The linear sampling method without sampling

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Abstract. We present a new implementation of the linear sampling method in which the set of discretized far-field equations for all sampling points is replaced by a single functional equation formulated in a Hilbert space defined as a direct sum of $L^2$ spaces. The squared norm of the regularized solution of such equation is used as indicator function and is analytically determined together with its Fourier transform. This provides some theoretical hints about the spatial resolution achievable by the method.

1. Introduction

The linear sampling method [7] provides an approach for the visualization of scatterers from measurements of the far-field pattern under fixed-frequency scattering conditions. The mathematical basis of this method [4] is given by the far-field equation, which is a Fredholm integral equation of the first kind defined for each point in $\mathbb{R}^2$ or $\mathbb{R}^3$ and in which the data function is a known analytic function and the integral kernel is the measured (and therefore noisy) far-field pattern. A basic theorem, from now on named the general theorem, states that an approximate solution for the far-field equation exists whose $L^2$-norm blows up to infinity for all points approaching the boundary of the scatterer from inside and stays arbitrarily large outside. The linear sampling method applies the regularization theory for linear inverse problems in order to provide an approximate solution of the far-field equation according to the following algorithm [10]: given $P \times Q$ measurements of the far-field pattern at $P$ observation angles and for $Q$ incident fixed-frequency fields:

- take a grid containing the scatterer;

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for each grid point, determine the Tikhonov regularized solution [13] of the linear system obtained as a discretization of the far-field equation over the incidence and observation angles;

• for each grid point, choose the optimal regularization parameter by means of the generalized discrepancy principle [14];

• for each grid point, map the Euclidean norm of the optimal regularized solution.

The linear sampling method is a qualitative method for the solution of inverse scattering problems based on the observation that the scatterer profile can be detected by all grid points where the Euclidean norm of the optimal regularized solution is mostly large. Of course a visualization of the scatterer profile can be obtained by mapping the values of an appropriate monotonically increasing or decreasing function $I : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ of the norm itself. The composition of $I$ with the euclidean norm of the optimal regularized solution represents the so-called indicator function, defined over the grid. The scatterer profile is given by the grid points where the indicator function is respectively small or large, depending on the decreasing or increasing monotonicity of $I$.

The state-of-the-art for the general theorem presents some theoretical open problems [5] which, although of notable mathematical interest, can be considered mainly technical. A typical example is the determination of the formal connections between the Tikhonov regularized solution of the far-field equation and the approximate solution whose properties are described by the general theorem (this connection is discussed in [1] for certain scalar cases). On the other hand, there are some general issues concerning the implementation and performances of the linear sampling method which, if solved, could notably improve the effectiveness of the method and widen its applicability in the field of inverse scattering problems of interest in applied sciences. Three of these issues are touched by the following questions:

(i) is it possible to give a characterization of the indicator function in terms of its physical meaning or analytical properties?

(ii) which is the spatial resolution power achievable by means of the linear sampling method?

(iii) once the indicator function is available, which general criterion can suggest the thresholding level for its values? Or, in other terms, what can be considered respectively “large” or “small” for the indicator function, depending on the increasing or decreasing monotonicity of $I$?

Interesting results concerning item (i) are in [6], where a physical interpretation of the euclidean norm of the optimal regularized solution is given. As far as item (ii) is concerned, we can mention [12], in which an interesting discussion of super-resolution related to the factorization method [11] (considered as a modified version of the linear
sampling method) is proposed. Finally, an heuristic answer to question (iii) is presented in [9] together with some numerical validations.

The present paper deals with questions (i) and (ii). In particular, we will present a new implementation of the method whereby the regularization parameter does not depend any longer on the sampling point and an analytical representation for any indicator function is therefore possible. Then, for sake of simplicity, we choose a particular indicator function whose analytical expression allows one to show that it is band-limited and, consequently, to obtain some theoretical information about the spatial resolution achievable by the method. The paper focuses on scattering conditions modelled by means of two-dimensional inverse scattering problems, although generalization to three-dimensional cases should be straightforward. Moreover the problem of the choice of the cut-off for the indicator function (item (iii)) will be addressed by explicitly accounting for the knowledge of the scatterers to be reconstructed. In fact, in all numerical experiments the optimal visualization profile will be given by the level curve of the indicator function containing an area equal to the one contained by the theoretical profile.

The plan of the paper is as follows: in Section 2 the standard implementation of the linear sampling method is outlined. In Section 3 we describe the new implementation, which does not require any sampling. Section 4 shows the band-limitedness of the chosen indicator function in the case of this new implementation, while Section 5 presents some discussions and numerical simulations concerning the spatial resolution achievable by the method. Finally our conclusions will be offered in Section 6.

2. The linear sampling method

Let us consider the far-field equation

\[ \int_{\Omega} u_\infty(\hat{x};d)g_z(d)ds(d) = \Phi_\infty(\hat{x}, z). \]  

(2.1)

Here $\Omega = \{ x \in \mathbb{R}^2 , |x| = 1 \}$; $\hat{x}$ and $d$ are unit vectors denoting the observation and incidence directions respectively; $u_\infty(\hat{x};d)$ is the far-field pattern associated with the scattered field; $z$ is a point in $\mathbb{R}^2$; $g_z(\cdot)$ is a function in $L^2(\Omega)$ for each $z$; finally, $\Phi_\infty(\hat{x}, z)$ is the far-field pattern of the fundamental solution $\Phi(x, z)$ of the Helmholtz equation in $\mathbb{R}^2$ and is given by

\[ \Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x} \cdot z}. \]  

(2.2)

where $k$ is the wavenumber. We introduce the far-field operator $F : L^2(\Omega) \to L^2(\Omega)$ defined as

\[ (Fg(\cdot))(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d)g(d)ds(d). \]  

(2.3)
Then, for the case of a perfect conductor with Dirichlet conditions at the boundary, the general theorem at the basis of the linear sampling method is [4]:

**Theorem 2.1 (general theorem):** Let us assume that $k^2$ is not an eigenvalue for the negative Laplacian in the scatterer $D$. Then, if $F$ is the far-field operator (2.3), we have that

