge \to \mathbf{Top}$ $(j \in J)$ is a cospan u equipped with a family $p_j\colon u \to u_j$ of 1-maps satisfying the usual universal property; it is simply computed componentwise: $X^- = \Pi\, X_i^-$, and so on.

Similarly, a sum $\Sigma\, u_j$ of cospans is computed componentwise, and the (co)equaliser of two 1-maps $f, g\colon u \to v$, in the category of diagrams $\mathbf{Top}^{\wedge}$, consists of the three (co)equalisers of the components.

A *sub-cospan* of $u = (u^-\colon X^- \to X^0 \leftarrow X^+ : u^+)$ will be a regular subobject of u (an equaliser of two 1-maps, as above). Therefore, it amounts to assigning three subspaces (Y^-, Y^0, Y^+) such that

$$Y^t \subset X^t, \qquad u^\alpha(Y^\alpha) \subset Y^0 \qquad (t \in \wedge;\;\; \alpha = \pm), \tag{9}$$

and we say that the sub-cospan is *open* (resp. *closed*) in u if so are the three subspaces $Y^t \subset X^t$. The sub-cospans of u form a complete lattice, which is a sublattice of $\mathcal{P}(X^-)\times \mathcal{P}(X^0)\times\mathcal{P}(X^+)$.

Finally, let us recall that decomposing a space X into a categorical sum $X = \Sigma\, X_j$ (in **Top**) amounts to giving a partition of the space X into a family of *clopens* (closed and open subspaces). Similarly, to give a decomposition $u = \Sigma\, u_j$ into a sum of cospans amounts to give a *clopen partition* (u_j) of u, i.e. a cover of u by disjoint sub-cospans, closed and open in u.

1.4. Standard homotopy pushouts. We recall now a fundamental notion of homotopy theory, introduced by Mather [18].

Let $f\colon A \to X$, $g\colon A \to Y$ form a span in **Top**. The *standard homotopy pushout from* f *to* g is a four-tuple $(P; u, v; \lambda)$ as in the left diagram below, where $\lambda\colon uf \to vg\colon A \to P$ is a homotopy satisfying the following universal property (as for *cocomma squares* of

categories), which determines it *up to homeomorphism*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "v"] \\ X \arrow[r, "u"'] \arrow[ur, phantom, "\lambda"] & P \end{tikzcd} \qquad\qquad \begin{tikzcd} & A \arrow[r, "g"] \arrow[d, "d^+"'] & Y \arrow[dd, "v"] \\ A \arrow[r, "d^-"] \arrow[d, "f"'] & IA \arrow[dr, "\lambda"] & \\ X \arrow[rr, "u"'] & & P \end{tikzcd} \tag{10}$$

(a) for every $\lambda'\colon u'f \to v'g\colon A \to W$, there is precisely one map $h\colon P \to W$ such that $u' = hu, v' = hv, \lambda' = h\lambda$.

(Writing $h\lambda$ we are using the obvious *whisker composition* of homotopies and maps). In **Top**, the solution always exists and can be constructed as the ordinary colimit of the right-hand diagram above. This construction is based on the cylinder $IA = A\times[0,1]$ and its faces

$$d^-, d^+\colon A \to IA, \qquad d^-(a) = (a,0), \qquad d^+(a) = (a,1) \qquad (a \in A). \tag{11}$$

Therefore, the space P is a pasting of X and Y with the cylinder IA, and can be realised as a quotient of their topological sum, under the equivalence relation which gives the following identifications:

$$P = (X + IA + Y)/\!\sim, \qquad [f(a)] = [a,0], \qquad [g(a)] = [a,1] \qquad (a \in A). \tag{12}$$

As a crucial feature, this construction always has strong properties of homotopy invariance (e.g., see [9], Section 3), which an ordinary pushout - generally - does not have. Notice also that the cylinder IA is itself the standard homotopy pushout from $\mathrm{id}A$ to $\mathrm{id}A$.

1.5. General homotopy pushouts. As in Mather's original paper, *a homotopy pushout* of f, g will be any space P' homotopy equivalent to the *standard homotopy pushout* P.

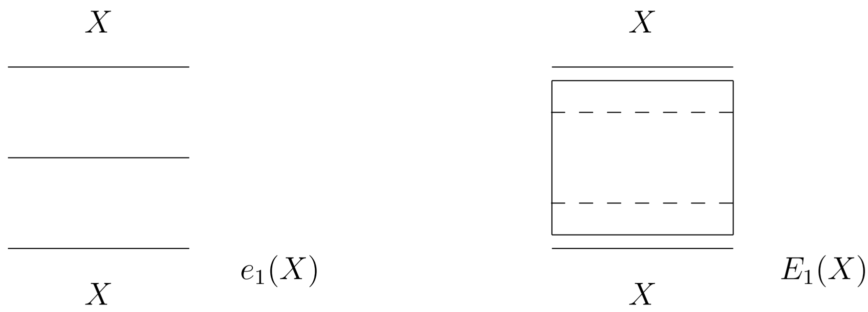
With the notation of 1.4, the triple (u, v, λ) of P, composed with an equivalence-map $P \to P'$, gives maps $u'\colon X \to P'$, $v'\colon Y \to P'$ and a homotopy $\lambda'\colon u'f \to v'g\colon A \to P'$ satisfying the following *weak universal property*, of mere existence:

(a) for every $\lambda''\colon u''f \to v''g\colon A \to P''$, there is *some* map $k\colon P' \to P''$ such that $u'' \simeq ku'$ and $v'' \simeq kv'$.

It follows easily that every homotopy pushout becomes a *weak pushout* in the homotopy category Ho**Top** = **Top**/ $\simeq$ (of spaces and homotopy classes of maps). Furthermore, every (contravariant) cohomotopy functor $[-, S]\colon$ **Top** $\to$ **Set** takes a homotopy pushout of spaces to a *weak pullback* of sets, a fact which will be used below.

1.6. Degenerate and cylindrical cospans. Before going on to define collarable maps, in the next section, it will be useful to compare the degenerate cospan $e_1(X) = (\mathrm{id}X, \mathrm{id}X)$ of a space X with the *cylindrical cospan*

$$E_1(X) = (d^-\colon X \to IX \leftarrow X : d^+), \qquad d^-(x) = (x,0), \qquad d^+(x) = (x,1), \tag{13}$$



which are produced by the degenerate span $X \leftarrow X \rightarrow X$, as an ordinary pushout and a standard homotopy pushout, respectively. (They coincide for $X = \emptyset$.)

Assuming that $X \neq \emptyset$, let us note that these cospans are *not* isomorphic, i.e. cannot be linked by an invertible special transversal map. (They are just *weakly equivalent* cospans, as defined in 2.8).

In fact, we have a special transversal 1-map $p\colon E_1(X) \rightarrow e_1(X)$, as in the left diagram below

$$
\begin{tikzcd}
X \arrow[r, "d^-"] \arrow[d, "1"'] \arrow[dr, phantom, "="] & IX \arrow[d, "p"] \arrow[dr, phantom, "="] & X \arrow[l, "d^+"'] \arrow[d, "1"] \\
X \arrow[r, "1"'] & X & X \arrow[l, "1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "f^-"'] \arrow[dr, phantom, "?"] & X \arrow[d, "f^0"] \arrow[dr, phantom, "?"] & X \arrow[l, "1"'] \arrow[d, "f^+"] \\
X \arrow[r, "d^-"'] & IX & X \arrow[l, "d^+"]
\end{tikzcd}
\tag{14}
$$

but there is no transversal map f going backwards (special or not): in the right diagram above, any point $x \in X$ would give $(f^-(x), 0) = f^0(x) = (f^+(x), 1)$.

Now, the cylinder cospans admits 'collars' (suggested by dashed lines, in the figure above), while $e_1(X)$ does not (for $X \neq \emptyset$). However, working with the cylindrical cospans as weak identities for the collarable cospans gives various problems. In Part III, extending our construction to *cubical collared cospans*, of any degree, we will see that the cubical relation $e_1e_1 = e_2e_1$ cannot be fulfilled with cylindrical cospans: *we would not even get a cubical set*. Already here (as a minor point), if we want the new structure to be a weak double subcategory of $\mathbb{C}\mathrm{osp}(\mathbf{Top})$, we must keep the same degeneracies. This is why *we will use a generalised notion of collarable cospan, including the degenerate ones* as 'trivially collarable'.

Finally, one can think of a third alternative, which presents other technical problems. One can modify $\mathbb{C}\mathrm{osp}(\mathbf{Top})$ with a 'better' concatenation of consecutive cospans, say $u \otimes_1 v$, realised by means of *standard homotopy pushouts* (instead of using ordinary pushouts and restricting cospans, as we do below). Now, *associativity works well* for the new concatenation, up to invertible special transversal maps, as one can easily see. But *degenerate cospans do not work as weak identities*: for a space X, we get $e_1(X) \otimes_1 e_1(X) = E_1(X)$, which is not isomorphic to $e_1(X)$. On the other hand, we have already seen that, using the cylindrical cospans instead of the degenerate ones, as weak identities, gives other problems in the higher cubical structure.

2. Collarable cospans and collarable pushouts

We introduce here the weak double subcategory of collarable cospans $\mathbb{C}\mathrm{blc}(\mathbf{Top}) \subset \mathbb{C}\mathrm{osp}(\mathbf{Top})$. They are concatenated by means of pushouts of collarable maps, which are homotopy pushouts (Thm. 2.5). This will be crucial for applying cohomotopy functors (Section 3), or (co)homology and homotopy functors (Section 4).

