Inverse Scattering and Edge Detection: the Threshold Problem for the Linear Sampling Method

M. Piana\textsuperscript{1}, M. Brignone\textsuperscript{2}, R. Aramini\textsuperscript{3} and J. Coyle\textsuperscript{4}

\textsuperscript{1} Dipartimento di Informatica
Università di Verona, Ca’ Vignal 2, Strada le Grazie 15, 37134 Verona, Italy
michele.piana@univr.it

\textsuperscript{2} Dipartimento di Matematica
Università di Genova, via Dodecaneso 35, 16146 Genova, Italy
brignone@dima.unige.it

\textsuperscript{3} Dipartimento di Matematica
Università di Trento, via Sommarive 14, 38050 Povo di Trento, Italy
aramini@science.unitn.it

\textsuperscript{4} Dipartimento di Matematica
Monmouth University, West Long Branch, New Jersey, USA
jcoyle@monmouth.edu

Abstract: The linear sampling method is a fast inversion technique for visualizing the profile of a scatterer from measurements of the far-field pattern. The mathematical result at the basis of this method is that, given a spatial grid of points covering the scatterer, for each grid point there is an approximate solution of a linear integral equation whose $L^2$ norm blows up when the point approaches the boundary of the scatterer from inside and stays large when the point is outside the scatterer. According to a recent formulation, the linear sampling method requires the regularized solution of a single functional equation where the unique optimal value of the regularization parameter is fixed by means of the generalized discrepancy principle. Within this framework in this paper we first show that the introduction of an incompatibility measure in the generalized discrepancy function provides more reliable maps of the scatterer. Then we apply active contour techniques to the visualization maps in order to optimally extract the profile of the scatterer.

1. Introduction

The linear sampling method [6] provides an approach for the visualization of scatterers from measurements of the far-field pattern under fixed-frequency scattering conditions. The mathematical basis of this method [3] is given by the far-field equation, which is a Fredholm integral equation of the first kind defined for each point in $\mathbb{R}^2$ or $\mathbb{R}^3$ and in which the data function is a known analytic function and the integral kernel is the measured (and therefore noisy) far-field pattern. A general theorem [3] states that an approximate solution for the far-field equation exists whose $L^2$-norm blows up to infinity for all points approaching the boundary of the scatterer from inside and stays arbitrarily large outside. This result suggests the following visualization algorithm [8]: for each point of a grid containing the scatterer:
• compute the Tikhonov regularized solution of the far-field equation, whereby the optimal
regularization parameter is determined by means of the generalized discrepancy principle
[11];
• plot the $L^2$-norm of the optimal regularized solution.

Then a visualization map of the scatterer will be given by all points where this norm is mostly
large. A new implementation of the linear sampling method, proposed in [1], involves the Tikhonov
regularized solution of a single functional equation where the unique optimal value of the regu-
larization parameter is again fixed by means of the generalized discrepancy principle.

An intriguing problem for the implementation of the linear sampling method is the formulation
of an automatic threshold criterion for the pixel content of the visualization map. In the present
paper this issue is addressed by applying an edge detection technique based on deformable models
[5, 10] which explicitly exploits the new 'no-sampling' implementation of the method. In particular
we show that this approach is effective only if the quality of the visualization map is enhanced
by applying the generalized discrepancy principle with an appropriately corrected form of the
generalized discrepancy function.

2. The linear sampling method

Let us consider the far–field operator $F : L^2(\Omega) \rightarrow L^2(\Omega)$ defined as

$$(Fg(\cdot))(\hat{x}) := \int_{\Omega} u_{\infty}(\hat{x}, d)g(d)ds(d) \quad \forall g(\cdot) \in L^2(\Omega).$$

Here $\Omega = \{x \in \mathbb{R}^2, |x| = 1\}$; $\hat{x}$ and $d$ are unit vectors denoting the observation and incidence
directions respectively; $u_{\infty}(\hat{x}; d)$ is the far-field pattern associated with the scattered field; $z$ is a
point in $\mathbb{R}^2$ (for sake of simplicity we consider only the 2D case). Then we introduce the far–field
equation

$$(Fg_z(\cdot))(\hat{x}) = \Phi_{\infty}(\hat{x}, z),$$

where $g_z(\cdot)$ is a function in $L^2(\Omega)$ for each $z$ and $\Phi_{\infty}(\hat{x}, z)$ is the far-field pattern of the fundamental
solution $\Phi(x, z)$ of the Helmholtz equation in $\mathbb{R}^2$ and is given by

$$\Phi_{\infty}(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z},$$

where $k$ is the wavenumber.

The general theorem at the basis of linear sampling method [3] assures that there exists an
approximate solution to (2) whose $L^2-$norm grows up to infinity when $z$ approaches the boundary
of the scatterer.

