POST-PROCESSING OF THE LINEAR SAMPLING METHOD BY MEANS OF DEFORMABLE MODELS

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Abstract. The linear sampling method is a qualitative procedure for the visualization of both impenetrable and inhomogeneous scatterers, which requires the regularized solution of a linear ill-posed integral equation of the first kind. An open issue in this technique is the one of determining the optimal scatterer profile from the visualization maps in an automatic manner. In the present paper this problem is addressed in two steps. First, linear sampling is optimized by using a new regularization algorithm for the solution of the integral equation, which provides more accurate maps for different levels of the noise affecting the data. Then an edge detection technique based on active contours is applied to the optimized maps. Our computation exploits a recently introduced implementation of the linear sampling method which enhances both the accuracy and the numerical effectiveness of the approach.

Key words. Inverse scattering, linear sampling method, regularization, active contours.

1. Introduction. The linear sampling method [11, 14] is a qualitative procedure for the solution of acoustical and electromagnetic inverse scattering problems in the resonance region. It is based on what we will call the general theorem [3, 4], which is concerned with a linear ill-posed integral equation of the first kind, named the far-field equation, whose integral kernel is the far-field pattern of the scattered field and whose data is a known analytical function. According to the general theorem, for each point in the physical space an approximate solution of the far-field equation exists, such that the $L^2$ norm of this solution blows up to infinity when the point approaches the boundary of the scatterer from inside, while it stays arbitrarily large outside. This behavior inspires a visualization algorithm [14] based on the following steps: for each point of a computational grid containing the object

1. (approximately) solve a discretized version of the far-field equation;
2. plot the indicator function, which is an appropriate monotonic function of the Euclidean norm of this approximate solution: then the scatterer profile will be highlighted by all points in the physical space whereby the indicator function is significantly large or small with respect to the surrounding points.

In step 1 of the method, it is crucial to account for the ill-posedness of the linear inverse problem of solving the far-field equation. This is done by applying a regularization procedure like Tikhonov method, where the integral equation is replaced by a convex minimum problem and the optimal trade-off between the stability of the solution and its reliability in reproducing the data function is realized through a judicious choice of a regularization parameter. In most traditional sampling implementations [5, 9, 10, 14, 23], the regularization algorithm is applied for each point in a computational grid containing the scatterer, so that a different optimal regularization parameter and a different regularized solution are determined for each one of these sampling points. More recently [6] heuristic recipes have been utilized for

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reducing this multiple regularization approach to a single procedure for the choice of the regularization parameter. In a new rigorous formulation of the method [2] the set of far-field equations parameterized over the grid has been replaced by a single functional equation formulated in a functional space which is the direct sum of many $L^2$ spaces. This “no-sampling” approach does not require to sample over the region containing the object, so that a single regularization procedure is applied to the functional equation and a single regularization parameter is determined by means of some optimality criterion. This implies two essential advantages: first, the computation of the method is now notably faster; second, the indicator function now has an explicit analytical form whose Fourier transform can be rather easily computed. From this Fourier analysis a quantitative estimate, based on the Shannon theorem [17], of the spatial resolution achievable by the method is now possible.

Step 2 of the linear sampling method contains a still open issue, which holds for both the “sampling” and the “no-sampling” implementation of the algorithm. In fact, according to the general theorem, the Euclidean norm of the approximate solution to the far-field equation becomes large close to the scatterer boundary, but an objective recipe to decide what “large” means in this context is mandatory in an actual application of the method to real inverse scattering problems. To this purpose, the most common technique [13] consists in using a (heuristic) threshold criterion in order to automatically decide which level curve of the indicator function best approaches the profile of the scatterer; on the other hand, level curves are only a subset of the class of (plane) curves which can be regarded as good, or simply admissible, approximations of the true boundary. Anyway, the problem of detecting edges in the visualization maps provided by the linear sampling method is deeply related to the fact that these images are the result of a regularization procedure and to the effectiveness of the specific regularization method applied. Indeed, regularization implies smoothing on the variations of the pixel content and this limits the performances of any edge detection algorithm. At least in the case of conducting scatterers, such performances are further weakened by the fact that, at high frequencies, maps from linear sampling present notable artefacts inside and around the obstacle [22]. Therefore, in the case of linear sampling, the effectiveness of edge detection for the automatic choice of an optimal scatterer profile and the effectiveness of regularization for the solution of the discretized far-field equation are two interplaying issues.

The main goal of the present paper is to describe a post-processing procedure for the detection of the scatterer profile in the visualization maps provided by the linear sampling method. This aim is achieved by means of two steps. First, we point out that the regularization method based on Morozov’s discrepancy principle, typically applied for the regularized solution of the discretized far-field equation, is not optimal for certain levels of the noise affecting the scattering data. In particular, for small noise amounts, the regularization parameter picked up by the standard implementation of the generalized discrepancy principle [14, 24] provides inaccurate reconstructions, while a notable improvement is obtained including an incompatibility measure [24] in the discrepancy function. From this result we formulate a new regularization algorithm which is effective for all levels of the noise in the far-field pattern.

Second, an edge detection algorithm based on active contours will be applied to the visualization maps provided by optimized linear sampling. Active contours [7, 8, 18] are based on the iterative minimization of a functional representing the energy of a deformable model which is fit to the profile. They have proved their
effectiveness in addressing several issues in learning, filtering and problems in visual clutter, and found applications from computer graphics, user-interface design, medical imaging and robotics. As most edge detection techniques, also these deformable models require at some stage the computation of the gradient of the visualization map. In the new “no-sampling” formulation of the linear sampling method, the pixel content distribution, given by the point values of the indicator function, is analytically known. It follows that also the gradient of the image can be analytically determined and thus the accuracy of the contour computation is increased.

The next section of this paper is devoted to describe the linear sampling method in its “no-sampling” formulation. In Section 3 we discuss the application of the generalized discrepancy principle and provide a new regularization algorithm based on the introduction of an incompatibility measure in the discrepancy function. Then Section 4 contains a description of deformable models for edge detection and how they can be applied to linear sampling. Our conclusions are offered in Section 5.

Throughout the paper we use the Dirichlet two-dimensional obstacle problem as a test example. However, all the results discussed in the following hold for more general two-dimensional scattering frameworks. Active surfaces can be applied to three-dimensional volumes provided by linear sampling [26]. Even in this case the use of the new “no-sampling” implementation is in principle advantageous, thanks to its higher computational effectiveness.

2. The linear sampling method. We consider the Dirichlet two-dimensional direct scattering problem of determining the total field $u$:

\[
\begin{align*}
\triangle u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
u &= 0 \quad \text{on } \Gamma, \\
u &= u^i + u^s \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - iku^s \right) &= 0,
\end{align*}
\]  

where $k$ is the wavenumber, $d \in \Omega := \{v \in \mathbb{R}^2 : |v| = 1\}$ is the direction vector of the incident plane wave $u^i(x) = e^{ikx \cdot d}$, limit (2.4) is known as the Sommerfeld radiation condition for the scattered field $u^s$ and $D$ denotes the bounded scatterer with boundary $\Gamma$. The asymptotic behavior of $u^s$ can be written as

\[
u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} u_\infty(\hat{x}, d) + \mathcal{O}\left(|x|^{-3/2}\right) \quad \text{as } |x| \to \infty,
\]

where $u_\infty$ is the so-called far-field pattern, and $\hat{x} = \frac{x}{|x|} \in \Omega$.