2.1. Collarable maps. The idea of a topological cospan which 'admits collars', inspired by cobordisms between manifolds, is rather intuitive. Examples can be seen in fig. 1 of the Introduction, in the right-hand figure of (13) and, below, in all the figures of (44), (75), (77). We have already seen above that it is convenient to generalise it, to include degenerate cospans (1.6); similarly, we must use a generalised notion of collarable map.

Let us begin by recalling that a closed injective map $f\colon X \to Y$ in **Top** is always a topological embedding, in the sense that X has the pre-image topology (the less fine one which makes f continuous): indeed, if C is closed in X, then $C = f^{-1}(f(C))$ is the pre-image of a closed subset of Y. Such maps will be called *closed* (topological) *embeddings.*

We say that a map $f\colon X \to Y$ between topological spaces:

(i) is *0-collarable*, or *trivially collarable*, if it is a homeomorphism,

(ii) is *1-collarable*, or *admits a collar*, if it has a (continuous) extension F to the cylinder IX, which is a *closed embedding*

$$F\colon IX \to Y, \qquad\qquad f = F(-,0)\colon X \to Y, \tag{15}$$

so that the subset $F(X\times[0,1[\,)$ *is open in* Y.

The only map which is both 0- and 1-collarable is $\mathrm{id}(\emptyset)$ (because F is injective and $F(X\times\mathbf{I}) \subset Y = f(X) = F(X\times\{0\})$).

Now, a *collarable map* $f\colon X \to Y$ will be a continuous mapping which can be decomposed in a sum of two maps:

$$f = f_0 + f_1\colon (X_0 + X_1) \to (Y_0 + Y_1) \qquad (\textit{collarable decomposition}), \tag{16}$$

where f_0 is 0-collarable and f_1 is 1-collarable.

Each component f_i and f itself are closed embeddings. The collarable decomposition of f is uniquely determined (if it exists): given a second, based on the decomposition $X = X'_0 + X'_1$, f must be both 0- and 1-collarable on the subspaces $X_0 \cap X'_1$ and $X'_0 \cap X_1$, which must be empty. Collarable maps, as here defined, do not form a subcategory of **Top**: the composition of two 1-collarable maps is *2-collarable*, in a graded category (see 2.9).

A 1-*collared* map is a map *equipped* with a collar. A *collared map* is a sum $f = f_0 + f_1$ of topological maps, where f_0 is a homeomorphism (considered as *0-collared*) and f_1 is *1-collared.*

2.2. COLLARABLE COSPANS. Similarly, we say that a topological cospan

$$u = (u^- \colon X^- \to X^0 \leftarrow X^+ : u^+), \tag{17}$$

(i) is *0-collarable*, or *trivially collarable*, if it is a pair of homeomorphisms,

(ii) is *1-collarable*, or *admits a collar cospan*, (or, simply, a *collar*), if there exists a cospan (U^-, U^+) formed of a pair of collars of its maps having disjoint images.

In other words, in the second case, we have two disjoint closed embeddings

$$U = (U^- \colon IX^- \to X^0 \leftarrow IX^+ : U^+), \qquad u^\alpha = U^\alpha(-, 0) \colon X^\alpha \to X^0, \tag{18}$$

where $U^\alpha(X^\alpha \times [0, 1[)$ is open in X^0. (Examples have been recalled at the beginning of 2.1.)

Furthermore, we say that the cospan u is *collarable* if it *admits a collarable decomposition*, i.e. can be decomposed into a binary sum (1.3):

$$\begin{array}{ll} u = u_0 + u_1 = (X_0^- + X_1^- \to X_0^0 + X_1^0 \leftarrow X_0^+ + X_1^+), & \\ u_0 = (u_0^- \colon X_0^- \to X_0^0 \leftarrow X_0^+ : u_0^+), & u_1 = (u_1^- \colon X_1^- \to X_1^0 \leftarrow X_1^+ : u_1^+), \end{array} \tag{19}$$

where u_0 is 0-collarable and u_1 is 1-collarable (with collar cospan (U^-, U^+)).

Again, the only cospan which is both 0- and 1-collarable is the empty cospan $e_1(\emptyset)$, and the collarable decomposition of a cospan is uniquely determined.

Cubical faces and degeneracy are inherited from $\mathbb{C}\mathrm{osp}(\mathbf{Top})$

$$\partial_1^\alpha u = X^\alpha, \qquad e_1(X) = (\mathrm{id} \colon X \to X \leftarrow X : \mathrm{id}) \qquad (\alpha = \pm 1). \tag{20}$$

We prove below (2.4) that collarable cospans are closed under the concatenation $u +_1 v$ of cospans. Topological spaces and collarable cospans form thus a *transversally full* weak double subcategory

$$\mathbb{C}\mathrm{blc}(\mathbf{Top}) \subset \mathbb{C}\mathrm{osp}(\mathbf{Top}). \tag{21}$$

Also here, we will write $u = (u^-, u^+) \colon X^- \to\!\!\!\!\to X^+$ to specify the cubical faces of u. Notice that a topological map, even if collarable, cannot be viewed as a collarable cospan, in general. The category of topological spaces is *transversally embedded* in $\mathbb{C}\mathrm{blc}(\mathbf{Top})$, sending a map $f \colon X \to Y$ to the same transversal 0-map.

Collared cospans, where the 1-collared component is *equipped* with assigned collars, will be studied in Part III.

2.3. STUDYING CONCATENATION. Let us be given a concatenation $w = u +_1 v$ of two collarable cospans, with the following collarable decompositions (2.2):

$$\begin{array}{lll} u = (u^-, u^+) \colon X^- \to\!\!\!\!\to X^+, & u = u_0 + u_1, & ((u_1^-, u_1^+) \text{ has collar } (U^-, U^+)), \\ v = (v^-, v^+) \colon Y^- \to\!\!\!\!\to Y^+, & v = v_0 + v_1, & (v_1^-, v_1^+) \text{ has collar } (V^-, V^+), \\ X^+ = A = Y^-. & & \end{array} \tag{22}$$

In order to prove that w is collarable, we remark that the middle space A inherits a decomposition into four clopens (1.3)

$$A_{ij} = X_i^+ \cap Y_j^-, \qquad A = A_{00} + (A_{10} + A_{01} + A_{11}) \qquad\qquad (i, j = 0, 1), \tag{23}$$

which we reorganise as above into two, A_{00} (on which both cospans are *trivially* collarable) and its complement.

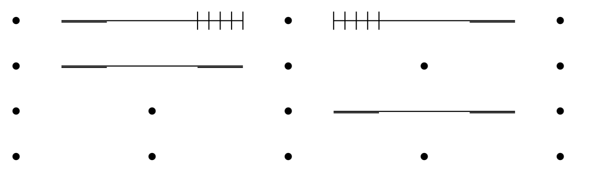
This decomposes the trivially collarable cospan u_0 into a sum $u' + u''$ of two trivially collarable cospans with $\partial_1^+ u' = A_{00}$; analogously, $v_0 = v' + v''$ with $\partial_1^- v' = A_{00}$. The decompositions $u = u' + (u'' + u_1)$ and $v = v' + (v'' + v_1)$ give a decomposition of $w = u +_1 v$ as

$$u +_1 v = (u' +_1 v') + ((u'' + u_1) +_1 (v'' + v_1)), \tag{24}$$

where the first component $w_0 = u' +_1 v'$ is 0-collarable. We prove below (Thm. 2.4) *that the second is 1-collarable.*

The argument can be better understood looking at the following example, where the spaces X^-, A and Y^+ are discrete (with four points, each), and each A_{ij} is a singleton

$$X^- \;\rightarrow\; X^0 \;\leftarrow\; A \;\rightarrow\; Y^0 \;\leftarrow\; Y^+ \tag{25}$$



(A_{11})
(A_{10})
(A_{01})
(A_{00})

The bottom row contains the component w_0 of w, constructed by concatenating the restrictions of u_0 and v_0 to A_{00}. All the rest yields the component w_1, whose 'left' and 'right' collars are suggested by thick segments; notice that the collar of w_1^- is produced *by the collars of* u_1^- *and part of the ones of* v_1^-; on the other hand, the two dashed collars, which are pasted at A_{11} in the pushout, do not intervene in the collars of the composite. (One can get a similar example for cobordisms between 1-manifolds, replacing the singletons with circles and the 1-dimensional spaces with spheres with holes).

We speak of a *1-collarable concatenation* when $X_0^+ \cap Y_0^- = \emptyset$, so that the trivial part is empty and $u +_1 v$ admits a collar; in this case, concatenation is computed by a pushout which will be said to be *1-collarable* - and will be proved to be *homeomorphic* to a standard homotopy pushout (Thm. 2.5).

Thus, a concatenation of collarable cospans decomposes into the sum of two concatenations: the first is computed by a *trivial pushout* of homeomorphisms (preserved by any functor), and the second by a *1-collarable pushout*, which will be proved to be respected - in suitable ways - by most (co)homotopy and (co)homology functors.

2.4. Theorem. [Concatenating collarable cospans] *Collarable cospans of topological spaces are stable under concatenation in* Cosp(**Top**). *As a consequence, the same holds for their transversal maps.*

More precisely, a concatenation $w = u +_1 v$ of collarable cospans has a collarable decomposition as in (24); in particular, w is 1-collarable if and only if the clopen $X_0^+ \cap Y_0^-$, on which u and v are both 0-collarable, is empty.

PROOF. We have to prove that the second component of the last term in (24) admits a collar. To simplify notation, *we will assume that* $A_{00} = X_0^+ \cap Y_0^-$ *is empty* (in figure (25), this amounts to discard the bottom row) and prove that then the concatenation $w = u +_1 v$ admits a collar. Note that, in A, we have now: $X_0^+ \subset Y_1^-$ and $Y_0^- \subset X_1^+$.