1) if $z \in D$, then for every $\epsilon > 0$ there exists a solution $g_z(\cdot) \in L^2(\Omega)$ of the inequality

$$
\|(Fg_z(\cdot))(\cdot) - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon
$$

such that

$$
\lim_{z \to \partial D} \|g_z(\cdot)\|_{L^2(\Omega)} = \infty
$$

and

$$
\lim_{z \to \partial D} \|v_{g_z(\cdot)}(\cdot)\|_{H^1(D)} = \infty,
$$

where $v_{g_z(\cdot)}(\cdot)$ is the Herglotz wave function with kernel $g_z(\cdot)$;

2) if $z \notin D$, then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_z(\cdot) \in L^2(\Omega)$ of the inequality

$$
\|(Fg_z(\cdot))(\cdot) - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon + \delta
$$

such that

$$
\lim_{\delta \to 0} \|g_z(\cdot)\|_{L^2(\Omega)} = \infty
$$

and

$$
\lim_{\delta \to 0} \|v_{g_z(\cdot)}(\cdot)\|_{H^1(D)} = \infty
$$

Analogous theorems hold for different boundary conditions in the obstacle case and for inhomogeneous scattering with both TE and TM polarization.

In real experiments, the far-field pattern is measured for $P$ observation angles $\{\varphi_i\}_{i=0}^{P-1}$ and $Q$ incidence angles $\{\theta_j\}_{j=0}^{Q-1}$, i.e. for observation directions $\{\hat{x}_i = (\cos \varphi_i, \sin \varphi_i)\}_{i=0}^{P-1}$ and incidence directions $\{d_j = (\cos \theta_j, \sin \theta_j)\}_{j=0}^{Q-1}$. In the following we will assume $P = Q = N$, the generalization to rectangular problems being straightforward. Furthermore, for sake of simplicity, we will take

$$
\varphi_i = \frac{2\pi i}{N}, \quad \theta_j = \frac{2\pi j}{N}, \quad i, j = 0, \ldots, N - 1.
$$

These values are placed into the far-field matrix $F$, whose elements are defined as

$$
F_{ij} := u_\infty(\hat{x}_i, d_j).
$$
In practical applications the far-field matrix is affected by the measurement noise, and therefore only a noisy version $F_h$ of the far-field matrix is at disposal, such that

$$F_h = F + H,$$

(2.12)

where $H$ is the noise matrix. If, as usual, we denote with $\| \cdot \|$ the operatorial norm of a linear continuous operator and regard the matrix $H$ as a linear continuous operator in $\mathbf{C}^N$, we assume to know that $\|H\| \leq h$.

Then a grid $\mathcal{Z} := \{z_i\}_{i=0}^{L-1}$ of $L$ sampling points in $\mathbb{R}^2$ is created, where the scattering object is located. For each $z_l = r_l(\cos \psi_l, \sin \psi_l) \in \mathcal{Z}$, we perform a discretization of $\Phi_\infty(x, z)$ by defining the column vector

$$\Phi_\infty(z_l) := \frac{e^{i\pi} e^{-ikr_l \cos(\varphi_0 - \psi_l)}}{\sqrt{8\pi k}} \left[ e^{-ikr_l \cos(\varphi_0 - \psi_1)}, \ldots, e^{-ikr_l \cos(\varphi_{N-1} - \psi_1)} \right]^T \cdot$$

(2.13)

Analogously, for each $z_l \in \mathcal{Z}$, the solution vector $g(z_l)$ is an element of $\mathbf{C}^N$ with the $i$-th component given by $g_i(z_l) = g_{z_l}(d_l)$. Therefore the discretized noisy version of the far-field equation (2.1) is given, for each $z_l \in \mathcal{Z}$, by the square linear system

$$F_h g(z_l) = \frac{N}{2\pi} \Phi_\infty(z_l) \cdot$$

(2.14)

This linear system is ill-conditioned and the numerical instabilities due to the presence of noise can be reduced by applying Tikhonov regularization method, i.e. by determining

$$g_{\alpha(z_l)}(z_l) = \arg\min \left\{ \left\| F_h g(z_l) - \frac{N}{2\pi} \Phi_\infty(z_l) \right\|_{\mathbf{C}^N}^2 + \alpha(z_l) \| g(z_l) \|_{\mathbf{C}^N}^2 \right\},$$

(2.15)

where we have denoted with $\| \cdot \|_{\mathbf{C}^N}$ the euclidean norm on $\mathbf{C}^N$. The optimal regularized solution corresponds to the value $\alpha^*(z_l)$ of the regularization parameter $\alpha(z_l)$ fixed by means of the generalized discrepancy principle. In general, the linear sampling method visualizes the scattering profile by plotting, as indicator function, $I(\| g_{\alpha^*(z_l)}(z_l) \|_{\mathbf{C}^N})$ for each $z_l \in \mathcal{Z}$, where $I: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a suitable monotonic continuous function. Here we take $I(t) = t^2$, i.e. we choose the map $z_l \mapsto \| g_{\alpha^*(z_l)}(z_l) \|_{\mathbf{C}^N}^2$ as our indicator function: it is defined over the grid $\mathcal{Z}$ and it will be denoted with $\Psi(z_l)$. An explicit form for this function can be determined by using the Singular Value Decomposition (SVD) $\{\sigma_p^h, u_p^h, v_p^h\}_{p=0}^{r_h-1}$ of the far-field matrix $F_h$, where $r_h$ is the rank of $F_h$. In fact, if $\langle \cdot, \cdot \rangle_{\mathbf{C}^N}$ is the scalar product in $\mathbf{C}^N$, then the Tikhonov regularized solution is given by

$$g_{\alpha^*(z_l)}(z_l) = \frac{N}{2\pi} \sum_{p=0}^{r_h-1} \frac{\sigma_p^h}{(\sigma_p^h)^2 + \alpha^*(z_l)}(\Phi_\infty(z_l), u_p^h)_{\mathbf{C}^N} u_p^h$$