Following the implementation of the linear sampling method suggested in [1], we consider the
Hilbert space $[L^2(T_A^B)]^N := \bigoplus_{j=0}^{N-1} L^2(T_A^B)$ and the linear operator $F_h : [L^2(T_A^B)]^N \rightarrow
[L^2(T_A^B)]^N$ given by

$$[F_h g(\cdot)](\cdot) := \left\{ \sum_{j=0}^{N-1} (F_h)_{ij} g_j(\cdot) \right\}^{N-1}_{i=0} \forall g(\cdot) := (g_0(\cdot), \ldots, g_{N-1}(\cdot)) \in [L^2(T_A^B)]^N,$$
where \( T^B_A := (-A, A) \times (-B, B) \) is a rectangle containing the scatterer, \( N \) denotes the number of incidence and observation angles, \((F_h)_{ij}\) are the elements of the noisy far-field matrix and finally \( h \) is an upper bound for the noise.

From a practical viewpoint, the introduction of the operator \( F_h \) allows one to express the infinitely many algebraic systems of the traditional implementation of the linear sampling method as the single functional equation in \([L^2(T^B_A)]^N\)

\[
[F_h g(\cdot)](\cdot) = \Phi_{\infty}(\cdot),
\]

where \( \Phi_{\infty}(\cdot) := (\Phi_{\infty}(\hat{x}_0, \cdot), \ldots, \Phi_{\infty}(\hat{x}_{N-1}, \cdot)) \); here \( \hat{x}_i = (\cos(2\pi i/N), \sin(2\pi i/N)) \) for any \( i = 0, \ldots, N-1 \). If \( r_h \) is the rank of \( F_h \) and \( \{s^h_p, u^h_p, v^h_p\} \) is the singular system of \( F_h \), the regularized solution of problem (5) is given by

\[
g_\alpha(\cdot) = \sum_{p=0}^{r_h-1} \frac{s^h_p}{(s^h_p)^2 + \alpha} (\Phi_{\infty}(\cdot), v^h_p)_{2,N} u^h_p, \tag{6}
\]

where \((\cdot, \cdot)_{2,N}\) is the scalar product in \([L^2(T^B_A)]^N\) and \( \alpha \) is a real positive number. This regularization parameter can be determined, according to the generalized discrepancy principle as the zero of the generalized discrepancy function [11]

\[
\rho(\alpha) := ||[F_h g_\alpha(\cdot)](\cdot) - \Phi_{\infty}(\cdot)||^2_{2,N} - h^2 ||g_\alpha(\cdot)||^2_{2,N} - [\mu_h(\Phi_{\infty}(\cdot), F_h)]^2, \tag{7}
\]

where

\[
\mu_h(\Phi_{\infty}(\cdot), F_h) := \inf_{g \in [L^2(T^B_A)]^N} ||[F_h g(\cdot)](\cdot) - \Phi_{\infty}(\cdot)||_{2,N} \tag{8}
\]

is the incompatibility measure and \( \| \cdot \|_{2,N} \) is the norm induced by the scalar product \((\cdot, \cdot)_{2,N}\). If \( \alpha^* \) is the optimal value provided by this criterion, then \( g_{\alpha^*}(\cdot) \) is the optimal regularized solution of problem (5) and an indicator function

\[
\Xi(z) := J(\Psi(z)) \tag{9}
\]

can be introduced, where \( J : (0, +\infty) \to \mathbb{R} \) denotes an appropriate monotonic function and

\[
\Psi(z) := ||g_{\alpha^*}(z)||^2_{C^N}. \tag{10}
\]

A visualization of the scatterer will be obtained by plotting (9) for all point \( z \) on a computational grid.

Numerical tests show that, at least for low noise levels this approach provides undersmoothing maps. However, a more accurate choice for the regularization parameter [11] can be obtained by using the new discrepancy function

\[
\hat{\rho}(\alpha) := ||[F_h g_\alpha(\cdot)](\cdot) - \Phi_{\infty}(\cdot)||^2_{2,N} - [h||g_\alpha(\cdot)||_{2,N} + \hat{\mu}_h(\Phi_{\infty}(\cdot), F_h)]^2, \tag{11}
\]

where \( \hat{\mu}_h(\Phi_{\infty}(\cdot), F_h) \) is a new incompatibility measure defined as

\[
\hat{\mu}_h(\Phi_{\infty}(\cdot), F_h) := \inf_{g \in [L^2(T^B_A)]^N} (||[F_h g(\cdot)](\cdot) - \Phi_{\infty}(\cdot)||_{2,N} + h||g(\cdot)||_{2,N}). \tag{12}
\]

It can be shown that the zero of (11)-(12) provides a larger value for the regularization parameter, and this allows one to obtain better maps.
3. Deformable models

A traditional deformable contour is a closed curve \( \gamma : [0,1] \rightarrow \mathbb{R}^2 \) that moves through the domain of an image and minimize the functional

\[
E(\gamma) := \int_0^1 \frac{1}{2} (\alpha(s) |\gamma'(s)|^2 + \beta(s) |\gamma''(s)|^2) + E_{\text{ext}}(\gamma(s)) \, ds .
\] (13)