The cospan $w = (w^-, w^+)$ is computed by a (distinguished) pushout in **Top**, over the common face $A = X^+ = Y^-$

$$\begin{tikzcd}[column sep=small]
 & & Z^0 & & \\
 & X^0 \arrow[ur, "j^-"] & & Y^0 \arrow[ul, "j^+"'] & \\
X^- \arrow[ur, "u^-"] & & A \arrow[ul, "u^+"'] \arrow[ur, "v^-"] \arrow[uu, phantom, "\lrcorner" rotate=-45] & & Y^+ \arrow[ul, "v^+"']
\end{tikzcd}
\qquad w^- = j^-u^-, \;\; w^+ = j^+v^+. \tag{26}$$

The maps u^+ and v^- are closed embeddings, and the set Z^0 (the pasting of X^0 and Y^0 over A) is determined as

$$Z^0 = j^-(X^0) \cup j^+(Y^0), \qquad j^-(X^0) \cap j^+(Y^0) = j^-u^+(A) = j^+v^-(A). \tag{27}$$

Therefore, the mappings j^α are injective and Z^0 has the finest topology which makes them continuous. Moreover, both j^α are closed embeddings: if C is closed in X^0, then the set $j^-(C)$ has pre-image C in X^0 and $v^-((u^+)^{-1}(C))$ in Y^0; since both are closed, so is $j^-(C)$ in Z^0.

We construct now a collar cospan $W = (W^-, W^+)$ for w. Following again the guideline of figure (25), the extension $W^-\colon IX^- \to Z^0$ of w^- is defined separately on the components of $IX^- = IX_1^- + IX_0^-$, using the collars U^- and V^-, respectively, and is a closed embedding

$$\begin{tikzcd}[column sep=small]
 & & & & Z^0 & & \\
 & IX_1^- \arrow[rr, "U^-"] & & X^0 \arrow[ur, "j^-"] & & Y^0 \arrow[ul, "j^+"'] & IY_1^+ \arrow[l, "V^+"'] \\
X_1^- \arrow[ur, "d^-"] \arrow[rr] & & X^- \arrow[ur, "u^-"] & X_0^0 \arrow[u] & A \arrow[ul, "u^+"'] \arrow[ur, "v^-"] \arrow[u, phantom, "\lrcorner" rotate=-45] & IY_1^- \arrow[u, "V^-"'] & \\
 & & X_0^- \arrow[u] \arrow[ur, "a"'] & & X_0^+ \arrow[ul, "b"] \arrow[u] \arrow[r, phantom, "\subset"] & IY_1^- \arrow[u, "V^-"'] &
\end{tikzcd}
\qquad
\begin{aligned}
w^- &= j^-u^-, \\
w^+ &= j^+v^+, \\
a &= u_0^-, \\
b &= u_0^+,
\end{aligned} \tag{28}$$

$$\begin{aligned}
&W_1^- = j^-U^-\colon IX_1^- \to Z^0, \qquad W_1^-(x,0) = j^-U^-(x,0) = j^-u^-(x) = w^-(x), \\
&W_0^- = j^+.V^-.Ib^{-1}.Ia\colon IX_0^+ \to IX_0^0 \to IX_0^+ \subset IY_1^- \to Y^0 \to Z^0, \\
&W_0^-(x,0) = j^+.v^-.b^{-1}a(x) = j^-.u^+.b^{-1}a(x) = j^-u^-(x) = w^-(x).
\end{aligned} \tag{29}$$

The extension W^+ is defined in the symmetric way, using V^+ on IY_1^+ and U^+ on IY_0^+.
The images of W^α are disjoint, because:

$$
\begin{aligned}
&\mathrm{Im}(W^-) = \mathrm{Im}(j^-U^-) \cup \mathrm{Im}(j^+V^-), \qquad \mathrm{Im}(W^+) = \mathrm{Im}(j^+V^+) \cup \mathrm{Im}(j^-U^+),\\
&\mathrm{Im}(j^-U^-) \cap \mathrm{Im}(j^+V^+) \subset \mathrm{Im}(j^-U^-) \cap (\mathrm{Im}(j^-) \cap \mathrm{Im}(j^+))\\
&\qquad = \mathrm{Im}(j^-U^-) \cap \mathrm{Im}(j^-u^+) = \emptyset,\\
&\mathrm{Im}(j^-U^-) \cap \mathrm{Im}(j^-U^+) = j^-(\mathrm{Im}U^- \cap \mathrm{Im}U^+) = \emptyset,
\end{aligned}
\tag{30}
$$

and symmetrically.

Finally $W^-(X^- {\times} [0,1[)$ is open in Z^0, since its pre-images in X^0 and Y^0 are, respectively

$$
U^-(X_1^- \times [0,1[) \cup b(X_0^+ \times [0,1[), \qquad \emptyset \cup V^-(X_0^+ \times [0,1[). \tag{31}
$$

■

2.5. Theorem. [Collarable pushouts and homotopy] *(a) A pushout in* **Top** *of two collarable maps f and g is a homotopy pushout and satisfies the* weak *universal property 1.5(a), only concerning existence.*

(b) More particularly, a 1-collarable *pushout (2.3), i.e. the pushout of two collarable maps f and g whose clopens A_0', A_0'' (where both f and g restrict to homeomorphisms)* do not meet*, is* homeomorphic *to the standard homotopy pushout.*

In fact, the canonical map p from the standard homotopy pushout $(P; u, v; \lambda)$ to the ordinary pushout (B, u'', v''), which collapses the cylinder IA, is a homotopy equivalence, with a quasi inverse $h\colon B \to P$, which is a homeomorphism

$$
\begin{tikzcd}
& X \arrow[d, "u"'] \arrow[dr, "u''"] \arrow[drr, "u'"] & & \\
A \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\lambda"] & P \arrow[r, dashed, "p"] & B \arrow[r, dashed, "h"] & P \\
& Y \arrow[u, "v"] \arrow[ur, "v''"'] \arrow[urr, "v'"'] & &
\end{tikzcd}
\tag{32}
$$

(c) The same, as in (b), holds for the pushout of a closed embedding *f along a* 1-collarable *map g (and symmetrically).*

Note. This statement has some similarity with a classical fact, stating that the ordinary pushout of any map along a cofibration is a homotopy pushout.

Proof. Point (a) is a consequence of (b), since a pushout of homeomorphisms is trivially a homotopy pushout. We prove (c), the proof of (b) being similar. The map g has a collar $G\colon IA \to Y$.

The standard homotopy pushout $(P; u, v; \lambda)$ is constructed as in (12), with u, v closed embeddings

$$
P = (X + IA + Y)/\!\sim, \qquad [f(a)] = [a, 0], \qquad [g(a)] = [a, 1] \qquad (a \in A). \tag{33}
$$

We define $u' \colon X \to P$ as the obvious composite

$$u' = (X \subset X + IA + Y \to (X + IA + Y)/\!\sim). \tag{34}$$

Now, Y has a closed cover (Y_1, Y_2, Y_3) with two possibly non-empty intersections

$$\begin{array}{lll} Y_1 = G(A\times[0,1/2]), & Y_2 = G(A\times[1/2,1]), & Y_3 = (Y \setminus G(A\times[0,1[), \\ Y_1 \cap Y_2 = F(A\times\{1/2\}), & & Y_2 \cap Y_3 = G(A\times\{1\}). \end{array} \tag{35}$$

We define $v' \colon Y \to P$ as the pasting of the following closed embeddings

$$\begin{array}{lll} v_1 \colon Y_1 \to P, & v_1G(a,t) = [a, 2t] & (0 \leqslant t \leqslant 1/2), \\ v_2 \colon Y_2 \to P, & v_2G(a,t) = [G(a, 2t-1) & (1/2 \leqslant t \leqslant 1), \\ v_3 \colon Y_3 \to P, & v_3(y) = [y], & \end{array} \tag{36}$$

which agree on the intersections, because

$$v_1G(a,1/2) = [a,1] = [g(a)] = [G(a,0)], \qquad v_2G(a,1) = [G(a,1)] = v_3(G(a,1)). \tag{37}$$

Now, u' and v' agree on A

$$u'(f(a)) = [f(a)] = [a,0] = v'G(a,0) = v'(g(a), \tag{38}$$

and define a closed embedding $h \colon B \to P$ which forms commutative triangles in (32). It is surjective, hence a homeomorphism. Finally, it is easy to prove that p (which is not injective) and h are inverse up to homotopy. ■

2.6. Cobordisms. We will use the weak double subcategory $\mathbb{C}\mathrm{ob}(n) \subset \mathbb{C}\mathrm{blc}(\mathbf{Top})$ of *n-dimensional manifolds and their cobordisms.* Explicitly:

(a) a 0-cube is any n-dimensional compact manifold (without boundary);

(b) a 1-cube, called a (generalised) *cobordism*, is a cospan between 0-cubes which can be decomposed as $u = u_0 + u_1$ in a *trivial part* u_0, consisting of a pair of homeomorphisms of compact n-manifolds, together with a *cobordism part* $u_1 = (u_1^- : X_1^- \to X_1^0 \leftarrow X_1^+ : u_1^+)$, where X_1^0 is a compact (n+1)-manifold with boundary and the maps $u_1^\alpha \colon X_1^\alpha \to X_1^0$ are disjoint closed embeddings whose images cover the boundary of X_1^0; then, the existence of the collar-pair (U^-, U^+) is a well-known consequence (e.g., see [22], 6.2 and [4]);

(c) transversal n-maps are all the natural transformations (in $\mathbf{Top}$) between such n-cubes ($n = 0, 1$); notice that 1-maps are *double* cells.