(2.16)

and therefore the indicator function can be written as

$$\Psi(z_l) := \left\| g_{\alpha^*(z_l)}(z_l) \right\|_{\mathbf{C}^N}^2 = \frac{N^2}{4\pi^2} \sum_{p=0}^{r_h-1} \frac{(\sigma_p^h)^2}{((\sigma_p^h)^2 + \alpha^*(z_l))^2} \left| \left(\Phi_\infty(z_l), v_p^h \right)_{\mathbf{C}^N} \right|^2$$

(2.17)
We point out that in this implementation the optimal regularization parameter explicitly depends on the grid point \( z_l \) and therefore must be fixed \( L \) times by means of the generalized discrepancy principle, i.e., for each \( z_l \), by finding the zero of the generalized discrepancy function

\[
\rho(\alpha(z_l)) = \left\| F_h g_{\alpha(z_l)}(z_l) - \frac{N}{2\pi} \Phi_\infty(z_l) \right\|^2_{\mathbb{Q}^N} - h^2 \left\| g_{\alpha(z_l)}(z_l) \right\|^2_{\mathbb{Q}^N}.
\]  

(2.18)

Remark 2.2: The reason why the indicator function (2.17) is chosen, is because it leads to feasible analytical results, as explained in the following sections. However, in Section 5 we will motivate this choice in a more general framework.

3. A new implementation of the linear sampling method

The indicator function (2.17) is known only on the grid \( Z \). On the other hand, the knowledge of its analytic form on \( \mathbb{R}^2 \) or, better, over a rectangle \( T_{AB} := (-A, A) \times (-B, B) \) containing the scatterer, would open new perspectives on both the computational effectiveness of the method and the quantitative assessment of its performances in terms of spatial resolution, as we shall see in the following. Then we are interested in regarding expression (2.17) as a sampled version of a function \( \Psi(z) \) defined over \( T_{AB} \); nevertheless, this is not at all a straightforward task, since the dependence of expression (2.17) on \( z_l \), and therefore on any \( z \), is explicit for \( \Phi_\infty(z_l) \), but only implicit, and in general not known explicitly, for \( \alpha^*(z_l) \).

In order to overcome this drawback, in the present section we derive a new implementation of the algorithm, again based on the general Theorem 2.1, whereby the optimal value of the regularization parameter does not depend on \( z \in T_{AB} \). The starting point is to replace the finite set of equations (2.14) by an infinite set of equations

\[
F_h g(z) = \frac{N}{2\pi} \Phi_\infty(z) \quad \forall z \in T_A^B.
\]  

(3.1)

In this framework \( T_A^B \) can be regarded as a continuous grid whereby the generic sampling point \( z_l \) has become a continuous variable \( z \in T_A^B \). Then we want to modify the approach to the method from the pointwise algebraic setting represented in equation (3.1) to a unifying functional context, whereby regularization consists of a single procedure, which gives rise to a single value of the regularization parameter.

This result can be accomplished within the following mathematical framework. Let us consider the direct sum of Hilbert spaces:

\[
\left[ L^2(T_A^B) \right]^N := \underbrace{L^2(T_A^B) \oplus \ldots \oplus L^2(T_A^B)}_{N \text{ times}}.
\]  

(3.2)
where $L^2(T_A^B)$ denotes the usual set of Lebesgue square-integrable functions defined for almost all (f.a.a.) $z \in T_A^B$ and with values in $\mathbb{C}$. If $f(\cdot) = \{f_i(\cdot)\}_{i=0}^{N-1}$, $g(\cdot) = \{g_i(\cdot)\}_{i=0}^{N-1}$ are two elements of $[L^2(T_A^B)]^N$, then this is a Hilbert space equipped with the scalar product

$$(f(\cdot), g(\cdot))_{2,N} := \sum_{i=0}^{N-1} (f_i(\cdot), g_i(\cdot))_2 \quad \forall f(\cdot), g(\cdot) \in [L^2(T_A^B)]^N,$$  \hspace{1cm} (3.3)$$

and the induced norm

$$\|f(\cdot)\|_{2,N} = \sqrt{\int_{T_A^B} \|f(z)\|_{L^2}^2 \, dz}.$$  \hspace{1cm} (3.4)$$

For future purpose, we observe that the following implication holds:

$$\|g(\cdot)\|_{2,N} \leq 1 \Rightarrow \|g_i(\cdot)\|_2 \leq 1 \quad \forall i = 0, \ldots, N-1.$$  \hspace{1cm} (3.5)$$

We now introduce the

**Definition 3.1:** The linear operator $F_h : [L^2(T_A^B)]^N \rightarrow [L^2(T_A^B)]^N$ is given by

$$[F_h g(\cdot)](\cdot) := \left\{ \sum_{j=0}^{N-1} (F_h)_{ij} g_j(\cdot) \right\}_{i=0}^{N-1} \quad \forall g(\cdot) \in [L^2(T_A^B)]^N,$$  \hspace{1cm} (3.6)$$

where the $(F_h)_{ij}$ are the elements of the noisy far-field matrix.

The following properties for $F_h$ hold:

**Theorem 3.2:** The linear operator $F_h$ is continuous and its kernel $\mathcal{N}(F_h)$ is

$$\mathcal{N}(F_h) = \left\{ f(\cdot) \in [L^2(T_A^B)]^N \mid f(z) \in \mathcal{N}(F_h) \text{ f.a.a. } z \in T_A^B \right\},$$  \hspace{1cm} (3.7)$$

where we have denoted with $\mathcal{N}(F_h)$ the kernel of the linear operator $F_h$. Furthermore, if $g(\cdot) \in [L^2(T_A^B)]^N$ is such that $g(z) \in \mathcal{N}(F_h)^\perp$ f.a.a. $z \in T_A^B$, then $g(\cdot) \in \mathcal{N}(F_h)^\perp$, where the orthogonality must be intended with respect to the corresponding scalar product.