Here \( \alpha \) and \( \beta \) are weights that control the importance of the first- and second-order terms respectively, and \( E_{\text{ext}} \) denotes the external energy derived from the image map: in this application, we choose

\[
E_{\text{ext}} = -|\nabla \Xi|^2 .
\] (14)

Minimizing \( E(\gamma) \) is a variational problem, which turns out to be equivalent to finding the solution of the Euler equation [9]

\[
(\alpha(s) \gamma'(s))' - (\beta(s) \gamma''(s))'' - \nabla E_{\text{ext}}(\gamma(s)) = 0 .
\] (15)

In order to determine a solution to (15), we assume that the curve \( \gamma \) becomes dynamic, by regarding it as a function of time \( t \) as well as \( s \), i.e. \( \gamma = \gamma(s,t) \). We now consider the equation

\[
\frac{\partial}{\partial t} \gamma(s,t) = (\alpha(s) \gamma'(s,t))' - (\beta(s) \gamma''(s,t))'' - \nabla E_{\text{ext}}(\gamma(s,t)) .
\] (16)

When the solution \( \gamma \) stabilizes, the term \( \frac{\partial}{\partial t} \) goes to zero and we have a solution of (15). Therefore a numerical solution to (15) can be found by discretizing equation (16) and solving iteratively the discretized system [10]. An accurate implementation of this iterative procedure must account for the following two issues:

- discretization may introduce numerical instabilities which can be reduced by using [?]

\[
\frac{\partial}{\partial t} \gamma(s,t) = (\alpha(s) \gamma'(s,t))' - (\beta(s) \gamma''(s,t))'' + F ,
\] (17)

where

\[
F = -\kappa \frac{\nabla E_{\text{ext}}}{||\nabla E_{\text{ext}}||} .
\] (18)

- In the framework of the ‘no-sampling’ implementation of the linear sampling method, \( \nabla E_{\text{ext}} \) can be computed analytically, thus notably increasing the accuracy of the computation.

4. Numerical applications

We consider the reconstruction of an impenetrable kite whose profile is described by the parametric equation

\[
x(t) := (1.5 \cdot \sin t, \cos t + 0.65 \cdot \cos(2t) - 0.65), \quad t \in [0,2\pi] ,
\] (19)

and assume that the total field is zero on this boundary (Dirichlet condition). The wavenumber is set equal to 1 and the far–field matrix \( \mathbf{F} \) is computed by using the Nyström method [7] for
Figure 1: Reconstruction of an impenetrable kite in the case of Dirichlet boundary conditions, by means of the linear sampling method. The wavenumber is $k = 1$, while the far-field matrix is computed for 36 incidence and observation angles and affected by 2% Gaussian noise. We plot as indicator function $\log\left(\|g_{\alpha^*}(z)\|_{CN}^2\right)$, where $\alpha^*$ is fixed by finding the zero of the generalized discrepancy function as given by (7)-(8) (panel a) or by (11)-(12) (panel b).

Figure 2: Implementation of the deformable contour technique to extract the boundary of the impenetrable kite of Figure 1. The circle of radius 2 is the initial guess; the white line is the true boundary of the scatterer; the blue line represents the curve obtained by iteratively solving equations (17)-(18).
\( N = 36 \) incidence and observation angles, and \( 2\% \) Gaussian noise is added by means of a \( 36 \times 36 \) noise matrix \( \mathbf{H} \) summed to \( \mathbf{F} \). The optimal value of the regularization parameter is obtained by using both generalized discrepancy functions (7)-(8) and (11)-(12). Then the indicator function \( \Xi(z) := \log(\| \mathbf{g}_{\alpha}(z) \|_2^2) \) is determined for any \( z \in T_A^B := [-3,3] \times [-3,3] \) (see Figure 1). The profile of the scatterer is extracted from the map in Figure 1(b) by applying the deformable model described in the previous section, i.e. by iteratively solving (17)-(18) for \( \alpha(s) = 0.01, \beta(s) = 0 \) and \( \kappa = 11.8 \). The result is shown in Figure 2: the red line denotes the initial guess, the white line is the true boundary of the scatterer, while the blue line represents the reconstructed profile.

5. Conclusions

In this work we adopt a no–sampling implementation of the linear sampling method, whereby the traditional scheme is replaced by a unifying functional approach, which requires a single regularization procedure. The advantage of this approach is that the optimal value of the regularization parameter, chosen by means of the generalized discrepancy principle, does not depend on the sampling point, then all the derivatives of the indicator function, which play a central role in the edge–detection techniques, are analytically known. In this framework we observe that the choice of the regularization parameter as a zero of the generalized discrepancy function (11) and the use of a suitable deformable contour provides a reconstructed boundary closer to the true profile than the one obtained using a traditional Canny edge–detection algorithm as in [2].

References