One has only to check that, in the 1-collarable pushout (26) for two consecutive cobordisms, the central space Z^0 is again an $(n+1)$-manifold, with boundary covered by the new collars W^α. But all this is well known.

2.7. Pairs of spaces. We will also use the category $\mathbf{Top}_2$, of *relative pairs* (X, A) *of topological spaces*, in the usual sense of Algebraic Topology: A is a subspace of X; a map $f\colon (X, A) \to (Y, B)$ is a continuous mapping $X \to A$ which takes A into B.

This category gives rise to a weak double category $\mathbb{C}\mathrm{osp}(\mathbf{Top}_2)$ of relative cospans and to a weak double subcategory $\mathbb{C}\mathrm{blc}(\mathbf{Top}_2)$ of relative collarable cospans.

For the latter, we give the following two definitions. A *collarable map of relative pairs* $f\colon (X, A) \to (Y, B)$ is a relative map *which has a collarable decomposition*

$$f = f_0 + f_1\colon (X_0, A_0) + (X_1, A_1) \to (Y_0, B_0) + (Y_1, B_1), \tag{39}$$

into an isomorphism f_0 of $\mathbf{Top}_2$ and a map f_1 *which admits a (relative) collar*, i.e. a closed embedding $F\colon X_1 \times I \to Y_1$ such that:

$$\begin{array}{ll} f_1 = F(-, 0)\colon X_1 \to Y_1, & F(X_1 \times [0, 1[) \text{ is open in } Y, \\ F(A_1 \times I) \text{ is closed in } B_1, & F(A_1 \times [0, 1[) \text{ is open in } B_1. \end{array} \tag{40}$$

A *collarable cospan* of relative pairs is a cospan in $\mathbf{Top}_2$

$$u = (u^- : (X^-, A^-) \to (X^0, A^0) \leftarrow (X^+, \mathbf{A}^+) : u^+), \tag{41}$$

which has a collarable decomposition, in the obvious sense extending the absolute case, 2.2.

2.8. Weak equivalences and homotopy invariance. Recall that every space X has a degenerate cospan $e_1(X)$ and a cylindrical cospan $E_1(X)$ (13), which are 0- and 1-collarable, respectively. We want to make them 'equivalent', also because, within cobordisms of n-manifolds, the cylindrical cospan is generally used as a weak identity, instead of the degenerate one. (Notice that this latter one is not 'represented' by an (n+1)-manifold with boundary.)

We say that a transversal 1-map $f = (f^-, f^0, f^+)\colon X \to Y$ between topological cospans is a *weak equivalence* if it is special (i.e., f^- and f^+ are identities) and its central component f^0 is a homotopy equivalence. Thus, the canonical map $p\colon E_1(X) \to e_1(X)$ (14) is a weak equivalence. We have already seen that it cannot be considered as a 'homotopy equivalence' in $\mathbb{C}\mathrm{osp}(\mathbf{Top})$, because there are no arrows backwards, which is why this notion is too restrictive (and we do not even define it).

We say that two cospans X, Y with the same cubical faces are *weakly equivalent* if there exists a finite sequence of weak equivalences connecting them: $X \to X_1 \leftarrow X_2 \to \ldots \to X_n = Y$.

Plainly, weak equivalence is closed under sum of cospans. Thus, *every collarable cospan is weakly equivalent to a cospan which admits a collar.*

A weak double functor $F\colon \mathbb{C}\mathrm{osp}(\mathbf{Top}) \to \mathbb{A}$, with values in an arbitrary weak double category, will be said to be *homotopy invariant* if

(i) it identifies parallel transversal 0-maps $X \to Y$ (between topological spaces) which are homotopic;

(ii) it sends weak equivalences $f \colon X \to Y$ between topological cospans to invertible (special) cells of $\mathbb{A}$, and therefore weakly equivalent cospans to isomorphic 1-cubes of $\mathbb{A}$. (This means isomorphic cospans when $\mathbb{A}$ is a weak double category of cospans, and equal relations when $\mathbb{A}$ is a strict double category of relations.)

We use the same terminology for weak double functors defined on substructures of $\mathbb{C}\mathrm{osp}(\mathbf{Top})$, like the previously considered $\mathbb{C}\mathrm{blc}(\mathbf{Top})$ and $\mathbb{C}\mathrm{ob}(n)$ (2.2, 2.6). Weak equivalences and homotopy invariance for cospans of relative pairs are similarly defined.

2.9. Higher collars. Extending the definitions of 2.1, one can form a graded category $\mathrm{Col}_*(\mathbf{Top})$ of topological spaces and *n-collarable maps*, which will not be used in this Part.

After degree 0 (which means a homeomorphism), we say that a continuous mapping $f \colon X \to Y$ *admits an n-collar* if there is an extension to the n-cylinder $I^n X = X \times [0,1]^n$, which is a closed embedding

$$F \colon I^n X \to Y, \qquad\qquad f = F(-, 0, ..., 0) \colon X \to Y, \tag{42}$$

so that the subset $F(X \times [0,1[^n)$ is open in Y.

It is easy to verify that the composite gf with an m-collarable map $g \colon Y \to Z$ is collarable, of degree $m+n$. Indeed, leaving out the trivial cases, if $G \colon I^m Y \to Z$ is an m-collar of g, then gf can be extended to

$$G.I^m F \colon I^{m+n} X \to I^m X \to Y, \tag{43}$$

and $G.(I^m F(X \times [0,1[^{m+n}) = G.(F(X \times [0,1[^n) \times [0,1[^m)$ is open in Z.

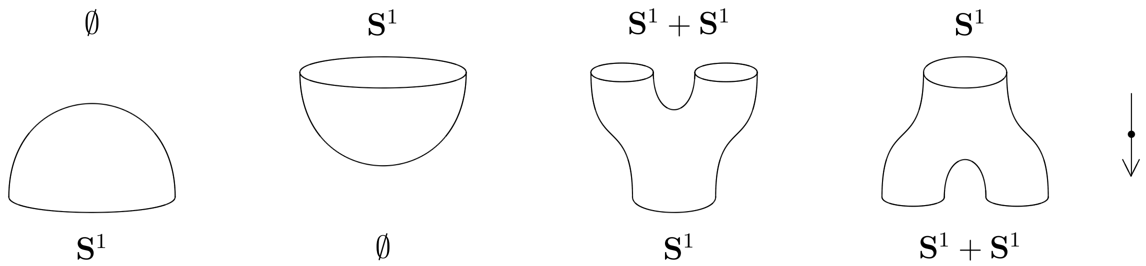
3. Cohomotopy functors and the linearised TQFT

The circle $\mathbf{S}^1$ is a Frobenius object in the monoidal category of (isomorphism classes of) collarable cospans. Furthermore, Borsuk's cohomotopy functors induce functors from collarable cospans to spans of sets. Restricting to manifolds and their cobordisms, we get spans of finite sets, and - by linearisation - topological quantum field theories. It would be interesting to *characterise* the TQFTs which can be obtained in this way.

3.1. The Frobenius structure on the circle. Let us consider the circle $\mathbf{S}^1$ in the weak double category $\mathbb{C}\mathrm{ob}(1) \subset \mathbb{C}\mathrm{blc}(\mathbf{Top})$ of 1-dimensional manifolds and their cobordisms (2.6), or rather in the associated involutive category $\mathrm{Cob}_{iso}(1) \subset \mathrm{Cblc}_{iso}(\mathbf{Top})$ of isomorphism-classes of such cobordisms (cf. 1.1). These categories are equipped with the monoidal structure induced by the sum in the previous double structures.

$\mathbf{S}^1$ is an involutive Frobenius object in $\mathrm{Cob}_{iso}(1)$, in the sense of [16], 3.6, when equipped with the following cospans

$$\begin{array}{lll} \eta \colon \emptyset \to \mathbf{S}^1, & \varepsilon = \eta^\sharp \colon \mathbf{S}^1 \to \emptyset, & (\textit{cups}), \\ \mu \colon \mathbf{S}^1 + \mathbf{S}^1 \to \mathbf{S}^1, & \delta = \mu^\sharp \colon \mathbf{S}^1 \to \mathbf{S}^1 + \mathbf{S}^1 & (\textit{trousers}), \end{array} \tag{44}$$



The axioms are satisfied (strictly), in $\mathrm{Cob}_{iso}(1)$ and $\mathrm{Cblc}_{iso}(\mathbf{Top})$:

$$\begin{aligned} &\mu(\eta+1) = 1 = \mu(1+\eta), && (\varepsilon+1)\delta = 1 = (1+\varepsilon)\delta, \\ &(1+\mu)(\delta+1) = \delta\mu = (\mu+1)(1+\delta). && \end{aligned} \tag{45}$$

3.2. Cohomotopy functors. Every topological space S defines a Borsuk *cohomotopy functor*, which is homotopy invariant (in the classical sense)

$$\pi^S = [-, S] \colon \mathbf{Top} \to \mathbf{Set}^{\mathrm{op}}, \qquad\qquad \pi^S(X) = [X, S]. \tag{46}$$

In particular, the spheres give the functors $\pi^n = [-, \mathbf{S}^n]$. Restricting to suitable spaces X (e.g., CW spaces of dimension up to $2n-1$), the set $\pi^n(X)$ has a canonical abelian group structure [3, 21, 14]. More generally, an arbitrary Moore space $K'(A, n)$ is also used as the classifying space S, to give $\pi^n(X; A)$, the *n-th cohomotopy group with coefficients in the abelian group A* (always under restrictions on X) [20].

