The use of constraints for solving inverse scattering problems: physical optics and the linear sampling method

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Abstract. Physical optics is an asymptotic scattering regime where the wavelength of the incident field is much smaller than the linear dimensions of the scatterer. In such conditions the inverse scattering problem of determining the object profile from measurements of the far-field pattern can be formulated in terms of a linear integral equation of the first kind. We apply an iterative algorithm with convex projections to scattering data corresponding to different conducting objects and discuss the complementarity between this physical optics approach and the reconstruction algorithm based on the linear sampling method. Finally, the two approaches are combined to increase the reconstruction accuracy.

1. Introduction

The inverse scattering problem considered in the present paper is concerned with the restoration of the boundary of a perfectly reflecting object (a perfect conductor in electromagnetic scattering or a sound-soft obstacle in acoustic scattering) from a knowledge of the incident time-harmonic plane wave and of the far-field pattern associated with the scattered field. The boundary integral equation formulation of the direct problem uniquely defines a non-linear operator mapping the boundary of the sound-soft scatterer onto the far-field pattern of the scattered wave [9]. This boundary-to-far-field operator is Frechet differentiable and locally compact [17], which illustrates the ill-posedness of the inverse problem. Numerical ill-conditioning implied by ill-posedness can be acknowledged by applying appropriate regularization methods able to reduce the artificial oscillations, in a way consistent with the scattering data. Unfortunately a reliable and systematic framework for regularization algorithms is available only for linear ill-posed inverse problems, non-linear problems typically being treated by means of computationally heavy optimization schemes. In the case of inverse scattering problems, linearity can be restored asymptotically, by considering high or low
frequency regimes, or exactly, by introducing appropriate factorizations of the linear far-field operator, defined as the integral operator whose kernel is given by the far-field pattern.

Physical optics approximation \([2, 3, 19]\) leads to a linearized inverse scattering problem, when the wavelength of the incident wave is much smaller than the linear dimensions of the perfectly conducting scatterer. Indeed, under physical optics conditions, the Fourier transform of the characteristic function of the scatterer can, at least in principle, be obtained from measurements of the far-field pattern corresponding to back scattering, for all incident directions and all wavenumbers. The obvious drawback of this approach is that it is reliable only when this small wavelength condition is fulfilled. It follows that the linear inverse problem involved by the physical optics approximation is an out-of-band extrapolation problem where the band is a circular corona with inner radius imposed by the small wavelength constraint and outer radius imposed by experimental limitations. Physical optics is a classical problem in inverse scattering theory and reconstruction algorithms based on the analysis of the directional derivative of the characteristic function of the target \([20]\) or on the theory of the Radon transform \([13]\) have been shown to produce satisfactory results in the case of simple geometries, even for limited aperture data. However, only the application of restoration methods based on the regularization theory for ill-posed inverse problems allows one to obtain notable super-resolution effects, when appropriate constraints on the solution are applied during the inversion procedure \([1]\).

The contents of the present paper are 1) the study of how these constraints can be implemented in the reconstruction of a two-dimensional object profile from scattering data and 2) a comparison between the inversion approach based on the physical optics approximation and the visualization procedure based on the linear sampling method. The linear sampling method \([10]\) is an exact linear method, where the restoration of the scatterer is accomplished by solving a set of linear ill-posed Fredholm equations of the first kind and the resulting algorithm can be implemented by using classical regularization techniques \([12, 24]\). The linear sampling method is a low resolution visualization procedure, since the solution of the linear integral equations is neither positive nor compactly supported and therefore super-resolving constrained algorithms cannot be applied for its implementation. However, its main atout is that it can work even under resonance regimes, i.e. when the non-linearity of the problem cannot be reduced by means of physically-based linearizations. Furthermore, in the paper we will show that its notable computational effectiveness can be exploited to significantly help the implementation of constraints in the physical optics approximation approach.

The plan of the paper is as follows. Section 2 will review the derivation of the Bojarski's identity in two dimensions. Section 3 will formulate the physical optics approximation problem as a linear integral equation involving the Slepian operator and
will describe the regularization algorithm we will use in the applications. In Section 4 a test problem will be considered, studying the effects of convex constraints on the reconstruction quality. Section 5 will be devoted to the comparison between the physical optics approximation approach and the linear sampling method and examples of a combined use of the two procedures will be discussed. Finally, Section 6 will contain some concluding remarks.