**Proof.** From implication (3.5) and the Schwartz’s inequality, we easily find that if $\|g(\cdot)\|_{2,N} = 1$, then

$$\|[F_h g(\cdot)](\cdot)\|_{2,N} \leq \max_{i,j} |(F_h)_{ij}| \sqrt{(N^3 - N^2 + N)}.$$  \hspace{1cm} (3.8)$$

The characterization (3.7) is obvious. Finally, let us take any $f(\cdot) \in \mathcal{N}(F_h)$. From (3.7), $f(z) \in \mathcal{N}(F_h)$ f.a.a. $z \in T_A^B$ and therefore if $g(\cdot) \in [L^2(T_A^B)]^N$ is such that $g(z) \in \mathcal{N}(F_h)^\perp$ f.a.a. $z \in T_A^B$, then

$$(f(\cdot), g(\cdot))_{2,N} = \int_{T_A^B} \sum_{i=0}^{N-1} f_i(z) g_i(z) \, dz = \int_{T_A^B} (f(z), g(z))_{\mathbb{C}^N} \, dz = 0.$$  \hspace{1cm} (3.9)$$
Remark 3.3: The operator $F_h$ is not compact. This can be easily seen by introducing the set of orthonormal (in $[L^2(T_A^B)]^N$) functions

$$b_n(\cdot) := \frac{1}{\sqrt{N}} \{e_{i+nN}(\cdot)\}_{i=0}^{N-1} \quad \forall n \in \mathbb{N},$$

(3.10)

where the $\{e_i(\cdot)\}_{i=0}^{\infty}$ are a countable infinite orthonormal set of $L^2(T_A^B)$, and by showing that, for $m \neq n$,

$$\|[F_h b_n(\cdot)(\cdot) - F_h b_m(\cdot)(\cdot)]\|_2^2 = \frac{2}{N} \sum_{i=0}^{N-1} \sum_{p=0}^{N-1} |(F_h)_{ip}|^2,$$

(3.11)

so that the sequence $\{[F_h b_n(\cdot)(\cdot)]\}_{n=0}^{\infty}$ does not satisfy the Cauchy criterion. The non-compactness of the operator $F_h$ is not in contradiction with the well-known compactness of the usual far-field operator $F$ defined in (2.3) (see [8]): indeed $F_h$ acts upon $N$-tuples of functions of the spatial variable $z$, while $F$ acts upon functions of the angular variable $\theta$.

However, by utilizing the Singular Value Decomposition of the far-field matrix, the representation

$$[F_h g(\cdot)(\cdot)] = \left\{ \sum_{p=0}^{v_h-1} \sigma_p v_{p,i}^{h} (g(\cdot), u_{p}^{h})_{\mathcal{C}^N} \right\}_{i=0}^{N-1} \quad \forall g(\cdot) \in [L^2(T_A^B)]^N$$

(3.12)

for $F_h$ holds, where $(g(\cdot), u_{p}^{h})_{\mathcal{C}^N}$ is the element in $L^2(T_A^B)$ defined as

$$(g(\cdot), u_{p}^{h})_{\mathcal{C}^N} := \sum_{i=0}^{N-1} \bar{a}_{p,i} g_i(\cdot),$$

(3.13)

and the $u_{p,i}^{h}$ are the components of the vector $u_{p}^{h}$. A simple computation using (3.12) leads to the property that

$$\|F_h\| = \sigma_0^h.$$  

(3.14)

This property has an important consequence: in fact it implies that $\|F_h\| = \|F_h\|$ and therefore that the bound $h$ on the noise affecting the matrix $F_h$ is the same of the noise affecting the operator $F_h$.

From a practical viewpoint, the introduction of the operator $F_h$ allows one to express the infinitely many algebraic systems (3.1) as the single functional equation in $[L^2(T_A^B)]^N$

$$[F_h g(\cdot)(\cdot)] = \frac{N}{2\pi} \Phi_\infty(\cdot),$$

(3.15)
The generalized solution

where \( \Phi_\infty(\cdot) \) is the element in \( [L^2(T_A^B)]^N \) trivially obtained from \( \Phi_\infty(z) \) simply regarding \( z \) as a variable on \( T_A^B \) instead of a fixed point in \( \mathbb{R}^2 \). The regularization of this equation occurs in a way which is independent from \( z \) and therefore provides a single value of the regularization parameter (we point out, however, that such a value will depend on the choice of the rectangle \( T_A^B \)). At this stage there is a final computational open issue to address, which is concerned with how to determine the regularized solution of equation (3.15) in practice. But this problem is solved by the following theorem, which shows that the regularized solution in the new functional context is obtained from the regularized solution of (3.1) when \( z \) is thought as an independent variable.

**Theorem 3.4:** The generalized and regularized solutions of problem (3.15) are given by

\[
g_h^\dagger(\cdot) = N \frac{r_h^{-1}}{2\pi} \sum_{p=0}^{N} \frac{(\Phi_\infty(\cdot), \Phi_p^h)_{q^N}}{\sigma_p^h} \Phi_p^h;
\]

and

\[
g_\alpha(\cdot) = N \frac{r_h^{-1}}{2\pi} \sum_{p=0}^{N} \frac{\sigma_p^h}{(\sigma_p^h)^2 + \alpha} (\Phi_\infty(\cdot), \Phi_p^h)_{q^N} \Phi_p^h;
\]

**Proof.** The generalized solution \( g_h(\cdot) \) of equation (3.1) is the unique least-squares solution in \( \mathcal{N}(F_h)^\perp \). Therefore, for any \( g(\cdot) \in [L^2(T_A^B)]^N \) and f.a.a. \( z \in T_A^B \)

\[
\|F_h g_h(z) - \frac{N}{2\pi} \Phi_\infty(z)\|_{q^N}^2 \leq \|F_h g(z) - \frac{N}{2\pi} \Phi_\infty(z)\|_{q^N}^2.
\]

From (3.18), we immediately get:

\[
\int_{T_A^B} \left\|F_h g_h(z) - \frac{N}{2\pi} \Phi_\infty(z)\right\|_{q^N}^2 dz \leq \int_{T_A^B} \left\|F_h g(z) - \frac{N}{2\pi} \Phi_\infty(z)\right\|_{q^N}^2 dz.
\]