The simplest non trivial example is $\pi^0(X) = [X, \mathbf{S}^0]$, which can be identified with the set of *clopen* subsets of X; then, $\pi^0(f) = f^*$ acts as counterimage.

We will also consider the following functors, for $n \geqslant 2$

$$[-, P_n] \colon \mathbf{Top} \to \mathbf{Set}^{\mathrm{op}}, \qquad\qquad [-, \mathbf{P}^2] = [-, P_2] \colon \mathbf{Top} \to \mathbf{Set}^{\mathrm{op}}, \tag{47}$$

where $P_2 = \mathbf{P}^2$ is the real projective plane, and more generally P_n (called a *pseudo-projective plane*) is the quotient of the disc $\mathbf{B}^2$ modulo the obvious action of $\mathbf{Z}_n$ on the boundary (identifying each n-tuple of vertices of a regular n-gon inscribed in the boundary).

3.3. Theorem. [Collarable cospans and cohomotopy] *Let us be given a pushout of collarable maps*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "v"] \\ X \arrow[r, "u"'] & B \end{tikzcd} \tag{48}$$

(a) Every cohomotopy functor $\pi^S = [-, S] \colon \mathbf{Top} \to \mathbf{Set}^{\mathrm{op}}$ turns it into a weak pullback of sets, and extends therefore to a strict double functor $\mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{R}\mathrm{el}(\mathbf{Set}^{\mathrm{op}})$, which is homotopy invariant (as defined in 2.8). We will write it as

$$\pi^S = [-, S] \colon \mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{R}\mathrm{el}(\mathbf{Set}) \qquad\qquad \textit{(contravariant on the strict maps).} \tag{49}$$

In particular, this holds for $\pi^n = [-, |S^n]$.

(b) In the same way, we obtain a homotopy invariant colax *double functor with values in spans*

$$\pi^S = [-, S] \colon \mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{S}\mathrm{p}(\mathbf{Set}) \qquad \textit{(contravariant on the strict maps).} \tag{50}$$

(c) If the classifying space S *is discrete (as happens for* $\pi_0 = [-, \mathbf{S}^0]$*), every pushout (1) is transformed into a pullback of sets, which yields a homotopy invariant pseudo double functor*

$$\pi^S = [-, S] \colon \mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{S}\mathrm{p}(\mathbf{Set}) \qquad \textit{(contravariant on the strict maps).} \tag{51}$$

Moving to the associated categories $\mathrm{Cblc}_{iso}(\mathbf{Top})$ *and* $\mathrm{Sp}_{iso}(\mathbf{Set})$ *(1.1) and restricting to the full subcategory* $\mathrm{fco}\mathbf{Top}$ *of topological spaces with a finite number of clopens, then applying the linearisation functor defined in the Appendix (see 5.2), we get a functor with values in finitely generated modules*

$$\mathrm{Cblc}_{iso}(\mathrm{fco}\mathbf{Top}) \to \mathrm{Sp}_{iso}(\mathrm{f}\mathbf{Set}) \to \mathrm{f}\mathbf{Mod}, \tag{52}$$

(which will be used below to construct a TQFT on manifolds). It is homotopy invariant, with respect to the weak-equivalence relation induced on the quotient $\mathrm{Cblc}_{iso}(\mathrm{fco}\mathbf{Top})$.

Proof. First, let us remark that, in (c), the pseudo double functor π^S is not strict: we cannot expect it to preserve the *choices* of pushouts and pullbacks we are using to define concatenation in these pseudo double categories. In (a), this fact does not appear because the composition of relations does not depend on the choice of pushouts or pullbacks we may use.

Point (a) is a straightforward consequence of Thms. 2.4, 2.5: a concatenation of collarable cospans decomposes into two: the first is computed by a *trivial pushout*, preserved by any functor, and the second by a 1-collarable pushout, which is a homotopy pushout. Now, it suffices to apply Thm 5.5, of the Appendix.

Point (b) is obvious. As to (c), let $h, k \colon B \to S$ be two maps whose homotopy classes coincide on X and Y. Since S is discrete, this means that $hu = hv$ and $ku = kv$. Therefore, $h = k$. ∎

3.4. An elementary example. Consider the involution-preserving functor

$$\pi^0 = [-, \mathbf{S}^0] \colon \mathrm{Cblc}_{iso}(\mathbf{Top}) \to \mathrm{Sp}_{iso}(\mathbf{Set});$$

it is monoidal, in the sense that it takes sums of spaces to product of sets.

Applying it to the involutive Frobenius object $\mathbf{S}^1$ (3.1), we get an involutive Frobenius object in the *category* $\mathrm{Sp}_{iso}(\mathbf{Set})$ of isomorphism-classes of spans. Writing $2 = \{0, 1\}$ the Boolean algebra of clopens of $\mathbf{S}^1$, we have:

$$\eta \colon \{0\} \to \{0, 1\}, \qquad \mu \colon \{00, 01, 10, 11\} \to \{0, 1\} \qquad (\varepsilon = \eta^\sharp, \delta = \mu^\sharp). \tag{53}$$

By elementary computation on cups and trousers, we see that ε and δ are mappings:

$$\varepsilon\colon 2 \to 1, \qquad \delta\colon 2 \to 2{\times}2 \qquad\qquad \text{(the Cartesian diagonal).} \tag{54}$$

This Frobenius object has a peculiar property, which will obviously be preserved in its linearisation:

$$\mu\delta = \mathrm{id}\colon 2 \to 2. \tag{55}$$

Applying now the (monoidal) linearisation functor $\mathrm{Sp}_{iso}(\mathrm{f}\mathbf{Set}) \to \mathrm{f}\mathbf{Vct}$ on a field $\mathbf{k}$ (5.2), we get an ordinary Frobenius algebra on $\mathbf{k}^2$ (whose canonical basis we write e_0, e_1)

$$\begin{array}{ll}
\eta\colon \mathbf{k} \to \mathbf{k}^2, & \eta(a) = (a, a),\\
\mu\colon \mathbf{k}^2 \otimes \mathbf{k}^2 \to \mathbf{k}^2, & \mu((a,b)\otimes(c,d)) = \mu((ae_0 + be_1)\otimes(ce_0 + de_1))\\
 & = \mu(ace_{00} + ade_{11} + bce_{10} + bde_{11}) = (ac, bd),\\
\varepsilon\colon \mathbf{k}^2 \to \mathbf{k}, & \varepsilon(a,b) = a + b,\\
\delta\colon \mathbf{k}^2 \to \mathbf{k}^2 \otimes \mathbf{k}^2, & \delta(a,b) = ae_{00} + be_{11} = ae_0 \otimes e_0 + be_1 \otimes e_1.
\end{array} \tag{56}$$

The (plainly non-degenerate) pairing is:

$$\beta = \varepsilon\mu\colon \mathbf{k}^2 \otimes \mathbf{k}^2 \to \mathbf{k}, \qquad \beta((a,b)\otimes(c,d)) = ac + bd. \tag{57}$$

3.5. The Frobenius algebra associated to a pseudo-projective plane. Consider the *colax* double functor $\pi^* = [-, \mathbf{P}^2]\colon \mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{S}\mathrm{p}(\mathbf{Set})$ (contravariant on the strict maps). It would be interesting to prove that it is actually a *pseudo* double functor, or at least that this holds true when restricting to $\mathbb{C}\mathrm{ob}(1)$.

In any way, applying π^*, we do get a Frobenius object in $\mathrm{Sp}(\mathbf{Set})$, on the set (group) $[\mathbf{S}^1, \mathbf{P}^2] = \mathbf{Z}_2$. Then, the cup is contractible and any map to $\mathbf{P}^2$ is nullhomotopic; the trousers T have the homotopy type of the space 'eight' T' (two tangent circles) and the obvious quotient-map $\mathbf{S}^1 + \mathbf{S}^1 \to T'$ yields an isomorphism $\pi^*(T) \to \mathbf{Z}_2 {\times} \mathbf{Z}_2$ which will be used to identify these sets.

Therefore we get the following involutive Frobenius object in $\mathrm{Sp}_{iso}(\mathbf{Set})$

$$\eta\colon \{0\} \to \mathbf{Z}_2, \qquad \mu = \mathrm{sum}\colon \mathbf{Z}_2{\times}\mathbf{Z}_2 \to \mathbf{Z}_2 \qquad (\varepsilon = \eta^\sharp, \delta = \mu^\sharp). \tag{58}$$

When we linearise this Frobenius object on the field $\mathbf{k}$, we get the vector space $\mathbf{k}^2$, with Frobenius structure:

$$\begin{array}{ll}
(a,b).(c,d) = (ac + bd, ad + bc), & \delta(a,b) = a(e_{00} + e_{11}) + b(e_{01} + e_{10}),\\
\eta(a) = (a, 0), & \varepsilon(a,b) = a.
\end{array} \tag{59}$$

The 'handle operator' $\mu\delta\colon \mathbf{S}^1 \to \mathbf{S}^1$ and the torus $\varepsilon\mu\delta\eta\colon \emptyset \to \emptyset$ give the following linear maps

$$\mu\delta\colon \mathbf{k}^2 \to \mathbf{k}^2, \qquad \mu\delta(a,b) = 2(a,b), \qquad \varepsilon\mu\delta\eta\colon \mathbf{k} \to \mathbf{k}, \qquad \varepsilon\mu\delta\eta(a) = 2a. \tag{60}$$

Notice that the multiplicative part *is precisely the group-algebra* $\mathbf{k}[\mathbf{Z}_2]$. In fact, the space 'eight' T' is the sum of two pointed circles $(\mathbf{S}^1, x_0)$, and, in the cospan

$$\mu = (\mathbf{S}^1 + \mathbf{S}^1 \to T \leftarrow \mathbf{S}^1),$$

the second map can be viewed as the usual 'comultiplication' of the H-cospace structure on the pointed circle. Now, the functor π^* sends the first map to an isomorphism (which we have made into an identity), and the second to the multiplication of $\pi^*(\mathbf{S}^1) = [\mathbf{S}^1, \mathbf{P}^2] = \pi_1(\mathbf{P}^2)$.