2. Bojarski’s identity

In the case of time-harmonic acoustic waves or time-harmonic electromagnetic waves with Transverse Magnetic (TM) polarization, the scattering from infinitely long cylindrical sound-soft obstacles leads to exterior Dirichlet problems for the Helmholtz equation in $\mathbb{R}^2$. More precisely, given an entire function $u^i$ representing an incident field, the direct scattering problem for the two-dimensional sound-soft obstacle $D$ with smooth boundary $\partial D$ is the problem of finding the scalar field

$$ u(x) = u^i(x) + u^s(x) \quad x \in \mathbb{R}^2, $$

solving

$$ \Delta_2 u(x) + k^2 u(x) = 0 \quad x \in \mathbb{R}^2 \setminus D, $$

such that the Dirichlet boundary condition

$$ u(x) = 0 \quad x \in \partial D $$

and the Sommerfeld radiation condition

$$ \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i k u^s \right), $$

with $r = |x|$, are satisfied. Here $k$ is the wavenumber and, for sake of simplicity (and without loosing in generality) we will assume in the following that

$$ u^i(x) = e^{ikd \cdot x}, $$

where $d \in \Omega = \{x \in \mathbb{R}^2, |x| = 1\}$ denotes the incident direction. The radiation condition (4) implies, through Green’s formula, that the scattered field can be asymptotically factorized in the form

$$ u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}) + O(r^{-3/2}), $$

where $\hat{x}$ is the unit vector indicating the observation direction and $u_\infty(\hat{x})$ is the far-field pattern. In the inverse problem, measurements of the far-field pattern represent the data of the problem. In the case of a $C^2$ boundary it can be proved that the far-field pattern
is completely known when the normal derivative of the total field on the boundary is
known, i.e. (Huygens' principle, [9])

\[ u_\infty(\hat{x}) = -\frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \frac{\partial u(y)}{\partial n(y)} e^{-ik\hat{x} \cdot y} ds(y) \]  \hspace{1cm} (7)

Equation (7) together with Green's theory lead to a two-dimensional version of the
Bojarski's identity:

**Theorem 2.1** Let \( \partial D \) be a \( C^2 \) convex boundary and let us consider its decomposition
\( \partial D = \partial D_+ + \partial D_- \) such that

\[ \partial D_+ = \{ x \in \partial D, \nu(x) \cdot d < 0 \} \]  \hspace{1cm} (8)

and

\[ \partial D_- = \{ x \in \partial D, \nu(x) \cdot d > 0 \} \]  \hspace{1cm} (9)

If the boundary conditions

\[ \frac{\partial u}{\partial n} = 0 \quad x \in \partial D_+ \]  \hspace{1cm} (10)

and

\[ \frac{\partial u}{\partial n} = 2 \frac{\partial e^{ikd \cdot x}}{\partial n} \quad x \in \partial D_- \]  \hspace{1cm} (11)

hold, then

\[ \frac{\sqrt{8\pi k}}{4k^2} [e^{-i\pi/4}u_\infty(-d; d, k) + e^{-i\pi/4}u_\infty(d; -d, k)] = \hat{\chi}_D(-2kd) \]  \hspace{1cm} (12)

where at the right hand side, \( \hat{\chi}_D(-2kd) \) is the Fourier transform of the characteristic
function of \( D \) computed at \(-2kd\) and, at the left hand side, the dependence of the far-field
pattern on the incident direction and on the wavenumber has been pointed out.

**Proof** The proof is essentially analogous to the one for the 3D case [9]. From equation
(7) and the boundary conditions (10), (11) we have that

\[ e^{-i\pi/4}u_\infty(\hat{x}; d, k) = -\frac{1}{\sqrt{2\pi k}} ik \int_{\partial D_-} (\nu \cdot d) e^{ik(d-\hat{x}) \cdot y} ds(y) \]  \hspace{1cm} (13)

It follows that, for \( \hat{x} = -d \),

\[ e^{-i\pi/4}u_\infty(-d; d, k) = -\frac{1}{\sqrt{8\pi k}} \int_{\partial D_-} \frac{\partial e^{2ikd \cdot y}}{\partial n(y)} ds(y) \]  \hspace{1cm} (14)
On the other hand,
\[ e^{-i\pi/4}u_\infty(\hat{x}; -d, k) = -\frac{1}{\sqrt{2\pi k}}ik \int_{\partial D_+} (\nu \cdot (-d)) e^{-ik(d \hat{y})} ds(y) \]  
(15)
and therefore, for \( \hat{x} = d \),
\[ e^{-i\pi/4}u_\infty(d; -d, k) = -\frac{1}{\sqrt{8\pi k}} \int_{\partial D_+} \frac{\partial e^{2ikd}}{\partial \nu(y)} ds(y) . \]  
(16)
By summing up (14) and (16) and by applying the first Green’s theorem, the Bojarsi’s identity (12) straightforwardly follows.