We now observe that, for any \( g(\cdot) \in [L^2(T_A^B)]^N \), definition (3.6) implies that

\[
F_h g(z) = [F_h g(\cdot)](z);
\]

therefore relation (3.19) can be written as

\[
\int_{T_A^B} \left\|[F_h g_h(\cdot)](z) - \frac{N}{2\pi} \Phi_\infty(z)\right\|_{q^N}^2 dz \leq \int_{T_A^B} \left\|[F_h g(\cdot)](z) - \frac{N}{2\pi} \Phi_\infty(z)\right\|_{q^N}^2 dz,
\]

where \( g_h^\dagger(\cdot) \) is the element in \( [L^2(T_A^B)]^N \) obtained from \( g_h^\dagger(z) \) when \( z \) varies in \( T_A^B \). Then relation (3.4) leads to

\[
\left\|[F_h g_h(\cdot)](\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot)\right\|_{2,N}^2 \leq \left\|[F_h g(\cdot)](\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot)\right\|_{2,N}^2.
\]
Since this inequality holds $\forall g(\cdot) \in \left[L^2(T_B^R)\right]^N$, it immediately follows that

$$g_h^\dagger(\cdot) = \arg\min \left\{ \|F_h g(\cdot)(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot)\|_{2,N}^2 \right\}. \quad (3.23)$$

Moreover, since $g_h^\dagger(z) \in \mathcal{N}(F_h)^\perp \forall z \in T_B^R$, from Theorem 3.2 we have that $g_h^\dagger(\cdot) \in \mathcal{N}(F_h)^\perp$. Result (3.17) for the regularized solution follows analogously.

In expression (3.17), $\alpha$ is a generic real positive number which can be fixed by applying the generalized discrepancy principle, where the new generalized discrepancy function is now

$$\rho(\alpha) = \left\| F_h g_\alpha(\cdot) - \frac{N}{2\pi} \Phi_\infty(\cdot) \right\|_{2,N}^2 - h^2 \|g_\alpha(\cdot)\|_{2,N}^2. \quad (3.24)$$

If $\alpha^*$ is the optimal value provided by this criterion, then

$$g_{\alpha^*}(\cdot) = \frac{N}{2\pi} \sum_{p=0}^{r_h-1} \frac{\sigma^h_p}{(\sigma^h_p)^2 + \alpha^*} (\Phi_\infty(\cdot), v_p^h)_{\mathcal{C}^N} u_p^h \quad (3.25)$$

is the optimal regularized solution of the functional problem (3.15) and the new indicator function is

$$\Psi(z) := \|g_{\alpha^*}(z)\|_{\mathcal{C}^N}^2 = \frac{N^2}{4\pi^2} \sum_{p=0}^{r_h-1} \frac{(\sigma^h_p)^2}{[(\sigma^h_p)^2 + \alpha^*]^2} \left| (\Phi_\infty(z), v_p^h)_{\mathcal{C}^N} \right|^2. \quad (3.26)$$

Remark 3.5: We point out that in the analytic expression (3.26) no sampling is performed, since $\alpha^*$ does not depend on $z$ and all the other terms are known once the observed far-field matrix is at disposal. From a computational viewpoint, this removes the problem of deciding the parameters of the optimal grid containing the scatterer (number of points, sampling distance).

Remark 3.6: The mathematical framework for this new implementation of the linear sampling method naturally implies that the optimal regularization parameter is unique, independently of $z$. On the other hand, it is worthwhile noticing that, although in a completely different context, a sampling-point-independent choice of the regularization parameter is also suggested in [6], by giving a heuristic recipe based on physical considerations.
Theorem 3.4 provides a new implementation of the linear sampling method, where the contour of the scatterer is detected by all points where (3.26) becomes mostly large. In order to show that this implementation yields the same results as the traditional one based on a sampling in the space, in Figure 1 we consider the reconstruction of different impenetrable scatterers in the case of Dirichlet boundary conditions: panel (a) is concerned with an ellipse with center in \((0,0)\) and semiaxes equal to 1 and 2 respectively; panel (b) is concerned with a kite described by the parametric equation
\[
x(t) = 1.5 \cdot \sin t, \quad y(t) = \cos t + 0.65 \cdot \cos(2t) - 0.65, \quad t \in [0, 2\pi]
\]
panel (c) is concerned with a peanut of equation
\[
x(t) = f_0(t) \cos t, \quad y(t) = f_0(t) \sin t, \quad t \in [0, 2\pi]
\]
where
\[
f_0(t) = \sqrt{\cos^2 t + 4 \sin^2 t};
\]
finally panel (d) is concerned with two objects: a kite with the same equation (3.27) but centered in \((-4, -4)\) and rotate of 45° clockwise, together with an ellipse again with semiaxes 1 and 2 but centered in \((4, 4)\) and rotated of 45° counterclockwise. For all cases, the wavenumber is \(k = 1\), the far-field matrix \(F\) is computed by using the Nyström method [8] for \(N = 32\) incidence and observation angles, and 3% Gaussian noise is added by means of a suitable \(32 \times 32\) noise matrix \(H\) summed to \(F\) (see (2.12)). For all visualizations, we provide the profile of the true object and, superimposed, the reconstructed contours provided by the two implementations of the linear sampling method, where, for each implementation, the cut-off value has been fixed by making the area defined by the level curves of the indicator function equal to the one of the true scatterer. For both implementations, the regularization parameter has been fixed by applying the generalized discrepancy principle. Table 1 provides the minimum optimal value \(\alpha_m^* := \min_{z_l \in \mathcal{Z}} \{\alpha^*(z_l)\}\) and the maximum optimal value \(\alpha_M^* := \max_{z_l \in \mathcal{Z}} \{\alpha^*(z_l)\}\) of the regularization parameter in the traditional implementation and the (unique) optimal value of the regularization parameter \(\alpha^*\) in the new implementation. From the plots in Fig. 1 the differences between the two implementations turn out to be completely negligible. Analogous results occur when the noise level affecting the far-field pattern, the number of incidence/observation angles and the wavenumber are changed.