More generally, the pseudo-projective plane P_n (3.2) gives a Frobenius algebra on the vector space $\mathbf{k}^n$, whose multiplicative part is the group-algebra $\mathbf{k}[\mathbf{Z}_n]$

$$\eta\colon \{0\} \to \mathbf{Z}_n, \qquad \mu = \mathrm{sum}\colon \mathbf{Z}_{\mathrm{n}} \times \mathbf{Z}_{\mathrm{n}} \to \mathbf{Z}_{\mathrm{n}} \qquad (\varepsilon = \eta^\sharp,\ \ \delta = \mu^\sharp), \tag{61}$$

$$\begin{aligned} &(a_0, ..., a_{n-1}).(b_0, ..., b_{n-1}) = (\Sigma_{\,i+j=k}\ a_i b_j)_{k=0,...n-1}, \\ &\delta(a_0, ..., a_{n-1}) = a_0(\Sigma_{\,i+j=0}\ e_{ij}) + ... + a_{n-1}(\Sigma_{\,i+j=n-1}\ e_{ij}), \\ &\eta(a) = (a, 0, ..., 0), \qquad\qquad \varepsilon(a_0, ..., a_{n-1}) = a_0. \end{aligned} \tag{62}$$

4. Extending homology and homotopy functors to cospans

Absolute and relative (co)homology functors can be extended to collarable cospans, using the Mayer-Vietoris sequence. Similar extensions hold for the fundamental-groupoid functor.

4.1. Theorem. [Collarable pushouts and homology] *Let us be given a homology theory H_*, defined on all pairs of topological spaces (and satisfying the axioms of Eilenberg-Steenrod).*

(a) In a pushout $(B; u, v)$ of collarable maps f, g (as in (48)), the maps u, v and $uf = vg$ are closed embeddings, and we have a Mayer-Vietoris exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{(f_*, g_*)} H_n(X) \oplus H_n(Y) \xrightarrow{[u_*, -v_*]} H_n(B) \xrightarrow{d} H_{n-1}(A) \longrightarrow \cdots \tag{63}$$

(b) More generally, a pushout $((B, B_0); u, v)$ of collarable relative maps

$$f\colon (A, A_0) \to (X, X_0), \qquad g\colon (A, A_0) \to (Y, Y_0), \tag{64}$$

yields a Mayer-Vietoris exact sequence for relative homology

$$\begin{aligned} \cdots \longrightarrow H_n(A, A_0) &\xrightarrow{(f_*, g_*)} H_n(X, X_0) \oplus H_n(Y, Y_0) \xrightarrow{[u_*, -v_*]} H_n(B, B_0) \\ &\xrightarrow{d} H_{n-1}(A, A_0) \longrightarrow \cdots \end{aligned} \tag{65}$$

Proof. We prove (a), since the generalisation to the relative case is straightforward. By Thm. 2.5, we can assume that our pushout is 1-collarable, hence *homeomorphic* to a homotopy pushout, since the contribute of the trivial part plainly satisfies our thesis.

Let us replace the ordinary pushout with the standard homotopy pushout, in the usual construction $B = (X+IA+Y)/\sim$. Writing u', w, v' the closed embeddings of X, IA and Y in B, the latter is covered by the following closed neighbourhoods X', Y' of $u(X), v(Y)$

$$\begin{aligned} &X' = u'(X) \cup w(A\times[0,2/3]) \subset B, \qquad && Y' = v'(Y) \cup w(A\times[1/3,1]) \subset B, \\ &A' = X' \cap Y' = w(A\times[1/3,2/3]). && \end{aligned} \tag{66}$$

The embeddings $A \to A'$, $X \to X'$, $Y \to Y'$ are homotopy equivalences, and induce isomorphisms in homology. It suffices therefore to apply the exact sequence of the triad (B, X', Y'), provided we prove that the latter is a *proper* triad for H_*, in the sense of Eilenberg-Steenrod ([7], p, 34).

In other words, we want to show that the following induced homomorphisms are isomorphisms

$$H_n(X', A') \to H_n(B, Y'), \qquad\qquad H_n(Y', A') \to H_n(B, X'). \tag{67}$$

In fact, they are excision isomorphisms. Working on the first, we are exciding the subset $Y' \setminus A'$, since $B \setminus (Y' \setminus A') = X'$. The topological hypotheses are satisfied:
(i) the subset $Y' \setminus A' = v'(Y) \cup w(A\times]2/3,1])$ is open in B,
(ii) its closure is contained in the interior of Y':

$$\mathrm{cl}(Y' \setminus A') \ = \ v'(Y) \cup w(A \times [2/3,1]) \ \subset \ v'(Y) \cup w(A\times]1/3,1]) \ = \ \mathrm{int}(Y'). \tag{68}$$

■

4.2. Extending homology theories. Let us be given a homology theory H_* on $\mathbf{Top_2}$. We already know, by Thm. 4.1 (and definition 5.3(b)) that H_n takes any pushout of collarable maps to an exact square of abelian groups. Therefore, by 5.5, every H_n has a canonical extension to a homotopy invariant (strict) double functor

$$H_n \colon \mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{R}\mathrm{el}(\mathbf{Ab}), \tag{69}$$

with values in the strict double category of abelian groups and relations. Furthermore, if we restrict to the full substructure $\mathbb{C}\mathrm{blc}(\mathbf{Top}, H_{n-1}) \subset \mathbb{C}\mathrm{blc}(\mathbf{Top})$ determined by the topological spaces X whose reduced homology $\widetilde{H}_{n-1}(X)$ is trivial:

$$\widetilde{H}_{n-1}(X) = Ker(H_{n-1}(X) \to H_{n-1}\{*\}), \tag{70}$$

we get a stronger result: the functor H_n takes a pushout of collarable maps to a *pushout* of abelian groups (see 4.3).

Thus, H_n has a unique involution-preserving extension to a homotopy invariant pseudo double functor

$$H_n \colon \mathbb{C}\mathrm{blc}(\mathbf{Top}, H_{n-1}) \to \mathbb{C}\mathrm{osp}(\mathbf{Ab}). \tag{71}$$

Notice that the condition $\widetilde{H}_{n-1}(X) = 0$ *is only imposed on the cubical faces of cospans,* not *on their middle spaces.* Of course, composing extension (71) with the projection $\mathbb{C}\mathrm{osp}(\mathbf{Ab}) \to \mathbb{R}\mathrm{el}(\mathbf{Ab})$ we find again the previous extension of H_n; or, more precisely, a restriction of the latter.

4.3. Proposition. *Let us be given a concatenation $u +_1 v$ of collarable cospans, in the usual notation $u = (u^-\colon X^- \to X^0 \leftarrow X^+ : u^+)$, $v = (v^-\colon Y^- \to Y^0 \leftarrow Y^+ : v^+)$ and $X^+ = A = Y^-$.*

(a) Each of the following conditions ensures that H_n preserves the concatenation $u +_1 v$:

(i) the homomorphism $u^+_* = H_{n-1}(u^+)\colon H_{n-1}(A) \to H_{n-1}(X^0)$ is injective,

(ii) the homomorphism $v^-_* = H_{n-1}(v^-)\colon H_{n-1}(A) \to H_{n-1}(Y^0)$ is injective.

More generally, it suffices to know that this pair of homomorphisms is jointly monic.

(b) These conditions are necessarily satisfied if $\widetilde{H}_{n-1}(A) = 0$.

Proof. (a). In the composition $w = u +_1 v$ (8), the pushout gives rise to an exact sequence

$$H_n(A) \to H_n(X^0) \oplus H_n(Y^0) \to H_n(Z^0) \to H_{n-1}(A) \to H_{n-1}(X^0) \oplus H_n(Y^0). \qquad (72)$$

Saying that the pair (u^+_*, v^-_*) is jointly monic means that the last homomorphism is injective. Then the preceding one (the third) is null, and the second is surjective. Applying 5.3(d), we conclude that H_n preserves our pushout.

(b) For $n > 1$, our condition means $H_{n-1}(A) = 0$, which trivially implies (i) and (ii). For $n = 1$, it suffices to note that the terminal map $t\colon A \to \{*\}$ can be factored as $A \to X^0 \to \{*\}$; therefore, if $H_{n-1}(t)$ is injective, so must $H_{n-1}(u^+)$ be (and, similarly, $H_{n-1}(v^-)$). ∎

4.4. Extending cohomology theories. Let us be given a cohomology theory H^* on $\mathbf{Top}_2$.

We could redo and adapt what we have done for homology theories, but duality can be used as a shortcut, since H^* is a homology theory with values in the abelian category $\mathbf{Ab}^{\mathrm{op}}$. Moreover, $\mathbb{R}\mathrm{el}(\mathbf{Ab})$ and $\mathbb{R}\mathrm{el}(\mathbf{Ab}^{\mathrm{op}})$ have the same weak structure and a dual strict structure. Finally, the notion of an exact square in an abelian category (5.3) is self-dual.