In the inverse problem, we are interested in recovering properties of $D$ using the far-field pattern. In this context we first consider the far-field operator $F : L^2(\Omega) \to L^2(\Omega)$ defined as

\[
[Fg(\cdot)](\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d)g(d)ds(d) \quad \forall g(\cdot) \in L^2(\Omega).
\]

Now, for each $z = (z_1, z_2) \in \mathbb{R}^2$, we introduce the far-field equation

\[
[Fg_z(\cdot)](\hat{x}) = \Phi_\infty(\hat{x}, z),
\]

where $\Phi_\infty(\hat{x}, z)$ is the far-field pattern at $z$. The far-field equation (2.6) is of the same type as the direct scattering problem (2.1)–(2.4) and can be solved in an analogous way.
where \( g_i(\cdot) \) is a function in \( L^2(\Omega) \) for each \( z \), while \( \Phi_\infty(\hat{x}, z) \) is the far-field pattern of the fundamental solution \( \Phi(x, z) \) of the Helmholtz equation in \( \mathbb{R}^2 \) and is given by
\[
\Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x} \cdot z}.
\] (2.7)

The general theorem at the basis of the linear sampling method [3, 4] assures that, if \( k^2 \) is not a Dirichlet eigenvalue of the negative laplacian, there exists an approximate solution to (2.6) whose \( L^2 \)-norm blows up to infinity when \( z \) approaches the boundary of the scatterer from inside and stays arbitrarily large outside: so the idea arises to use a suitable monotonic map of such a norm as indicator function of the support of the scatterer. However, this is not an easy task, since \( F \) is compact and therefore solving (2.6)-(2.7) is an ill-posed problem.

In real experiments, the far-field pattern is measured for \( P \) observation angles \( \{\varphi_i\}_{i=0}^{P-1} \) and \( Q \) incidence angles \( \{\theta_j\}_{j=0}^{Q-1} \), i.e. for observation directions \( \hat{x}_i = (\cos \varphi_i, \sin \varphi_i) \) and incidence directions \( d_j = (\cos \theta_j, \sin \theta_j) \). In the following we shall assume \( P = Q = N \), the generalization to rectangular problems being straightforward. Furthermore, for sake of simplicity, we shall take
\[
\varphi_i = \frac{2\pi i}{N}, \quad \theta_j = \frac{2\pi j}{N}, \quad i, j = 0, \ldots, N - 1.
\] (2.8)

These values are placed into the far-field matrix \( F \), whose elements are defined as
\[
F_{ij} := u_\infty(\hat{x}_i, d_j), \quad i, j = 0, \ldots, N - 1.
\] (2.9)

In practical applications the far-field matrix is affected by the measurement noise, and therefore only a noisy version \( F_h \) of the far-field matrix is available, such that
\[
F_h = F + H,
\] (2.10)

where \( H \) is the noise matrix. If, as usual, we denote with \( \| \cdot \| \) the operatorial norm of a linear continuous operator and regard the matrix \( H \) as a linear continuous operator in \( \mathbb{C}^N \) (endowed with the norm \( \| \cdot \|_{\mathbb{C}^N} \) induced by the usual scalar product \( (\cdot, \cdot)_{\mathbb{C}^N} \)), we assume to know that \( \| H \| \leq h \).

Summarizing the implementation of the linear sampling method suggested in [2], we consider a known rectangle \( T \) containing the scatterer, and the Hilbert space \( [L^2(T)]^N := L^2(T) \oplus \cdots \oplus L^2(T) \) equipped with the scalar product
\[
(f(\cdot), g(\cdot))_{2,N} := \sum_{i=0}^{N-1} (f_i(\cdot), g_i(\cdot))_2
\] (2.11)

for all \( f(\cdot) := \{f_i(\cdot)\}_{i=0}^{N-1} \) and \( g(\cdot) := \{g_i(\cdot)\}_{i=0}^{N-1} \) elements of \([L^2(T)]^N\), as well as with the induced norm
\[
\|f(\cdot)\|_{2,N} = \sqrt{\int_T \|f(z)\|_{\mathbb{C}^N}^2 dz}.
\] (2.12)

Then we define the linear operator \( F_h : [L^2(T)]^N \to [L^2(T)]^N \) given by
\[
[F_h g(\cdot)](\cdot) := \left\{ \sum_{j=0}^{N-1} (F_h)_{ij} g_j(\cdot) \right\}_{i=0}^{N-1} \quad \forall g(\cdot) := \{g_j(\cdot)\}_{j=0}^{N-1} \in [L^2(T)]^N,
\] (2.13)
where \((F_h)_{ij}\) are the elements of the noisy far-field matrix. From a practical viewpoint, the introduction of the operator \(F_h\) allows one to express the several algebraic systems (one for each grid point) of the traditional implementation of the linear sampling method as the single functional equation in \([L^2(T)]^N\)

\[
[F_h \mathbf{g}(\cdot)](\cdot) = \frac{N}{2\pi} \Phi_\infty(\cdot), \tag{2.14}
\]

where \(\Phi_\infty(\cdot) := (\Phi_\infty(\bar{x}_0, \cdot), \ldots, \Phi_\infty(\bar{x}_{N-1}, \cdot))\) and again, for any \(j = 0, \ldots, N - 1, \bar{x}_j = (\cos(2\pi j/N), \sin(2\pi j/N))\).

Now, let us denote with \(\{\sigma_p^h, \mathbf{u}_p^h, \mathbf{v}_p^h\}^N_{p=0}\) the singular system of the noisy far-field matrix \(F_h\), where \(\{\mathbf{u}_p^h\}^N_{p=1}\) and \(\{\mathbf{v}_p^h\}^N_{p=1}\) are two orthonormal bases in all \(\mathbb{C}^N\) such that

\[
F_h \mathbf{u}_p^h = \sigma_p^h \mathbf{v}_p^h, \quad F_h^* \mathbf{v}_p^h = \sigma_p^h \mathbf{u}_p^h \quad \forall p = 0, \ldots, N - 1, \tag{2.15}
\]

\(F_h^*\) denotes the adjoint of the matrix \(F_h\) and \(\sigma_0^h \geq \sigma_1^h \geq \sigma_2^h \geq \ldots \geq \sigma_{N-1}^h \geq 0\). Therefore the rank of \(F_h\) is \(r_h \leq N\) if and only if \(\sigma_{r_h+1}^h = \ldots = \sigma_{N-1}^h = 0\). Although \(F_h\) is not compact \([2]\), the representation

\[
[F_h \mathbf{g}(\cdot)](\cdot) = \sum_{p=0}^{N-1} \sigma_p^h (\mathbf{g}(\cdot), \mathbf{u}_p^h)_{\mathbb{C}^N} \mathbf{v}_p^h \quad \forall \mathbf{g}(\cdot) \in [L^2(T)]^N, \tag{2.16}
\]

holds, where \((\mathbf{g}(\cdot), \mathbf{u}_p^h)_{\mathbb{C}^N}\) is the element in \(L^2(T)\) defined as

\[
(\mathbf{g}(\cdot), \mathbf{u}_p^h)_{\mathbb{C}^N} := \sum_{i=0}^{N-1} \bar{g}_i^h u_{p,i},
\]

and the \(u_{p,i}\) are the components of the vector \(\mathbf{u}_p^h\). From relation (2.16) it follows that

\[
\|F_h\| = \sigma_0^h. \tag{2.17}
\]

This property has an important consequence: in fact it implies that \(\|F_h\| = \|F_h\|\) and therefore that the bound \(h\) on the noise affecting the matrix \(F_h\) is the same as the one on the noise affecting the operator \(F_h\).