4. The indicator function is band-limited

The aim of the present section is to compute the Fourier transform of the indicator function (3.26) analytically continued onto all \(\mathbb{R}^2\) and still denoted with \(\Psi(z)\). By using
\[
\Phi_\infty(z) = \frac{e^{i\pi x}}{\sqrt{8\pi k}} \begin{bmatrix} e^{-ikz \cdot \hat{x}_0}, \ldots, e^{-ikz \cdot \hat{x}_{N-1}} \end{bmatrix}^T
\]

scattering object & $\alpha_m^*$ & $\alpha_M^*$ & $\alpha^*$ \\
--- & --- & --- & --- \\
ellipse & 1.79x10^{-5} & 1.27x10^{-3} & 2.60x10^{-5} \\
kite & 2.08x10^{-5} & 3.30x10^{-4} & 4.22x10^{-5} \\
peanut & 1.49x10^{-5} & 4.98x10^{-4} & 1.98x10^{-5} \\
kite+ellipse & 5.00x10^{-5} & 7.83x10^{-3} & 1.07x10^{-4} \\

Table 1. Minimum $\alpha_m^*$ and maximum $\alpha_M^*$ optimal value of the regularization parameter given by the generalized discrepancy principle in the traditional implementation; optimal value $\alpha^*$ of the regularization parameter given by the generalized discrepancy principle in the new implementation. The scattering experiment is the same of Figure 1.

in (3.26) we obtain

$$\Psi(z) = \frac{N^2}{32\pi^3 k} \sum_{p=0}^{r_h-1} \frac{(\sigma_p^h)^2}{((\sigma_p^h)^2 + \alpha^*)^2} \cdot \left| v_{p,0}^h e^{-ik(x \cos \varphi_0 + y \sin \varphi_0)} + \ldots + v_{p,N-1}^h e^{-ik(x \cos \varphi_{N-1} + y \sin \varphi_{N-1})} \right|^2,$$

where $z = (x, y) \in \mathbb{R}^2$. By introducing the polar coordinates of the $N$ components of each vector $v_p^h$, i.e.:

$$v_{p,0}^h = (\rho_{p,0}^h, \epsilon_{p,0}^h) = \rho_{p,0}^h e^{i\epsilon_{p,0}^h},$$

$$v_{p,N-1}^h = (\rho_{p,N-1}^h, \epsilon_{p,N-1}^h) = \rho_{p,N-1}^h e^{i\epsilon_{p,N-1}^h},$$

we get

$$\Psi(z) = \frac{N^2}{32\pi^3 k} \sum_{p=0}^{r_h-1} \frac{(\sigma_p^h)^2}{((\sigma_p^h)^2 + \alpha^*)^2} \sum_{i,j=0}^{N-1} \rho_{p,i}^h \rho_{p,j}^h \cos \left( \hat{\omega}_{x,ij} x + \hat{\omega}_{y,ij} y + \Delta \epsilon_{p,ij}^h \right),$$

where

$$\hat{\omega}_{x,ij} := k(\cos \varphi_i - \cos \varphi_j),$$

$$\hat{\omega}_{y,ij} := k(\sin \varphi_i - \sin \varphi_j),$$

$$\Delta \epsilon_{p,ij}^h := \epsilon_{p,i}^h - \epsilon_{p,j}^h.$$

By using the definition

$$\mathcal{F}(f)(\omega_x, \omega_y) := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-i(\omega_x x + \omega_y y)} dx dy$$

for the Fourier transform, it follows that the Fourier transform of function (4.33) is

$$\mathcal{F}(\Psi)(\omega_x, \omega_y) = \frac{N^2}{32\pi^3 k} \sum_{p=0}^{r_h-1} \frac{(\sigma_p^h)^2}{((\sigma_p^h)^2 + \alpha^*)^2} \sum_{i,j=0}^{N-1} \rho_{p,i}^h \rho_{p,j}^h.$$
\[ \left\{ 2\pi^2 \cos \Delta \epsilon_{p,ij}^h \left[ \delta(\omega_x - \hat{\omega}_{x,ij}) \delta(\omega_y - \hat{\omega}_{y,ij}) + \delta(\omega_x + \hat{\omega}_{x,ij}) \delta(\omega_y + \hat{\omega}_{y,ij}) \right] + 
+ 2i \pi^2 \sin \Delta \epsilon_{p,ij}^h \left[ \delta(\omega_x - \hat{\omega}_{x,ij}) \delta(\omega_y - \hat{\omega}_{y,ij}) - \delta(\omega_x + \hat{\omega}_{x,ij}) \delta(\omega_y + \hat{\omega}_{y,ij}) \right] \right\}. \]

This expression shows that the Fourier transform of the indicator function analytically extended to all \( \mathbb{R}^2 \) is a distribution, which can be named “Dirac brush”, whose support (which is, by definition, the band of \( \Psi(z) \)) is the compact set

\[ S = \left\{ (\hat{\omega}_{x,ij}, \hat{\omega}_{y,ij}) \right\}_{i,j=0}^{N-1}. \]  

(4.39)

From relations (4.34), (4.35) we easily have that

\[ \sqrt{(\hat{\omega}_{x,ij})^2 + (\hat{\omega}_{y,ij})^2} \leq 2k \quad \forall i, j = 0, \ldots, N - 1, \]  

(4.40)
i.e. the support of the Dirac brush is contained in a circle of radius \( 2k \) in the frequency space. Moreover, if we define

\[ \Omega_x := \max_{i,j}[\hat{\omega}_{x,ij}], \quad \Omega_y := \max_{i,j}[\hat{\omega}_{y,ij}], \]  

(4.41)

we can say that the indicator function (3.26) is \((\Omega_x, \Omega_y)\)-bandlimited. Hence, if we put

\[ \text{sinc}(t) := \begin{cases} \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0, \end{cases} \]  

(4.42)

the following Shannon-Nyquist representation of \( \Psi(z) \equiv \Psi(x, y) \) [2]

\[ \Psi(z) = \sum_{n_1, n_2 = -\infty}^{+\infty} \Psi(n_1 d_x, n_2 d_y) \text{sinc} \left[ \frac{x - n_1 d_x}{d_x} \right] \text{sinc} \left[ \frac{y - n_2 d_y}{d_y} \right] \]  

(4.43)

holds, provided that the sampling distances \( d_x \) and \( d_y \) along the \( x \)-axis and \( y \)-axis respectively satisfy the following conditions [2], [3]:

\[ d_x < \frac{\pi}{\Omega_x}, \quad d_y < \frac{\pi}{\Omega_y}, \]  

(4.44)

where \( \frac{\pi}{\Omega_x} \) and \( \frac{\pi}{\Omega_y} \) are called \textit{Nyquist distances}. For example, if \( N \) is a multiple of 4, \( \Omega_x = \Omega_y = 2k \) holds and then the Nyquist distances are:

\[ \frac{\pi}{\Omega_x} = \frac{\pi}{\Omega_y} = \frac{\lambda}{4}, \]  

(4.45)

where \( \lambda = \frac{2\pi}{k} \) is the wavelength. For \( N \) not a multiple of 4, \( \lambda/4 \) is a strict lower bound for the Nyquist distance.