Thus, we conclude that H^n takes any pushout of collarable maps to an exact square of abelian groups, and has canonical extensions to homotopy invariant double functors, contravariant on the strict maps

$$H^n\colon \mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{R}\mathrm{el}(\mathbf{Ab}), \qquad H^n\colon \mathbb{C}\mathrm{blc}(\mathbf{Top}_2) \to \mathbb{R}\mathrm{el}(\mathbf{Ab}). \qquad (73)$$

4.5. Theorem. [Extending the fundamental-groupoid functor] *Let us be given a pushout of collarable maps f, g, as in (48). Then the fundamental-groupoid functor $\Pi_1\colon \mathbf{Top} \to \mathbf{Gpd}$ turns it into a pushout of groupoids. Therefore, Π_1 has a canonical extension to a pseudo double functor*

$$\Pi_1\colon \mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{C}\mathrm{osp}(\mathbf{Gpd}), \qquad (74)$$

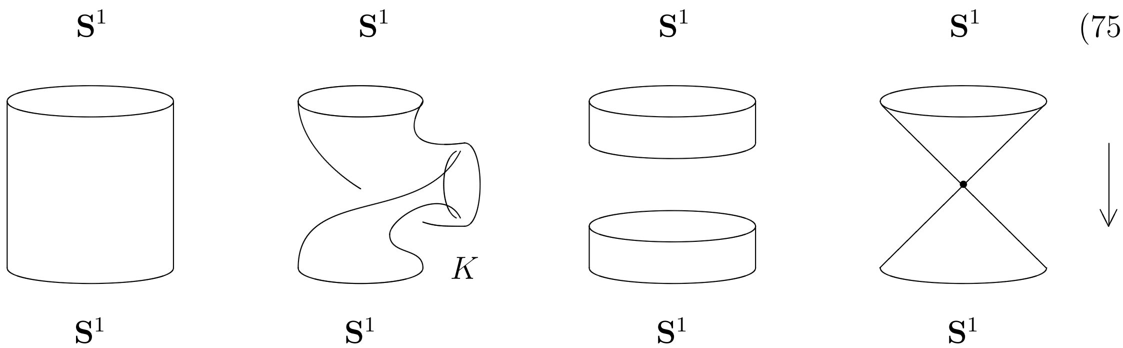
with values in the pseudo double category of small groupoids, their functors and their cospans.

Proof. As in the proof of 4.1, one replaces A, X, Y with homotopy equivalent spaces A', X', Y', so that B is covered by the interior of X', Y'. Then one applies R. Brown's version of the Seifert-van Kampen theorem for the fundamental groupoid [5, 6]. ■

4.6. Using absolute homology groups and relations. We end this section by a few elementary computations, to show how these invariants can work.

Consider the following collarable cospans $\mathbf{S}^1 \to \mathbf{S}^1$

(75)



The first is the cylinder $E_1(\mathbf{S}^1)$, while the second, K, results from 'reversed' embeddings of $\mathbf{S}^1$ into the cylinder (the coequaliser of such embeddings being the Klein bottle). The functor H_1 sends these cospans to the following relations $\mathbf{Z} \to \mathbf{Z}$, respectively

$$\mathrm{id}\colon \mathbf{Z} \to \mathbf{Z}, \qquad -\colon \mathbf{Z} \to \mathbf{Z}, \qquad \omega\colon \mathbf{Z} \to \mathbf{Z}, \qquad \Omega\colon \mathbf{Z} \to \mathbf{Z}, \tag{76}$$

(ω is the least endorelation, which only associates 0 to 0, while Ω is the greatest).

More interesting examples can be obtained on non-connected 1-manifolds, like topological sums of circles $\mathbf{S}^1 + ... + \mathbf{S}^1$, or, more generally, n-dimensional manifolds.

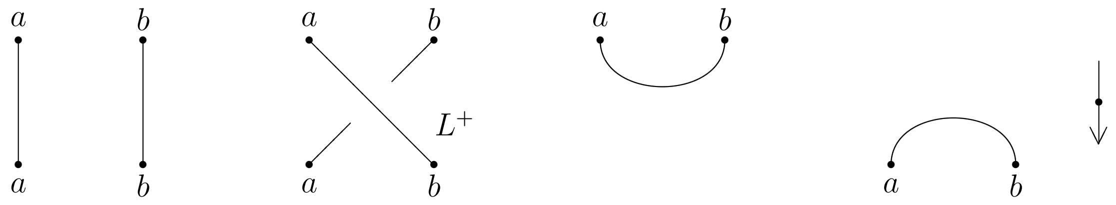
Working with Cosp(**Ab**), we get *lax* double functors $H_n\colon \mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{C}\mathrm{osp}(\mathbf{Ab})$ which are *pseudo* (i.e., preserve concatenation up to isomorphism) under suitable hypotheses (4.2, 4.3). This version can be used to distinguish the collarable cospans above, in figure (75), from other similar ones, having some non connected component in the middle space of the cospan (which would be ignored by the homology relation).

Similar results hold for the fundamental-groupoid functor (see 4.5).

4.7. Using relative homology groups and relations. Tangles *live in an ambient space*: say the disc $\mathbf{B}^2$ for domain and codomain, and the solid cylinder $B = \mathbf{B}^2 \times [0, 1]$ for the tangles themselves.

Relative homology (in these ambient spaces) allows one to distinguish among the following generators, viewed as relative cospans (the tangle is always written as T)

$$\begin{array}{ll} (\mathbf{B}^2, \mathbf{S}^0) \to (B, T) \leftarrow (\mathbf{B}^2, \mathbf{S}^0), & (\mathbf{B}^2, \mathbf{S}^0) \to (B, T) \leftarrow (\mathbf{B}^2, \mathbf{S}^0), \\ (\mathbf{B}^2, \mathbf{S}^0) \to (B, T) \leftarrow (\mathbf{B}^2, \emptyset), & (\mathbf{B}^2, \emptyset) \to (B, T) \leftarrow (\mathbf{B}^2, \mathbf{S}^0), \end{array} \tag{77}$$



In fact, easy computations show that they induce, via H_1, the following relations

$$\mathrm{id}\colon \mathbf{Z} \to \mathbf{Z}, \qquad -\colon \mathbf{Z} \to \mathbf{Z}, \qquad 0\colon \mathbf{Z} \to 0, \qquad 0^\sharp\colon 0 \to \mathbf{Z}. \tag{78}$$

Note, however, that we cannot distinguish in this way the tangle L^+ from L^- (which has the opposite choice of the arc which passes over). Thus, the invariants which we are obtaining are not sufficient to classify the isotopy classes of tangles.

On the other hand, again, relative-homology *cospans* (instead of relations) can detect components of the tangle which are not connected to domain and codomain, as in fig. 1 of the Introduction.

5. Appendix: Extending functors to cospans and relations

The abelian category of left R-modules, on a fixed (unitary) ring R is written **Mod**. In the preceding sections, we have used a field $\mathbf{k}$, and written **Vct** the corresponding category (but rings and modules can also be used there).

5.1. Theorem. [Extending functors to cospans] *(a) An arbitrary functor $F\colon \mathbf{X} \to \mathbf{Y}$ between categories with (a full choice of) pushouts has a canonical extension to a lax double functor*

$$F\colon \mathbb{C}\mathrm{osp}(\mathbf{X}) \to \mathbb{C}\mathrm{osp}(\mathbf{Y}), \tag{79}$$

which is a pseudo *double functor if F preserves pushouts (and is strict if F preserves the distinguished choices).*

(b) For $\mathbf{X} = \mathbf{Top}$, we have a lax extension $\mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{C}\mathrm{osp}(\mathbf{Y})$*, which is a pseudo double functor if F preserves pushouts* of collarable maps.

(c) For the contravariant case, take $\mathbf{Y} = \mathbf{C}^{\mathrm{op}}$, where $\mathbf{C}$ is a category with pullbacks. Now, we have a colax *extension (contravariant on the strict maps)*

$$F\colon \mathbb{C}\mathrm{osp}(\mathbf{X}) \to \mathbb{S}\mathrm{p}(\mathbf{C}), \tag{80}$$

which is pseudo if F transforms pushouts of $\mathbf{X}$ into pullbacks of $\mathbf{C}$.

Proof. We only have to prove (a). Let the extension F act on cospans and their transversal maps in the obvious way:

$$\begin{array}{ll} u = (u^-\colon X^- \to X^0 \leftarrow X^+ : u^+), & Fu = (Fu^-\colon FX^- \to FX^0 \leftarrow FX^+ : Fu^+), \\ f = (f^-, f^0, f^+)\colon u \to v, & Ff = (Ff^-, Ff^0, Ff^+). \end{array} \tag{81}$$

Degenerate cospans are strictly preserved, but a concatenation in $\mathbf{X}$ of consecutive cospans u, v (with notation as in (8)) gives rise to a *comparison* special transversal map $F[u, v]$

$$F[u, v]\colon Fu +_1 Fv \to F(u +_1 v), \tag{82}$$

$$\begin{tikzcd}[column sep=small] & & Z^0 & & \\ & X^0 \arrow[ur, dashed] & & Y^0 \arrow[ul, dashed] & \\ X^- \arrow[ur] & & A \arrow[ul] \arrow[ur] & & Y^+ \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & FZ^0 & & \\ & & \bullet \arrow[u] & & \\ & FX^0 \arrow[uur] \arrow[ur] & & FY^0 \arrow[uul] \arrow[ul] & \\ FX^- \arrow[ur] & & FA \arrow[ul] \arrow[ur] & & FY^+ \arrow[ul] \end{tikzcd}$$

Plainly, if F preserves pushouts (resp. distinguished pushouts), then these comparisons are invertible (resp. identities) and the extension is a pseudo (resp. strict) double functor. ■

5.2. Theorem. [Linearisation] *The free R-module functor $F\colon \mathrm{f}\mathbf{Set} \to \mathrm{f}\mathbf{Mod}$, from finite sets to finitely generated R-modules, can be extended to a monoidal functor $\mathrm{Sp}_{iso}(\mathrm{f}\mathbf{Set}) \to \mathrm{f}\mathbf{Mod}$, defined on the category of isomorphism classes of spans.*

Also the latter is monoidal: it takes Cartesian product of sets to tensor product of (free) modules.