Since the range of the operator \(F_h\) is closed \([1]\), the generalized inverse operator \(F_h^*\) is continuous, i.e. the problem of determining the generalized solution of the functional equation (2.14) is well-posed. Nevertheless, we know that such an inverse problem is, in general, ill-conditioned and therefore a regularization method is anyway necessary. Hence, the novelty in comparison with the traditional implementation is rather the fact that now the regularization of equation (2.14) occurs in a way which is independent from \(z\) and therefore provides a single value of the regularization parameter. In particular, Tikhonov regularization requires to compute the solution of the minimum problem

\[
\left\| [F_h \mathbf{g}(\cdot)](\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|^2_{2,N} + \alpha \|\mathbf{g}(\cdot)\|^2_{2,N} = \text{minimum}, \tag{2.18}
\]

which turns out to be

\[
\mathbf{g}_\alpha(\cdot) = \frac{N}{2\pi} \sum_{p=0}^{N-1} \frac{\sigma_p^h}{(\sigma_p^h)^2 + \alpha} (\Phi_\infty(\cdot), \mathbf{v}_p^h)_{\mathbb{C}^N} \mathbf{u}_p^h, \tag{2.19}
\]
must be taken into account: however, the same heuristic considerations as before
where \( \alpha \) is a real positive number independent of \( z \). Provided that an optimal value
\( \alpha^* \) for \( \alpha \) is chosen, we can then visualize the scatterer profile by plotting the value
of \( I(\|g_\alpha(z)\|_{CN}) \) for each \( z \in T \), where \( I : \mathbb{R}^+ \cup \{0\} \to \mathbb{R} \) is a suitable monotonic
continuous function; in other terms, the indicator function is defined as:
\[
\Psi_I : T \longrightarrow \mathbb{R}
\]
\[
z \mapsto \Psi_I(z) := I(\|g_\alpha(z)\|_{CN}).
\]  
From an analytical viewpoint, the case \( I(t) = t^2 \) is particularly convenient [2] and
the corresponding indicator function \( \Psi_2(z) = \|g_\alpha(z)\|_{CN}^2 \) will be simply denoted
with \( \Psi(z) \). For future notational purposes, we also observe that any indicator func-
tion \( \Psi_I(z) = I(\|g_\alpha(z)\|_{CN}) \), as defined in (2.20), can be equivalently expressed as
\[ J(\Psi(z)) = J(\|g_\alpha(z)\|_{CN}^2), \]
provided that \( J = I \circ \sqrt{\cdot} \).

3. Regularization. The regularization of problem (2.14) and, in particular, the
choice of a suitable value for the regularization parameter \( \alpha \), is a difficult issue. The
main difficulty is that here the noise is on the model, i.e. on the operator \( F_h \), and
not on the data function at the right hand side of the functional equation. A possible
approach in this case is the application of Morozov’s generalized discrepancy principle
[24]: in our case, we can briefly outline it as follows. The generalized solution \( g^*_I(\cdot) \) of
the noise-free (i.e. for \( h = 0 \)) version of problem (2.14) is defined by the two minimum
conditions:
\[
\left\| F g^*_I(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N} = \inf_{g(\cdot) \in [L^2(T)]^N} \left\| F g(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N}, \quad (3.1)
\]
\[
\| g^*_I(\cdot) \|_{2,N} = \text{minimum}, \quad (3.2)
\]
where \( F \) is the noise-free version of \( F_h \). If the datum belongs to the range of the exact
operator, i.e. if \( \frac{N}{2\pi} \Phi_\infty(\cdot) \in \mathcal{R}(F) \), then condition (3.1) can be replaced by the simpler one:
\[
\left\| F g^*_I(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N} = 0. \quad (3.3)
\]
When dealing with the noisy operator \( F_h \) one should take into account that, for any
\( g(\cdot) \in [L^2(T)]^N \), the quantity \( |F_h g(\cdot)|(\cdot) \in [L^2(T)]^N \) is computed with an error
bounded by \( h \|g(\cdot)\|_{2,N} \); hence, a reasonable request for the Tikhonov regularized
solution \( g_\alpha(\cdot) \) given by (2.19) is that \( g_\alpha(\cdot) \) itself reproduces, when mapped by \( F_h \), the
datum \( \frac{N}{2\pi} \Phi_\infty(\cdot) \) up to the error affecting \( |F_h g(\cdot)|(\cdot) \). These heuristic considerations
lead to replace condition (3.3) with
\[
\left\| F_h g_\alpha(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N} = h \| g_\alpha(\cdot) \|_{2,N}, \quad (3.4)
\]
regarded as an equation in the unknown \( \alpha \). This is equivalent to finding a zero of the
generalized discrepancy function defined as:
\[
\rho_h(\alpha) := \left\| F_h g_\alpha(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N}^2 - h^2 \| g(\cdot) \|_{2,N}^2. \quad (3.5)
\]
On the other hand, if \( \frac{N}{2\pi} \Phi_\infty(\cdot) \notin \mathcal{R}(F) \), then the more general condition (3.1)
must be taken into account: however, the same heuristic considerations as before
suggest that it should be replaced, in the noisy case, by
\[
\left\| F_h g_\alpha(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N} = h \| g_\alpha(\cdot) \|_{2,N} + \inf_{g(\cdot) \in [L^2(T)]^N} \left\{ \left\| F_h g(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N} + h \| g(\cdot) \|_{2,N} \right\},
\]
regarded again as an equation in the unknown \( \alpha \). If we now define the incompatibility measure
\[
\tilde{\mu}_h \left[ \frac{N}{2\pi} \Phi_\infty(\cdot), F_h \right] := \inf_{g(\cdot) \in [L^2(T)]^N} \left\{ \left\| F_h g(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N} + h \| g(\cdot) \|_{2,N} \right\},
\]
equation (3.6) is equivalent to finding a zero of the modified generalized discrepancy function
\[
\hat{\rho}_h(\alpha) := \left\| F_h g_\alpha(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N}^2 - \left( h \| g_\alpha(\cdot) \|_{2,N} + \tilde{\mu}_h \left[ \frac{N}{2\pi} \Phi_\infty(\cdot), F_h \right] \right)^2.
\]
If the generalized solution \( g^1(\cdot) \) of the noise-free version of problem (2.14) is different from zero, if the noise is not too high and \( R(F_h) = [L^2(T)]^N \), then [24] both \( \rho_h(\alpha) \) and \( \hat{\rho}_h(\alpha) \) have the unique zeros \( \alpha^* \) and \( \hat{\alpha}^* \) respectively and it can be proved that the computation of \( \alpha^* \) and \( \hat{\alpha}^* \) provides a regularization algorithm, i.e.
\[
\lim_{h \to 0^+} \sup_{F_h} \{ \| g_\alpha(\cdot) - g^1(\cdot) \|_{2,N} \mid \| F_h - F \| \leq h \} = 0,
\]
\[
\lim_{h \to 0^+} \sup_{F_h} \{ \alpha^* \mid \| F_h - F \| \leq h \} = 0,
\]
and
\[
\lim_{h \to 0^+} \sup_{F_h} \{ \| g_\alpha^*(\cdot) - g^1(\cdot) \|_{2,N} \mid \| F_h - F \| \leq h \} = 0,
\]
\[
\lim_{h \to 0^+} \sup_{F_h} \{ \hat{\alpha}^* \mid \| F_h - F \| \leq h \} = 0.
\]
For a fixed noise level \( h \neq 0 \), the choice of \( \alpha^* \) and \( \hat{\alpha}^* \) defines two different regularization algorithms and deciding which the most effective one is may become a crucial issue to enhance the quality of the visualization maps provided by the linear sampling method. In particular, in the noise-free case, since
\[
\mathcal{N}(F) = \left\{ f(\cdot) \in [L^2(T)]^N \mid f(z) \in \mathcal{N}(F) \text{ f.a.a. } z \in T \right\},
\]
if \( \det F = 0 \), then the compatibility condition \( (N/2\pi) \Phi_\infty(\cdot) \in \mathcal{R}(F) \) may not hold. In our numerical simulations \( | \det F | \), although never equal to zero, is always very small (this is an obvious consequence of the compactness of the far-field operator \( F \)). Therefore, at low noise levels, the choice \( \alpha = \hat{\alpha}^* \) is numerically more appropriate. On the other hand, while \( h \) increases the random noise components make the rows of \( F_h \) more and more linearly independent, thus increasing the absolute value of the determinant. Hence, for large \( h \) values, the choice \( \alpha = \alpha^* \) should be more effective. For
intermediate noise amounts, a \textit{blended regularization}, in which the two values $\alpha^*$ and $\hat{\alpha}^*$ are mixed, is a somewhat natural choice. Following this approach, we introduce a selection criterion for $\alpha$ and prove that it defines a regularization algorithm in the rigorous sense. To this aim we first recall [21, 25] the following theorem.