We remark that the support (4.39) of distribution (4.38) is independent of the scatterer and only depends on the wavenumber and on the number \( N \) of the observation/incidence angles (of course, the Fourier transform of the indicator function does depend on the scatterer characteristics, in particular through the singular system of the far-field matrix). As examples, Figure 2 shows this support in the case of \( k = 5 \)
and \( N = 8, 16, 32, 64 \). In order to validate these results, we considered the scattering of \( N = 8 \) plane waves for \( k = 5 \) with the conducting kite (3.27) in the case of Dirichlet boundary conditions, for \( N = 8 \) observation angles. In Figure 3 we computed the numerical Fourier transform of the corresponding indicator function (panel (a)) and compared it with the theoretical support for the same \( N \) (panel (b)): the position of the peaks of the numerical Fourier transform coincides with the peaks of the Dirac brush.

5. Spatial resolution

Spatial resolution is the main concept in image formation theory and practice. In the present paper the results of the previous two sections provide some hints about how to estimate the ability of the method to recover close objects from the superposition of their noisy discretized far-field patterns. Indeed, the Shannon-Nyquist representation (4.43) implies that \( \Psi(z) \) cannot vary significantly on length scales smaller than the Nyquist distance \( \lambda/4 \), since such representation consists of a superposition of sinc-like functions that are peaked at a distance smaller then \( \lambda/4 \) from one another and are very smooth between adjacent sampling points. On the other hand, it should be pointed out that, in whatever implementation of the linear sampling method, the visualization of the scatterer profile is obtained by choosing a cut-off section for the 3D plot of \( \Psi(z) \) and the relation between the Nyquist distance for \( \Psi(z) \) and the spatial resolution achievable for the scatterer profile on this section is still an open issue. Therefore the value \( \lambda/4 \) is only an approximated estimate of the true resolution power achievable by the method. In particular, when the cut-off criterion consists in selecting the level curve of the indicator function containing an area equal to the one contained by the theoretical profile (i.e., the cut-off criterion adopted in this paper), numerical experience suggests that the Nyquist distance represents a pessimistic estimate of the spatial resolution in two dimensions, i.e. situations occur where the cut-off section produces quite featured profiles although \( \Psi(z) \) varies very smoothly, while the converse does not seem to happen. Analogous results occur when the cut-off criterion is chosen as in [9].

In order to give some examples illustrating our previous observations, we consider the following numerical experiments. Two conducting objects with Dirichlet boundary conditions, an ellipse and a peanut, are taken. The ellipse is centered in \((-2, 0)\) and has semiaxes equal to 1 and 2 respectively, while the peanut is described by equations (3.28)-(3.29), has center in \((2, 0)\) and is rotated of 45° counterclockwise. The far-field matrix \( \mathbf{F} \) is computed by means of the Nyström method in the case of \( N = 32 \) incidence and observation angles, and 0.1% Gaussian noise is added by means of a suitable \( 32 \times 32 \) noise matrix \( \mathbf{H} \) summed to \( \mathbf{F} \) (see (2.12)). The wavenumber is \( k = 1 \), i.e. \( \lambda/4 \approx 1.57 \). We define the distance \( d \) between the two objects as the distance between the closest points of the two boundaries. Then we considered four cases: \( d = 1.8, 1.6, 1.5, 0.8 \), i.e. for one
case the distance between the objects is bigger than the Nyquist distance, for two cases it is very close to the Nyquist distance and for one case it is smaller. We computed $\Psi(z)$ with $\alpha$ provided by the generalized discrepancy principle and determined the optimal level curves in such a way that the area described by the theoretical curves and the area described by the level curves are equal. The results of this experiment are plotted in Figure 4, where our theoretical assessment of the spatial resolution is confirmed by the computational outcome. However, as stated before, there are scattering conditions and noise levels for which this theoretical estimate is pessimistic. For example, in Figure 5 the noise affecting the far-field pattern is 1% Gaussian and the peanut and the ellipse are still distinguishable at a distance $d = 1.5$, smaller than the Nyquist distance $1.57$.

We point out that in the indicator function noise only affects the singular system of the far-field matrix and its presence does not change the band, i.e. does not change the support of the distribution $\mathcal{F}(\Psi)(\omega)$. Therefore this is a typical application where the presence of noise does not change, in principle, the theoretical spatial resolution of the method. On the other hand, the presence of noise affects the overall shape of the indicator function and thus contributes to a general deterioration of the visualization accuracy. As an example, in Figure 6 we compare the visualizations for the ellipse and the peanut placed at a distance $d = 1.8$, bigger than the Nyquist distance, when the far-field pattern is affected by 5% (Fig. 6(a)) and 30% (Fig. 6(b)) Gaussian noise respectively. In both cases the two objects are distinguishable, but in the second case the profiles are very deteriorated.