Proof. Let $u = (f, g) = (X \leftarrow A \to Y)$ be a span in $\mathrm{f}\mathbf{Set}$. Its linearised homomorphism is defined on the standard basis X of FX as

$$Fu\colon FX \to FY, \qquad (Fu)(x) = \sum_{a\in A,\, f(a)=x} g(a). \tag{83}$$

One can notice that Fu is represented by the matrix (n_{xy}), where n_{xy} is the number of points $a \in A$ such that $f(a) = x$ and $g(b) = y$. The reversed span gives thus the transpose matrix. (The extension could be said to be an involution-preserving functor with values in the category of free modules with an assigned basis.)

Indeed, given a consecutive span $v = (h, k) = (Y \leftarrow B \to Z)$, with composition $u + v$ over the pullback $C = \{(a, b) \mid g(a) = h(b)\}$:

$$(Fv)(Fu)(x) = \sum_{a\in A,\, b\in B,\, f(a)=x,\, h(b)=g(a)} k(b) = \sum_{(a,b)\in C,\, f(a)=x} k(b). \tag{84}$$

Notice that a similar procedure for *relations*, instead of spans, would not be consistent with composition. ■

5.3. Additive relations and exact squares. We are also interested in extending functors to double categories of relations, like $\mathbb{R}\mathrm{el}(\mathbf{Mod})$ or $\mathbb{R}\mathrm{el}(\mathbf{Set})$. Let us begin by recalling the first case.

A *relation* $a\colon A \to B$ between R-modules (also called an additive, or linear, relation) can be equivalently defined as:

(i) a submodule of the direct sum $A \oplus B$,

(i′) a quotient of the direct sum $A \oplus B$,

(ii) an equivalence class $[v]\colon A \to B$ of spans having the same image in $A \oplus B$ (or the same pushout),

(ii′) a class $[u]\colon A \to B$ of cospans giving the same quotient of $A \oplus B$ (or having the same pullback),

(iii) a jointly monic span $A \to B$,

(iii′) a jointly epic cospan $A \to B$.

Relations $A \to B$, with a fixed domain and codomain, form an ordered set, by means of the inclusion of submodules of $A \oplus B$, or by the existence of a special transversal map between representative (co)spans. The composition of (co)spans induces a (well known) composition of relations, which is strictly categorical. We have thus a category $\mathrm{Rel}(\mathbf{Mod})$, which is a quotient of the categories $\mathrm{Sp}_{iso}(\mathbf{Mod})$ and $\mathrm{Cosp}_{iso}(\mathbf{Mod})$.

Now, in order to extend functors to additive relations, we must know which squares in **Mod**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \tag{85}$$

become *bicommutative* in $\mathrm{Rel}(\mathbf{Mod})$, which means that:

$$hf = kg, \qquad\qquad gf^\sharp = k^\sharp h\colon B \to C. \tag{86}$$

This has been determined in a 1966 paper, by Hilton [13]. An arbitrary square (85) of R-homomorphisms has an associated sequence

$$0 \longrightarrow A \xrightarrow{(f,g)} B \oplus C \xrightarrow{[h,-k]} D \longrightarrow 0 \tag{87}$$

The square is:

(a) commutative if and only if the sequence is of order two,

(b) *exact* (by definition) if and only if the sequence is exact in the central object,

(c) a pullback if and only if the sequence is exact in the central and left objects,

(d) a pushout if and only if the sequence is exact in the central and right objects.

It is easy to see that the square (85) is exact if and only if:

(e) for all $b \in B$, $c \in C$ with $h(b) = k(c)$, there exists *some* $a \in A$ such that $f(a) = b$, $g(a) = c$,

if and only if the underlying square of mappings is a *weak pullback* in **Set** (i.e., it verifies there the existence part of the universal property).

(A self-dual diagrammatic characterisation of exact squares will be given below, in 5.6). It is now straightforward to verify that the square (85) becomes bicommutative in Rel(**Mod**) if and only if it is exact in **Mod**.

Of course, we also have a strict double category $\mathbb{R}$el(**Mod**) (studied in [11]), which is a quotient of the weak double categories of spans and cospans: objects are modules, 1-cubes are relations, 0-maps are homomorphisms and 1-maps (or double cells) are 'lax-commutative' squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\leqslant"] & A' \arrow[d, "b"] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \qquad ga \leqslant bf. \tag{88}$$

Notice that this double category is *flat* [11]: a double cell is determined by its boundary.

All this extends to arbitrary abelian categories, except the characterisation of exact squares as weak pullbacks, or by property (e), which is based on elements. It is still true that a square in an abelian category is exact if and only if it is bicommutative in the corresponding category of relations.

5.4. Relations of sets. For sets (or for a general regular category), these dual constructions of relations - as classes of spans or cospans - split into two distinct constructions.

Thus, a *relation* $a\colon A \to B$ between sets amounts, equivalently, to:

(i) a subset of the Cartesian product $A \times B$,

(ii) an equivalence class $[v]\colon A \to B$ of spans having the same image in $A \times B$,

(iii) a jointly monic span $v\colon A \to B$.

They are composed by pullback, and form a category Rel(**Set**), which is a quotient of Sp_{iso}(**Set**).

Again, we also have a larger structure, the (flat) double category $\mathbb{R}$el(**Set**), which is a quotient of the weak double category $\mathbb{S}$p(**Set**).

On the other hand, the dual construction, say Corel(**Set**) = Rel($\mathbf{Set}^{\mathrm{op}}$), is not used here. (A *corelation* $A \to B$ is a quotient of the sum $A + B$; they compose by pushouts. Corelations between sets, or finite sets, were studied in the 70's under the name of *transductors*, for applications to the theory of devices, since they can be viewed as formalising 'boxes' of electrical connections between two sets of terminals. See [19] and its references.)

Also here, it is straightforward to see that a square (85) of mapping of sets becomes bicommutative in Rel(**Set**) if and only if it satisfies the condition 5.3(e), if and only if it is a weak pullback in **Set**. More elegant, self-dual characterisations are given below (5.6), but are not used in this article.

5.5. Theorem. [Extending functors to relations] *Let* $\mathbf{Y}$ *be the category* **Mod** *of* R*-modules (resp.* **Set***). A functor* $F\colon \mathbf{X} \to \mathbf{Y}$ *defined on a category* $\mathbf{X}$ *with (a full choice of) pushouts has a canonical extension to a lax double functor*

$$F\colon \mathbb{C}\mathrm{osp}(\mathbf{X}) \to \mathbb{R}\mathrm{el}(\mathbf{Y}). \tag{89}$$

The latter is a strict double functor if F *sends pushouts of* $\mathbf{X}$ *to exact squares of modules (resp. weak pullbacks of sets); in this case, the extension is uniquely determined.*

For $\mathbf{X} = \mathbf{Top}$*, we have a similar extension* $\mathbb{C}\mathrm{blc}(\mathbf{Top}) \to \mathbb{R}\mathrm{el}(\mathbf{Mod})$*, which is a strict double functor if* F *sends pushouts of collarable maps to exact squares of modules (resp. weak pullbacks of sets).*

Proof. The existence of the extension is proved as in 5.1, taking into account the characterisation of bicommutative squares recalled above, in 5.3 (resp. 5.4).

Its uniqueness depends on the fact that the reversed relation $f^\sharp$ of a morphism $f: A \to B$ is *determined* as its adjoint morphism in the 2-categorical structure, by the inequalities $1 \leqslant f^\sharp f$ and $ff^\sharp \leqslant 1$. Now, writing $u^\sharp = (1\colon Y \to Y \leftarrow X : u)$ the reversed cospan of a map $u\colon X \to Y$ in $\mathbf{X}$, it is easy to see that there are special transversal maps in $\mathbb{C}\mathrm{osp}(\mathbf{X})$

$$e_1(X) \to u +_1 u^\sharp, \qquad\qquad u^\sharp +_1 u \to e_1(Y). \tag{90}$$

Therefore a double functor (89) which extends the original F must send $u^\sharp$ to the adjoint of Fu, and is thus uniquely determined on all cospans. ■

5.6. Semicartesian squares. We end by recalling a self-dual notion, which characterises the squares which become bicommutative within relations of sets or modules (or also in any abelian category).

A commutative square $(f, g; h, k)$ in an arbitrary category is said to be *semicartesian* [8] if:

(i) given a span (f', g') which commutes with (h, k) and a cospan (h', k') which commutes with (f, g), the outer diamond commutes: $h'f' = k'g'$

$$\begin{tikzcd}
 & & B \arrow[dr, "h"] \arrow[drr, "h'"] & & \\
\bullet \arrow[urr, "f'"] \arrow[drr, "g'"'] & A \arrow[ur, "f"] \arrow[dr, "g"'] & & D & \bullet \\
 & & C \arrow[ur, "k"'] \arrow[urr, "k'"'] & &
\end{tikzcd} \tag{91}$$

Plainly, any pullback and any pushout is semicartesian. When pullbacks and (or) pushouts exist, semicartesian squares can be characterised by various equivalent properties based on such (co)limits: e.g., the span (f, g) and the pullback of (h, k) commute with the same cospans, or have the same pushout.

In **Set**, a square is semicartesian if and only if it is a weak pullback. In an abelian category, a square is semicartesian if and only if it is exact in the sense of Hilton (5.3(b)).

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