**Theorem 3.1. (Ostrowski)** Let $A$ and $B$ two $N \times N$ matrices having complex-valued entries $A_{ij}, B_{ij}$ respectively, such that

$$|A_{ij}| < 1, \quad |B_{ij}| < 1 \quad \forall i, j = 0, \ldots, N - 1;$$

moreover, let $\epsilon$ be a real positive number. Then the eigenvalues $\lambda_q$ of $A$ and $\lambda'_p$ of $A + \epsilon B$ can be put in an one-to-one correspondence $q = q(p)$ in such a way that

$$|\lambda_{q(p)} - \lambda'_p| < (N + 2)(N^2 \epsilon)^{\frac{1}{N}} \quad \forall p = 0, \ldots, N - 1.$$  

Then we adapt this theorem to our framework, starting with the following lemma.

**Lemma 3.2.** Let $\sigma_q$ and $\sigma^h_p$ be respectively the $N$ singular values of the two far-field matrices $F$ and $F_h$, defined in (2.9) and (2.10) and such that $\|F_h - F\| \leq h$; moreover, let $\det F \neq 0$, so that $\sigma_q > 0 \forall q = 0, \ldots, N - 1$. Then there exists a one-to-one correspondence $q = q(p)$ between the singular values $\sigma_q$ and $\sigma^h_p$, as well as a real-valued function $f = f(h)$, depending on the noise bound $h$ but not on the particular matrix $F_h$, in such a way that for all $p = 0, \ldots, N - 1$

$$\sigma_{q(p)} - f(h) < \sigma^h_p < \sigma_{q(p)} + f(h),$$

with

$$\lim_{h \to 0^+} f(h) = 0.$$  

**Proof.** First of all, remembering equality (2.10) and the anti-linearity of hermitian conjugation, we easily get:

$$F_h^* F_h = (F^* + H^*) (F + H) = F^* F + F^* H + H^* F + H^* H.$$  

Now, for notational convenience, let us put

$$T_h := F_h^* F_h, \quad T := F^* F, \quad J := F^* H + H^* F + H^* H,$$

so that we can rewrite (3.18) as:

$$T_h = T + J.$$  

Then, denoting with $T_{ij}$ and $J_{ij}$ the elements of the two $N \times N$ matrices $T$ and $J$ respectively, we also define

$$M_T := \max_{i,j} |T_{ij}|, \quad \hat{T} := \frac{1}{2M_T} T,$$

$$M_J := \max_{i,j} |J_{ij}|, \quad \hat{J} := \frac{1}{2M_J} J,$$
so that
\[ |\hat{T}_{ij}| \leq \frac{1}{2} \quad \forall i,j = 0, \ldots, N-1, \quad (3.23) \]
\[ |\hat{J}_{ij}| \leq \frac{1}{2} \quad \forall i,j = 0, \ldots, N-1. \quad (3.24) \]
From (3.20), (3.21) and (3.22) we get
\[ T_h = T + J = 2M_T \hat{T} + 2M_J \hat{J} = 2M_T \left( \hat{T} + \frac{M_J}{M_T} \hat{J} \right). \quad (3.25) \]
Now, thanks to (3.23) and (3.24), the matrices \( \hat{T} \) and \( \hat{J} \) clearly satisfy the hypotheses of Theorem 3.1, where the roles of \( A, B \) and \( \epsilon \) are respectively played by \( \hat{T}, \hat{J} \) and \( \frac{M_J}{M_T} \).
Therefore there exists a one-to-one correspondence \( q = q(p) \) between the eigenvalues \( \lambda_q \) of \( \hat{T} \) and the eigenvalues \( \lambda'_p \) of \( \left( \hat{T} + \frac{M_J}{M_T} \hat{J} \right) \) such that
\[ |\lambda_{q(p)} - \lambda'_p| < (N + 2) \left( N^2 \frac{M_J}{M_T} \right)^\frac{1}{N} \quad \forall p = 0, \ldots, N-1. \quad (3.26) \]
Moreover, from the definition of \( \hat{T} \) in (3.21) and from (3.25), it follows that the eigenvalues \( \lambda_q, \lambda'_p \) of \( \hat{T} \), \( \left( \hat{T} + \frac{M_J}{M_T} \hat{J} \right) \) are respectively
\[ \lambda_q = \frac{1}{2M_T} (\sigma_q)^2, \quad \lambda'_p = \frac{1}{2M_T} (\sigma'_p)^2. \quad (3.27) \]
Then, replacing (3.27) in (3.26) we immediately find:
\[ \frac{1}{2M_T} |(\sigma_{q(p)})^2 - (\sigma'_p)^2| < (N + 2) \left( N^2 \frac{M_J}{M_T} \right)^\frac{1}{N} \quad \forall p = 0, \ldots, N-1. \quad (3.28) \]
On the other hand, for all \( p = 0, \ldots, N-1 \),
\[ |(\sigma_{q(p)})^2 - (\sigma'_p)^2| = |\sigma_{q(p)} + \sigma'_p| \cdot |\sigma_{q(p)} - \sigma'_p| \geq \sigma_{N-1} |\sigma_{q(p)} - \sigma'_p| \quad (3.29) \]
holds, where \( \sigma_{N-1} > 0 \) is the minimum singular value of \( \mathbf{F} \); then, taking into account both inequalities (3.28) and (3.29), we find that for all \( p = 0, \ldots, N-1 \)
\[ |\sigma_{q(p)} - \sigma'_p| < 2M_T \frac{(N + 2)}{\sigma_{N-1}} \left( N^2 \frac{M_J}{M_T} \right)^\frac{1}{N} \quad (3.30) \]
holds, or, equivalently,
\[ \sigma_{q(p)} - 2M_T \frac{(N + 2)}{\sigma_{N-1}} \left( N^2 \frac{M_J}{M_T} \right)^\frac{1}{N} < \sigma'_p < \sigma_{q(p)} + 2M_T \frac{(N + 2)}{\sigma_{N-1}} \left( N^2 \frac{M_J}{M_T} \right)^\frac{1}{N}. \quad (3.31) \]
Finally, remembering the definition of the matrix \( \mathbf{J} \) in (3.19) and of the number \( M_J \) in (3.22), as well as the noise bound \( \|\mathbf{H}\| \leq h \), one can easily get the following chain of inequalities:
\[ M_J = \max_{i,j} |J_{ij}| \leq \sup_{\|\mathbf{x}\|_{\mathbb{C}N} = 1} \|\mathbf{J}\mathbf{x}\|_{\mathbb{C}N} = \|\mathbf{J}\| \leq 2 \|\mathbf{F}\| h + h^2. \quad (3.32) \]
Hence, replacing $M_J$ in (3.31) with the upper bound given by (3.32) and defining

$$f(h) := 2M_T \frac{(N + 2)}{\sigma_{N - 1}} \left[ N^2 \frac{2\|F\| h + h^2}{M_T} \right]^{\frac{1}{2}},$$

we immediately get (3.16) and (3.17).

We finally observe that the first one of the two inequalities (3.16) is, in general, not trivial: indeed, the singular value $\sigma_h^p$ is non-negative, while, for $h$ small enough, the left-hand side is even positive, owing to the fact that $\sigma_q > 0 \ \forall q = 0, \ldots, N - 1$ and limit (3.17) holds.

Finally, we are ready to state our theorem about the blended regularization.