All these results have been obtained by using a specific form for the indicator function, i.e. for the monotonic function $I$. In principle, the resolution power of the method depends on the form of $I$. By instance, an easy way to indefinitely widen the band is to replace our indicator function $\Psi(z)$, given by (3.26), with $\Psi^n(z)$, with an arbitrarily large $n \in \mathbb{N} \setminus \{0\}$. However this implies that, in general, also the range of $\Psi^n(z)$ becomes arbitrarily large, so that an appropriate visualization of a 3D plot of $\Psi^n(z)$ would require a (not necessarily linear) rescaling of the Cartesian axis perpendicular to the $xOy$-plane (and this procedure would be essentially equivalent to utilize a different indicator function with a smaller $n$). From a 2D perspective, after a cut-off criterion for the indicator function is applied, we observe that, despite regularization, random oscillations due to the presence of noise still affect the regularized solution $g_{\alpha^*}(z)$ and, consequently, its Euclidean norm $\|g_{\alpha^*}(z)\|_{\mathcal{L}^N}$. Hence, for increasing values of $n$ the correspondently decreasing Nyquist distances tend to become comparable or smaller than the length scale on which such oscillations deteriorate the detected scatterer profile. In other terms, if $n$ is too large, the theoretically estimated resolution power (heuristically identified with the Nyquist distance itself) may become unrealistic, since it concerns length scales on which the smoothing effect of the regularization procedure is not completely satisfactory.
Anyway, in general, the implementation of the linear sampling method proposed in Section 3 allows one to compute the analytical form for all possible indicator functions. From this, it is possible to infer information on the Fourier transform and therefore on the achievable resolution. By instance, for the 'traditional' choice $\sqrt{\Psi(z)}$ for the indicator function, it is easy to show that the Nyquist distance is $\lambda/2$.

Summing up, in the present paper we have chosen $\Psi(z)$, given by (3.26), as indicator function, since it involves feasible analytic computations and consequently gives a detailed and fully worked out example of the potential applications of our “no-sampling” implementation. However, $\Psi(z)$ is not necessarily the best indicator function, as well as our “equal-areas” cut-off criterion is not necessarily the best way to detect the scatterer profile.

6. Conclusions

This paper presents a new viewpoint for the implementation of the linear sampling method when the far-field matrix is discretized according to the same number of equidistant incidence and observation angles. In this new framework the sampling procedure of the previous implementation is replaced by a single functional equation which is regularized by means of a single optimization procedure. The advantages of this approach are two-fold. First, at a computational level, pointwise regularization is avoided together with a notable number of time consuming zero-finding processes for the generalized discrepancy function (as pointed out in [10], in the traditional implementation the optimal values of the regularization parameter chosen by means of the generalized discrepancy principle significantly depend on the sampling points and in some cases can be even used as further indicator function for the visualization of the scatterer profile). This fact may have important implications in the case of three-dimensional anisotropic scattering, when the inversion requires a notable computational effort. Second, from the point of view of applications, an estimate of the resolution power of the method is now possible and is based on the Shannon-Nyquist representation of the indicator function.

We observe that the “no-sampling” approach to linear sampling can be extended to other visualization methods requiring the solution of many ill-conditioned linear systems parametrized over a sampling grid containing the scatterer, like for example the factorization method [11] or methods which use more strongly singular sources than $\Phi(\cdot, z)$. In particular, the same ‘no-sampling’ approach introduced in this paper can be applied to theoretically assess whether the use, for example, of a derivative of the fundamental solution at the source point $z$ provides a better spatial resolution. On the other hand, establishing the relation between the regularized solution of equation (3.15) and the approximate solution introduced by the general theorem is still an open
problem.

7. Acknowledgments

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References

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Figure 1. Reconstructions of four impenetrable scatterers (solid line), in the case of Dirichlet boundary conditions, by means of the linear sampling method. For all cases, the wavenumber is $k = 1$, while the far-field matrix is computed for $N = 32$ incidence and observation angles, and affected by 3% Gaussian noise. The linear sampling method is implemented “traditionally”, by using a sampling grid $Z$ (dashed line), and according to the new implementation, without any sampling and with a single optimal value for the regularization parameter (dotted line). In both implementations, the values of the regularization parameters are determined by means of the generalized discrepancy principle.
Figure 2. Support (4.39) of the Fourier transform (4.38) of the indicator function in the case $k = 5$ and (a) $N = 8$, (b) $N = 16$, (c) $N = 32$, (d) $N = 64$. 
Figure 3. Scattering of $N = 8$ plane waves with a conducting kite in the case of Dirichlet boundary conditions, for $k = 5$ and $N = 8$ observation angles. The numerical Fourier transform of the corresponding indicator function (panel (a)) is computed and is compared with the Dirac brush in the same scattering situation (panel (b)).
Figure 4. Validation of the theoretical assessment of the spatial resolution in the case of two scatterers: an ellipse and a peanut (dashed lines). The far-field pattern is computed in the case of $N = 32$ incidence and observation angles, and 0.1% Gaussian noise is added. The wavenumber is $k = 1$, i.e. $\lambda/4 \approx 1.57$. Four cases are considered in which the distances between the scatterers are (a) $d = 1.8$, (b) $d = 1.6$, (c) $d = 1.5$ and (d) $d = 0.8$. The values of the unique regularization parameter, determined by means of the generalized discrepancy principle, are respectively (a) $\alpha^* = 9.30 \cdot 10^{-6}$, (b) $\alpha^* = 8.48 \cdot 10^{-6}$, (c) $\alpha^* = 8.35 \cdot 10^{-6}$ and (d) $\alpha^* = 7.50 \cdot 10^{-6}$. 
Figure 5. Validation of the theoretical assessment of the spatial resolution in the case of two scatterers: an ellipse and a peanut (dashed lines). The far-field pattern is computed in the case of $N = 32$ incidence and observation angles, and 1% Gaussian noise is added. The wavenumber is $k = 1$, i.e. $\lambda/4 \simeq 1.57$. The case is considered in which the distance between the scatterers is $d = 1.5$. The value of the unique regularization parameter, determined by means of the generalized discrepancy principle, is $\alpha^* = 8.29 \cdot 10^{-6}$. 
Figure 6. Validation of the theoretical assessment of the spatial resolution in the case of two scatterers: an ellipse and a peanut (dashed lines). The wavenumber is \( k = 1 \), i.e. \( \lambda/4 \simeq 1.57 \). The far-field pattern is computed in the case of \( N = 32 \) incidence and observation angles; 5\% (panel (a)) and 30\% (panel (b)) Gaussian noise is added. The distance between the scatterers is \( d = 1.8 \). The values of the unique regularization parameter, determined by means of the generalized discrepancy principle, are respectively (a) \( \alpha^* = 2.47 \cdot 10^{-4} \) and (b) \( \alpha^* = 8.29 \cdot 10^{-3} \).