\[ \square \]

The boundary conditions (10) and (11) have an immediate physical interpretation. Let us consider a plane wave scattering according to an incident direction \( d \) against the infinite conducting plane described by the equation \( x \cdot \nu = 0 \), \( \nu \) being the outgoing unit vector in the same half plane containing the incident wave. Simple geometrical considerations imply that the derivative of the total field \( v \) is given by
\[ \frac{\partial v}{\partial \nu} = ik \nu \cdot d \{ e^{ikx \cdot d} + e^{ikx \cdot d} e^{-2i(\nu \cdot d)(k\nu)} \} = 2 e^{ikx \cdot d} \frac{\partial}{\partial \nu} . \]  
(17)
Therefore the boundary conditions (10) and (11) leading to the Bojarsi’s identity correspond to the physical situation whereby the wavelength of the incident wave is so small with respect to the dimension of the convex conducting scatterer that this one appears to the incident field as an infinite dimension conducting plane.

The main drawback of Bojarsi’s approach is that access on both sides of the scatterer must be allowed in order to reconstruct the characteristic function. Of course this is not possible in applications, such as Ground Penetrating Radar, where only limited aperture data are available. In the following, we will always consider full aperture data, leaving limited aperture data experiments to possible future research.

3. The inverse problem of physical optics

The Bojarsi’s identity (12) shows that the Fourier transform of the characteristic function of the obstacle can be obtained from measurements of the back scattering far-field for all incident directions and all wave numbers. In fact, with the definition \( \omega = -2kd \), equation (12) becomes
\[ \sqrt{4\pi |\omega|} \left[ e^{-i\pi/4}u_\infty \left( \frac{\omega}{|\omega|}, -\frac{\omega}{|\omega|}, \frac{|\omega|}{2} \right) + e^{-i\pi/4}u_\infty \left( -\frac{\omega}{|\omega|}, \frac{\omega}{|\omega|}, \frac{|\omega|}{2} \right) \right] = \hat{\chi}(\omega) , \]  
(18)
and in principle the problem of recovering the boundary of the scatterer corresponds to
the well-posed problem of computing the Fourier transform inversion of the function

$$h(\omega) = \sqrt{\frac{4\pi|\omega|}{|\omega|^2}} \left[ e^{-i\pi/4}u_\infty \left( \frac{\omega}{|\omega|}, \frac{\omega}{|\omega|}, \frac{|\omega|}{2} \right) + e^{-i\pi/4}u_\infty \left( -\frac{\omega}{|\omega|}, \frac{\omega}{|\omega|}, \frac{|\omega|}{2} \right) \right].$$

(19)

However equation (18) is a good description of the obstacle scattering only when the
wavelength $\lambda$ of the incident field is smaller than the linear dimension $D$ of the obstacle,
or, in terms of spatial frequencies, when condition

$$D|\omega| > 4\pi$$

(20)

holds. In other terms, the inverse problem of the physical optics approximation is the
one of determining the characteristic function of the scatterer from the knowledge of
its Fourier transform on a circular corona of spatial frequencies, where the size of the
inner radius is suggested by condition (20) and the size of the outer radius is imposed
by measurement limitations.

An integral formulation of this inverse problem is possible by introducing the linear
integral operator $A : L^2(D) \rightarrow L^2(\mathbb{R}^2)$, such that

$$(Af)(x) = \int_D H_B(x - x')f(x')dx', \quad x \in \mathbb{R}^2$$

(21)

where

$$H_B(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \chi_B(\omega)e^{i\omega\cdot x}d\omega$$

(22)

and $B$ is the circular corona of inner radius equal to $\omega_1$ and outer radius equal to
$\omega_2$. Therefore the inverse problem of the physical optics approximation is the one of
determining $f \in L^2(D)$ by solving

$$g = Af$$

(23)

where $g$ is a noisy version of the function in $L^2(\mathbb{R}^2)$ whose Fourier transform is

$$\hat{g}(\omega) = \left\{ \begin{array}{ll} h(\omega) & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{elsewhere} \end{array} \right. .$$

(24)

$A$ is a linear, injective operator whose range is the linear subspace of the band-limited
functions with band $B$. $A$ is also compact, since its Hilbert-Schmidt norm is given by

$$\|A\|_{HS} := \int_{\mathbb{R}^2} dx \int_D dx'|H(x - x')|^2 = \frac{1}{(2\pi)^2}m(B)m(D)$$