**Theorem 3.3.** Let $\alpha^*$ and $\hat{\alpha}^*$ be the zeros of (3.5) and (3.8) respectively. If $\text{det } F \neq 0$, then taking

$$\alpha^*_\beta := \beta \hat{\alpha}^* + (1 - \beta) \alpha^*,$$

with $\beta$ a real number in $[0, 1]$, defines a regularization algorithm, i.e.

$$\lim_{h \to 0^+} \sup_{F_h} \left\{ \|G_{\alpha^*_\beta}(\cdot) - g^1(\cdot)\|_{2,N} | \|F_h - F\| \leq h \right\} = 0, \quad (3.35)$$

$$\lim_{h \to 0^+} \sup_{F_h} \left\{ \alpha^*_\beta | \|F_h - F\| \leq h \right\} = 0. \quad (3.36)$$

**Proof.** Relation (3.36) is a straightforward consequence of the analogous limits (3.10), (3.12) and of definition (3.34).

As regards relation (3.35), we first observe that, by virtue of the triangle inequality,

$$\|G_{\alpha^*_\beta}(\cdot) - g^1(\cdot)\|_{2,N} \leq \|G_{\alpha^*_\beta}(\cdot) - g_{\alpha^*}(\cdot)\|_{2,N} + \|g_{\alpha^*}(\cdot) - g^1(\cdot)\|_{2,N} \quad (3.37)$$

holds. We already know that the second addendum at the right-hand side of inequality (3.37) satisfies relation (3.9), so it suffices to prove an analogous relation for the first addendum. Remembering representation (2.19), equality (2.12) and the orthonormality of the singular vectors $u^h_p$, we easily find that

$$\|G_{\alpha^*_\beta}(\cdot) - g_{\alpha^*}(\cdot)\|^2_{2,N} =$$

$$\frac{N}{2\pi} \sum_{p=0}^{N-1} \left[ \frac{\sigma^h_p}{(\sigma^h_p)^2 + \alpha^*_\beta} - \frac{\sigma^h_p}{(\sigma^h_p)^2 + \alpha^*} \right] \langle \Phi_\infty(\cdot), v^h_p \rangle_{CN} u^h_p \|^2_{2,N} =$$

$$= \frac{N^2}{4\pi^2} \sum_{p=0}^{N-1} \frac{\left[ \sigma^h_p (\alpha^* - \alpha^*_\beta) \right]^2}{(\sigma^h_p)^2 + \alpha^*_\beta} \left[ (\sigma^h_p)^2 + \alpha^* \right]^2 \int_T \left| \langle \Phi_\infty(z), v^h_p \rangle_{CN} \right|^2 dz. \quad (3.38)$$

The Cauchy-Schwarz inequality implies that

$$\int_T \left| \langle \Phi_\infty(z), v^h_p \rangle_{CN} \right|^2 dz \leq \int_T \|\Phi_\infty(z)\|_{CN}^2 dz. \quad (3.39)$$
On the other hand, it is not restrictive to assume that \( h \) is small enough to make the left-hand side of inequalities (3.16) positive for each \( p = 0, \ldots, N - 1 \); hence, applying the previous lemma 3.2 and remembering that \( \sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_{N-1} > 0 \), for each \( p = 0, \ldots, N - 1 \) we can bound the fractionary factor in (3.38) as follows:

\[
\frac{\left[ \sigma_p^h (\alpha^* - \alpha_p^*) \right]^2}{(\sigma_p^h + \alpha_p^*)^2 (\sigma_p^h + \alpha)^2} \leq \frac{\left[ \sigma_p (f(h)) (\alpha^* - \alpha_p^*) \right]^2}{[\sigma_p (f(h))]^8} \leq \frac{\left[ \sigma_0 + f(h) (\alpha^* - \alpha_p^*) \right]^2}{[\sigma_{N-1} - f(h)]^8}.
\]

(3.40)

Hence, taking into account relations (3.38), (3.39) and (3.40), we easily find:

\[
\left\| g_{\alpha_p^*} (\cdot) - g_{\alpha^*} (\cdot) \right\|_{2,N}^2 \leq \frac{N^3 \left\{ [\sigma_0 + f(h)] (\alpha^* - \alpha_p^*) \right\}^2}{4\pi^2 [\sigma_{N-1} - f(h)]^8} \int_T \| \Phi_\infty (z) \|_{\mathcal{C}^{\infty}}^2 dz.
\]

(3.41)

At the right-hand side of inequality (3.41), the only term depending on the particular noisy far-field matrix \( F_h \) (compatible with the noise bound \( \| F_h - F \| \leq h \)) is \((\alpha^* - \alpha_p^*)\), then we have:

\[
\sup_{F_h} \left\| g_{\alpha_p^*} (\cdot) - g_{\alpha^*} (\cdot) \right\|_{2,N}^2 \leq \frac{N^3 \left\{ [\sigma_0 + f(h)] (\alpha^* - \alpha_p^*) \right\}^2}{4\pi^2 [\sigma_{N-1} - f(h)]^8} \int_T \| \Phi_\infty (z) \|_{\mathcal{C}^{\infty}}^2 dz \sup_{F_h} (\alpha^* - \alpha_p^*)^2.
\]

(3.42)

If we now remember limits (3.10), (3.17) and (3.36), we easily get (3.35) from the previous bound (3.42).

**Remark 3.4.** The purpose of Theorem 3.3 is to show that blending the two regularization parameters \( \alpha^* \) and \( \hat{\alpha}^* \) provides a regularization algorithm. However, the theorem also has a computational implication since it entails that there exists a whole confidence range where the optimal regularization parameter can be chosen. The theorem does not provide a criterion to fix the blending parameter \( \beta \), but we point out that choosing \( \beta \) can be considered as a sort of refinement of Morozov’s generalized discrepancy principle (see [24] or, for a more elementary proof in less general conditions, Theorem 5.1 in [19]).

**3.1. Numerical experiments.** In order to illustrate both the new generalized discrepancy function (3.8) and the blended regularization (3.34), we consider the following experiments with a conducting kite, whose profile is shown in Figure 3.1 and given by the parametric equations

\[
x(t) = a_1 \sin(t), \quad y(t) = \cos(t) + a_2 [\cos(2t) - 1], \quad t \in [0, 2\pi],
\]

(3.43)

with \( a_1 = 1.5 \) and \( a_2 = 0.65 \). We employ incident plane waves with wavenumber \( k = 1 \) coming from \( N = 15 \) incidence directions uniformly distributed over \([0, 2\pi]\). The far-field pattern is measured at the same 15 angles and 1% Gaussian noise (corresponding to a value of \( h = 3.7 \cdot 10^{-2} \)) is added at each entry of the far-field matrix. Panel (a) of Figure 3.2 contains the three-dimensional plot of the indicator function \( J(\Psi(z)) \)
provided by the linear sampling method in its “no-sampling” implementation, when
the function $J$ is chosen as minus the logarithm function normalized between zero
and one (but other choices are possible). We assume $T = [-3,3] \times [-3,3]$, and the
regularization parameter is fixed as $\alpha^* = 8.8 \cdot 10^{-4}$, i.e. as the zero of the traditional
form (3.5) of the generalized discrepancy function; panel (b) represents the axial view
of the three-dimensional plot shown in panel (a). Panels (c) and (d) of Figure 3.2
are analogous to the previous ones, but this time the regularization parameter is
chosen as $\hat{\alpha}^* = 4.3 \cdot 10^{-2}$, i.e. as the zero of the modified generalized discrepancy
function (3.8): a comparison of Figure 3.1 with panels (b) and (d) of Figure 3.2
shows that in the latter case the kite is better highlighted. In agreement with our
previous heuristic considerations, this “low-noise” experimental condition favors
the application of the incompatibility measure: indeed, in such a case the value
of the determinant of the far-field matrix $F_h$ is small enough to create a numerically
not negligible incompatibility between the data vector $\mathbf{\Phi}_\infty(\cdot)$ and the range of $F_h$.

Then, in Figure 3.3 we consider the same numerical scattering experiment, but
this time the added noise is increased to 3% Gaussian (corresponding to a value of
$h = 1.0 \cdot 10^{-1}$). For panels (a) and (b) the traditional form (3.5) of the generalized
discrepancy function is used to fix $\alpha$ as $\alpha^* = 5.6 \cdot 10^{-3}$, while for panels (c) and (d) the
incompatibility measure is introduced, i.e. the new form (3.8) is adopted to fix $\alpha$ as
$\hat{\alpha}^* = 6.1 \cdot 10^{-3}$; finally, panels (e) and (f) show the result obtained by blending the two
values $\alpha^*$, $\hat{\alpha}^*$ of the regularization parameter and using $\beta = 0.05$ as blending factor
(see definition (3.34)), i.e. $\alpha_{0.05}^* = 3.6 \cdot 10^{-2}$. In these intermediate noise conditions,
blending the two values $\alpha^*$ and $\hat{\alpha}^*$ turns out to be more effective than using only one of
the two, as shown by a comparison of Figure 3.1 with panels (b), (d) and (f) of
Figure 3.3.

**Remark 3.5.** Our approach to regularization for the linear sampling method
represents a refinement of the use of the classical discrepancy principle. To better
explain this, we first point out that the “no-sampling” approach to linear sampling
requires to optimally fix just one value of the regularization parameter (instead of a set
of values parametrized by the sampling points) and that the numerical experiments
illustrated in Figure 3.2 and Figure 3.3 show that the classical form of the generalized
discrepancy is not the best way of fixing such parameter. On the other hand Theorem
3.3 has pointed out a range of values of the regularization parameter where the optimal
value should be picked up. Now the problem is to fix this value in the range, i.e. to
choose an optimal value for $\beta$. In our approach we have addressed this issue in a
computational fashion using a look-up table technique. We computed the far-field
pattern corresponding to many theoretical objects characterized by different linear
dimensions $a$ and many values of the wavenumber $k$. Furthermore we added noise
of many different intensities $h$ to the far-field patterns. Then we constructed a look-
up table associating the best value of $\beta$ (obtained by comparison with the known
theoretical object) to each pair $(ka, h)$. Therefore, given an unknown object whose
linear dimension can be approximately estimated (for example, by means of a first
run of the linear sampling code with a value of the regularization parameter given
by the classical discrepancy principle) and for known values of $k$ and $h$, we apply
the value of $\beta$ in the table corresponding to that specific pair $(ka, h)$. We finally
remark that using the “no-sampling” implementation these numerical experiments
are performed in an extremely short time. For example, the visualization in panel
(d) of Figure 3.2 is obtained after a CPU time of around 1s, while under the same
computational conditions, a traditionally implemented linear sampling code would
employ around 60s of CPU time in the case of a grid formed by 60x60 equidistant points (both experiments have been realized on a PC with 1.73 GHz clock).}

4. Deformable models. Deformable models provide an effective tool for detecting edges in an image. In the present application we assume that the visualized scatterer is a connected object. From a geometrical viewpoint, a deformable contour is a curve (although in the applications we have in mind the curve is closed, the general approach we are sketching does not require this hypothesis) \( \gamma^0 : [0, 1] \to \mathbb{R}^2 \), while a deformable model is a couple formed by a space \( A_d \) of admissible deformations of \( \gamma^0 \) and a functional \( E : A_d \to \mathbb{R} \) to be minimized. This functional represents the energy of the model and has the following form:

\[
E(\gamma) := \int_0^1 \left[ \frac{1}{2} \left( w_1(s) \| \gamma'(s) \|^2_{\mathbb{R}^2} + w_2(s) \| \gamma''(s) \|^2_{\mathbb{R}^2} \right) + E_{\text{ext}}(\gamma(s)) \right] ds, \quad (4.1)
\]

where

1. the maps \( w_1 : [0, 1] \to \mathbb{R} \) and \( w_2 : [0, 1] \to \mathbb{R} \) are weight functions that respectively control the importance of the first-order and second-order terms imposing the regularity of the curve; their choice determines the mechanical properties or, more precisely, the internal forces, i.e. elasticity and rigidity respectively, of the model;

2. \( E_{\text{ext}} \) denotes the potential energy associated to the external forces deriving from the image map and pushing the curve to the significant lines which correspond to the desired attributes (i.e., in our case, edges); in our LSM-oriented application, we choose \([7, 8, 18]\)

\[
E_{\text{ext}} := -\| \nabla J(\Psi) \|^2_{\mathbb{R}^2}, \quad (4.2)
\]

where \( J(\Psi) \) is the generic indicator function.

The functions \( w_1, w_2, \gamma \) and \( E_{\text{ext}} \) are assumed to be smooth enough for computational purposes. Moreover, we shall restrict the space \( A_d \) of admissible deformations by assigning the boundary conditions \( \gamma(0), \gamma'(0), \gamma(1) \) and \( \gamma'(1) \); we can also use periodic curves or, in general, other kinds of boundary conditions.

A necessary condition for \( \gamma \) to be a local minimum for the functional \( E(\gamma) \) is that [15] it satisfies the Euler (vectorial) equation:

\[
(w_1(s) \gamma'(s))' - (w_2(s) \gamma''(s))'' - \nabla E_{\text{ext}}(\gamma(s)) = 0 \quad (4.3)
\]
Fig. 3.2. Reconstruction of the conducting kite in Figure 3.1 in the case \( k = 1, N = 15 \) and 1% Gaussian noise affecting the entries of the far-field matrix; (a) and (b): three-dimensional and axial view of the indicator function when \( \alpha \) is chosen as the zero of (3.5); (c) and (d): three-dimensional and axial view of the indicator function when \( \alpha \) is chosen as the zero of (3.8).

with given boundary conditions \( \gamma(0), \gamma'(0), \gamma(1) \) and \( \gamma'(1) \). In order to determine the solution to (4.3), we replace it with the evolution problem

\[
\begin{align*}
\frac{\partial}{\partial t} \gamma(s, t) &= (w_1(s) \gamma'(s, t))' - (w_2(s) \gamma''(s, t))'' - G(\gamma(s, t)), \quad (a) \\
\gamma(s, 0) &= \gamma^0(s), \quad (b) \\
\gamma(0, t) &= \gamma^0(0), \quad \gamma(1, t) = \gamma^0(1), \quad (c) \\
\gamma'(0, t) &= (\gamma^0)'(0), \quad \gamma'(1, t) = (\gamma^0)'(0), \quad (d)
\end{align*}
\]

where the boundary or initial conditions (4.4)(b)-(d) impose that \( \gamma \) is “close” enough to the initial guess \( \gamma^0 \). The term \( G \) in (4.4)(a) is a modification of \( \nabla E_{\text{ext}} \), which is introduced to reduce the instabilities arising with the discretization. In our application we assume

\[
G(\gamma(s)) := -\kappa \frac{\nabla E_{\text{ext}}(\gamma(s))}{\sup_{\gamma} \|\nabla E_{\text{ext}}(\gamma(s))\|_{R^2}},
\]

for an appropriate \( \kappa \in \mathbb{R}^+ \), but other choices are possible [8]. A numerical solution to (4.3) can be found by discretizing, in both \( s \) and \( t \), equation (4.4)(a) and solving
iteratively (in $t$) the discretized system until a stable solution is found [18]. We discretize the time domain with the time points $t_j = j\Delta t$, for $j = 0, 1, \ldots$, and the space domain with $M$ equidistant knots $s_i = i\Delta s$, for $i = 0, \ldots, M - 1$ and a distance $\Delta s = 1/M$. Then, if we put $\gamma_i^j := \gamma(s_i, t_j)$, $w_{1i} := w_1(s_i)$, $w_{2i} := w_2(s_i)$, we can write the finite difference approximation of the equation (4.4)(a) as:

$$
\frac{\gamma_i^j - \gamma_i^{j-1}}{\Delta t} = \frac{1}{\Delta s^2}[w_{1i+1}(\gamma_i^{j+1} - \gamma_i^{j}) - w_{1i}(\gamma_i^{j} - \gamma_i^{j-1})] + \\
- \frac{1}{\Delta s^2}[w_{2i-1}(\gamma_i^{j-2} - 2\gamma_i^{j-1} + \gamma_i^{j}) - 2w_{2i}(\gamma_i^{j} - \gamma_i^{j-1} + \gamma_i^{j+1}) + \\
w_{2i+1}(\gamma_i^{j} - 2\gamma_i^{j-1} + \gamma_i^{j+2})] + G(\gamma_i^{j}).
$$

With the approximation

$$
G(\gamma_i^{j}) \simeq G(\gamma_i^{j-1}), \tag{4.7}
$$

equation (4.6) has solution

$$
\gamma_i^j = (I - A)^{-1}(\gamma_i^{j-1} + \Delta t G(\gamma_i^{j-1})), \tag{4.8}
$$

where $\gamma_i^j := (\gamma_i^{0}, \ldots, \gamma_i^{M-1})$ and $I - A$ is a diagonally dominant, banded and sparse matrix.

Numerical analysis considerations show that reliable results are obtained when $w_1$ and $w_2$ are constants and of the order of $(\Delta s)^2$ and $(\Delta s)^4$ respectively, while $\kappa\Delta t$ is smaller than $\Delta s$ [8]. A computational recipe for choosing $w_1$, $w_2$ and $\kappa$ in an automatic way for all numerical tests is the following:

$$
w_1 = b_1(\Delta s)^2, \tag{4.9}
$$

$$
w_2 = b_2(\Delta s)^4, \tag{4.10}
$$

$$
\kappa = c\frac{\Delta s}{\Delta t}, \tag{4.11}
$$

where $b_1$, $b_2$ and $c$ are arbitrarily chosen in the intervals:

$$
b_1, b_2 \in (1, 10), \quad c \in (0, 1). \tag{4.12}
$$

Finally, we point out that in the “no-sampling” implementation of the linear sampling method the function $G : T \rightarrow \mathbb{R}$ can be analytically determined. In fact

$$
\nabla E_{ext} = \nabla \left[-(J'(\Psi))^2 \left(\Psi_{z_1}^2 + \Psi_{z_2}^2\right)\right] = (f_1, f_2), \tag{4.13}
$$

with, for any $z = (z_1, z_2) \in \mathbb{R}^2$ and for any $m, n \in \{1, 2\}$,

$$
f_m := -2J'(\Psi) \left[J''(\Psi)\Psi_{zm}(\Psi_{z_1}^2 + \Psi_{z_2}^2) + J'(\Psi)(\Psi_{z_1}\Psi_{z_1zm} + \Psi_{z_2}\Psi_{z_2zm})\right], \tag{4.14}
$$

where an usual notation for partial derivatives is adopted, i.e.

$$
\Psi_{zm} := \frac{\partial \Psi}{\partial z_m}, \quad \Psi_{zmzm} := \frac{\partial^2 \Psi}{\partial z_m \partial z_n}. \tag{4.15}
$$
By using polar coordinates \((r_{hp,j}^h, \epsilon_{hp,j}^h)\) for the generic \(j\)-th component of each singular vector \(v_{hp}^h\), it is possible to write the explicit form of \(\Psi(z)\) (for any \(\alpha > 0\)) as

\[
\Psi(z_1, z_2) = \sum_{p,i,j=0}^{N-1} R_{p,ij}^h \cos \left( \hat{\omega}_{1,ij} z_1 + \hat{\omega}_{2,ij} z_2 + \Delta \epsilon_{p,ij}^h \right),
\]

where

\[
R_{p,ij}^h := \frac{N^2}{32\pi^3 k^2} \cdot \frac{(\sigma_p^h)^2}{[(\sigma_p^h)^2 + \alpha]^2} \cdot \rho_{p,i}^h \rho_{p,j}^h
\]

and

\[
\hat{\omega}_{1,ij} := k \left[ \cos \left( \frac{2\pi i}{N} \right) - \cos \left( \frac{2\pi j}{N} \right) \right],
\hat{\omega}_{2,ij} := k \left[ \sin \left( \frac{2\pi i}{N} \right) - \sin \left( \frac{2\pi j}{N} \right) \right],
\Delta \epsilon_{p,ij}^h := \epsilon_{p,i}^h - \epsilon_{p,j}^h.
\]

Therefore the partial derivatives in (4.15) can be analytically computed:

\[
\Psi_{zm}(z_1, z_2) = - \sum_{p,i,j=0}^{N-1} R_{p,ij}^h \hat{\omega}_{m,ij} \sin \left( \hat{\omega}_{1,ij} z_1 + \hat{\omega}_{2,ij} z_2 + \Delta \epsilon_{p,ij}^h \right),
\]

\[
\Psi_{zn}(z_1, z_2) = - \sum_{p,i,j=0}^{N-1} R_{p,ij}^h \hat{\omega}_{m,ij} \hat{\omega}_{n,ij} \cos \left( \hat{\omega}_{1,ij} z_1 + \hat{\omega}_{2,ij} z_2 + \Delta \epsilon_{p,ij}^h \right).
\]

Of course, relation (4.14) can be made much simpler if \(J\) is chosen as the identity in \(\mathbb{R}^+ \cup \{0\}\): this clearly corresponds to taking \(\Psi(z)\) as indicator function. In such a case, relations (4.14) becomes:

\[
f_m := -2 \left( \Psi_{z_1} \Psi_{zm} + \Psi_{z_2} \Psi_{zm} \right). \quad (4.17)
\]

In general, the active contour technique is rather robust with respect to the choice of \(J\), insofar as this choice produces reliable visualization maps.

### 4.1. Numerical applications.

Our aim is now to test the performances of the active contour technique outlined just above both on simulated and real data. For all simulations, we have chosen as initial guess a circle of radius 2, discretized in \(M = 40\) equidistant knots, while the parameters \(b_1\), \(b_2\) and \(c\) have been fixed as \(b_1 = 9\), \(b_2 = 3\) and \(c = 0.2\), consistently with (4.12); finally, the time-step has been selected as \(\Delta t = 2.0 \cdot 10^{-4}\). As a first example, in Figure 4.1, panels (a) and (b), we consider the same scattering framework as in Figure 3.3, i.e.: the scatterer is the concave kite (3.43) with \(a_1 = 1.5\) and \(a_2 = 0.65\); the wavenumber is \(k = 1\) with \(N = 15\) incidence/observation angles and 3\% Gaussian noise contaminating the entries of \(F\); the regularization parameter is chosen according to Theorem 3.3 and is \(\alpha_0^* = 3.6 \cdot 10^{-2}\). In the same figure, panels (c) and (d) show what happens when the wavenumber is increased up to \(k = 5\) and, correspondingly [9], \(N = 45\). The noise is 10\% Gaussian and \(\alpha\) is chosen as the zero of (3.5), i.e. \(\alpha^* = 1.2 \cdot 10^{-2}\). Then we perform an analogous experiment with the convex kite in Figure 4.2, described again by equation (3.43) with, this time, \(a_1 = 1.5\) and \(a_2 = 0.2\). The scattering parameters
are again $k = 5$, $N = 45$, the noise is 10% Gaussian and the regularization parameter is fixed as the zero of (3.5), i.e. $\alpha^* = 1.1 \cdot 10^{-2}$. The performance of linear sampling post-processed with active contours is given in Figure 4.3.

These figures show that, at small $k$, deformable models allow the reconstruction of the scatterer profile very reliably. We also point out that a wavenumber $k = 5$ involves frequencies which are high enough to affect the visualization maps provided by the linear sampling method with artefacts both inside and outside the scatterer (in the second case, mostly near the external concavities: see [22]). However, the inner artefacts produce no effect on the efficiency of the deformable model, since this is initialized with a contour running outside the visualized scatterer. On the other hand, our choice (4.5) for the form of the external force allows the active contours to overtake the ringing effects outside the object and finally to satisfactorily model the scatterer profile, or at least its convex hull.

We also applied our procedure to the case of a real data set. The numerical result in this paper is run using the Ipswich data provided by Electromagnetics Technology Division, AFRL/SNH, Hanscom AFB: they are single frequency scattered field data measured using a bistatic system; multiple views, corresponding to incidence angles, are obtained by rotating the target on a separate azimuthal positioner in fixed increments. A detailed discussion of the data and of the measurement process can be found in [20]. Here we only point out that using Ipswich real data entails two problems which do not occur in numerically simulated scattering experiments:

1. the measured far-field matrix $F_h$ is not at all complete, i.e. several entries are missing: indeed, it is not possible to measure far-field data at or even near the backscattering direction, since the receiver and transmitter cannot be physically coincident; moreover, independently of this restriction, not all the feasible measurements are actually made. As a result, given an incident angle, the Ipswich data files do not have a full view of the target;
2. the amount of noise affecting the matrix $F_h$ is a priori unknown, since the entries of the latter are given without any estimate of the error affecting them.

Both the previous problems can be rather satisfactorily faced by using the reciprocity relation of the noise-free far-field pattern [12]:

$$u_\infty(\theta_i, \phi_o) = u_\infty(\phi_o + \pi, \theta_i + \pi), \quad (4.18)$$

where $\theta_i$ is the incident angle and $\phi_o$ is the observation angle. It is easy to prove [16] that relation (4.18) implies, for an even and equal number of incidence and observation angles, the following block structure inside the noise-free far-field matrix:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad (4.19)$$

where $F_{12}$ and $F_{21}$ are symmetric and $F_{11} = F_{22}^T$. Hence, in order to complete the far-field matrix, also in the noisy case, we can use this property. The remaining unknown entries, which correspond to the backscattering measurements, can be filled in by averaging, in the same row, the entries that are in the column before and after the unknown entry.

The reciprocity relation (4.18) and the consequent structure of the far-field matrix can also be used to estimate the level of noise on the far-field measurements. To this end, for any square matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$
with an even number of rows (or columns), we define
\[
\tilde{M} = \begin{bmatrix} M_{22}^T & M_{12}^T \\ M_{21}^T & M_{11}^T \end{bmatrix};
\]
it is straightforward to show that
\[
\|M\| = \|\tilde{M}\|.
\] (4.20)

Now, in absence of noise, the block structure (4.19) of the far-field matrix \(F\) is clearly such that
\[
F = \tilde{F};
\] (4.21)
hence, when there is noise, in order to estimate \(h\), one might consider a value that depends on
\[
v = \|Fh - \tilde{F}h\|.
\] It is simple to observe that, by virtue of equalities (2.10), (4.20) and (4.21), we have
\[
v = \|H - \tilde{H}\| \leq \|H\| + \|\tilde{H}\| = 2\|H\| \leq 2h,
\]
and then
\[
h \geq v/2.
\] (4.22)

Of course, this procedure makes sense only if \(v \neq 0\), i.e. if there exists at least one pair of scattering measurements which, on the one hand, are (slightly) different owing to the presence of noise, but, on the other hand, should be equal according to the reciprocity relation.

Taking into account the previous results, we have applied the linear sampling method for the reconstruction of the aluminum triangle Ips009: the scattering data are obtained for a frequency of 10GHz (i.e. \(k \simeq 209\)). Choosing the metre as unit of measure, the scatterer is contained in a rectangle \(T = [-0.1, 0.1] \times [-0.1, 0.1]\). The value \(h = v/2 = 4.4 \cdot 10^{-3}\) has been used in (3.5) for the optimal choice of the regularization parameter as \(\alpha^* = 2.1 \cdot 10^{-5}\). Then, active contours have been applied in order to determine the optimal profile: we have chosen an initialization circle with radius 0.07, discretized in \(M = 40\) equidistant knots, and a time-step of \(\Delta t = 2.0 \cdot 10^{-4}\), while the parameters \(b_1, b_2\) and \(c\) have been fixed as \(b_1 = 9, b_2 = 3\) and \(c = 0.03\), consistently with (4.12). The result shown in panel (a) of Figure 4.4 is obtained by overplotting the initial guess (dotted line) and the final post-processed profile (solid line) on the axial view of the visualization map provided by the indicator function \(J(\Psi(z))\); for sake of clarity, in panel (b) we plot the reconstructed profile alone: it is worthwhile noticing that, despite the presence of several external artefacts, the resulting profile is not very different from a geometrical triangle.

**Remark 4.1.** In general, when the visualization map offered by the linear sampling method is characterized by internal and external artefacts, like in the reconstruction of Figure 4.4, panel (a), any plane section of the plot cannot provide a reliable approximation of the true boundary of the scatterer. However, at most it might happen that the indicator function, when evaluated along the profile selected by the edge detection technique, is actually constant and in this sense we could regard this profile as a level curve. □
Finally we point out that the approach based on active contours is more robust than thresholding approaches based on some heuristic recipe for the selection of particular level curves of the indicator function: in fact, active contours require the selection of three parameters, but in relations (4.12) the ranges of values for which the method converges are provided. Numerical simulations show that active contours have the same effectiveness for many different values of the parameters in the ranges, so that in this framework the question of how to fix the value of $b_1$, $b_2$ and $c$ in their respective intervals is almost irrelevant, at least in absence of notable artefacts. On the contrary, heuristic criteria for choosing the best level curve are very sensitive to the threshold value.

5. Conclusions. This paper proposes a post-processing procedure based on active contours for detecting the optimal scatterer profile in the visualization map given by the linear sampling method. In particular we discussed the fact that active contours are effective in this application only if the regularization procedure associated with linear sampling is optimized. This aim is accomplished here by introducing a blended regularization exploiting an incompatibility measure in the generalized discrepancy function. We believe that these results increase the possibility for the linear sampling method to be applied in the case of experimental frameworks. However, this or analogous post-processing techniques should be first certainly generalized to the case of not-connected scatterers with non-uniform background and possibly extended to three-dimensional anisotropic frameworks.

REFERENCES


Fig. 3.3. Reconstruction of the conducting kite in Figure 3.1 in the case $k = 1$, $N = 15$ and 3% Gaussian noise affecting the entries of the far-field matrix; (a) and (b): three-dimensional and axial view of the indicator function when $\alpha$ is chosen as the zero of (3.5); (c) and (d): three-dimensional and axial view of the indicator function when $\alpha$ is chosen as the zero of (3.8); (e) and (f): three-dimensional and axial view of the indicator function when $\alpha$ is chosen according to Theorem 3.3.
Fig. 4.1. Application of active contours to the visualization map provided by the linear sampling method in the reconstruction of the kite profile in Figure 3.1: (a) axial view with superimposed the initial guess and some iterations of the deformable model algorithm in the case $k = 1$, $N = 15$; (b) comparison between the original profile (dotted) and the one provided by the last iteration (solid) in the previous panel; (c) axial view with superimposed the initial guess and some iterations of the deformable model algorithm in the case $k = 5$, $N = 45$; (d) comparison between the original profile (dotted) and the one provided by the last iteration (solid) in the previous panel.

Fig. 4.2. True profile of the convex conducting kite (cf. (3.43) with $a_1 = 1.5$ and $a_2 = 0.2$) used to test the performances of the active contours.
Fig. 4.3. Application of active contours to the visualization map provided by the linear sampling method in the reconstruction of the kite profile in Figure 4.2: (a) axial view with superimposed the initial guess and some iterations of the deformable model algorithm in the case $k = 5$, $N = 45$, (b) comparison between the original profile (dotted) and the one provided by the last iteration (solid) in the previous panel.

Fig. 4.4. Panel (a): application of active contours to the visualization map provided by the linear sampling method in the case of a real scattering data set (Ipswich data, object Ips009); the initial guess is a circle of radius 0.07. Panel (b): the reconstructed profile.