(25)

where $m(B)$ and $m(D)$ are the (finite) Lebesgue measures of $B$ and $D$ respectively.
The singular system of $A$ is related to the eigenfunctions and eigenvalues of the Slepian
operator [23] $A^*A$ in $L^2(D)$, where $A^*$ is the adjoint of $A$ and the explicit form for $A^*A$ is

$$(A^*Af)(x) = \int_D H_B(x - x')f(x')dx', \quad x \in D .$$

(26)
We denote with \( \{\lambda_k\}_{k=1}^\infty \) and \( \{\psi_k\}_{k=1}^\infty \) the eigenvalues and the eigenfunctions of the Slepian operator respectively. The \( \lambda_k \) are all positive while the \( \psi_k \) form a basis in \( L^2(D) \). By exploiting the fact that the integral kernel \( H_B(x) \) can be defined in all \( \mathbb{R}^2 \), the functions \( \psi_k \) can be extended to functions defined on \( \mathbb{R}^2 \), such that

\[
\psi_k(x) = \frac{1}{\lambda_k} \int_D H_B(x - x') \psi_k(x') \, dx' , \quad x \in \mathbb{R}^2 .
\]  

(27)

Then the following theorem can be easily proved \cite{23,14}:

**Theorem 3.1** Let us introduce the real positive numbers \( \sigma_k \), \( k = 1, \ldots, \infty \) such that \( \sigma_k = \sqrt{\lambda_k} \), the set of functions \( \{u_k\}_{k=1}^\infty \subset L^2(D) \) such that

\[
u_k(x) = \frac{1}{\sigma_k} \psi_k(x) , \quad x \in D
\]  

(28)

and the set of functions \( \{v_k\}_{k=1}^\infty \subset L^2(\mathbb{R}^2) \), such that

\[
v_k(x) = \tilde{\psi}_k(x) , \quad x \in \mathbb{R}^2
\]  

(29)

where the functions \( \tilde{\psi}_k(x) \) are obtained by applying an orthonormalization procedure to the set of functions defined in (27). Then the set of triples \( \{\sigma_k; u_k, v_k\}_{k=1}^\infty \) gives the singular system of the operator \( A \).

As a consequence of compactness, the problem of solving equation (23), (21), (22) is ill-posed in the sense of Hadamard \cite{16}. The numerical oscillations due to the presence of noise on the scattering data can be reduced by applying appropriate regularization methods where the trade-off between instability and data fitting is obtained by optimally tuning a regularization parameter. A classical approach is given by the Tikhonov method \cite{25}, which requires to solve the minimum problem

\[
\|Af - g\|_{L^2(\mathbb{R}^2)}^2 + \mu \|f\|_{L^2(D)}^2 = \text{minimum}
\]  

(30)

and to select the regularized solution by optimally fixing the parameter \( \mu \). However it is well-known \cite{1} that the Tikhonov regularization method can only provide a band-limited approximation of the true solution. On the contrary out-of-band extrapolation can be obtained by using inversion methods which explicitly account for \textit{a priori} knowledges on the scatterer through projections onto convex sets in the solution space \( L^2(D) \). An effective example is given by the projected Landweber method defined by the algorithm

\[
f_{n+1} = P_C(f_n + \tau(A^*g - A^*Af_n))
\]  

(31)

with \( f_0 = 0 \) and the relaxation parameter \( \tau \) such that

\[
0 < \tau < \frac{2}{\|A\|^2} .
\]  

(32)
The mapping $P_C$ is the projection operator onto the convex subset $C$ of the solution space containing all the functions satisfying some *a priori* known properties of the solution. When $C$ is the subspace of compactly supported functions, it can be shown [1] that the projected Landweber algorithm coincides with the Gerchberg method for extrapolation of band-limited signals [15]; when $C$ is the subset of positive functions, super-resolution effects show up often after many iterations [18] although preconditioned versions of the algorithm provide a significant acceleration without deteriorating the reconstruction quality [21]. Even better reconstructions can be obtained by combining both positivity and compact support constraints.

4. A test example

The Slepian operator is uniquely defined once $D$ and $B$ are given. In the case where $D$ is a circle of radius $R$ and $B$ is a circular corona of inner radius $\omega_1$ and outer radius $\omega_2$, the following result holds:

**Proposition 4.1** When $D$ is a circle of radius $R$ and $B$ is a circular corona of inner radius $\omega_1$ and outer radius $\omega_2$, the singular system of the operator $A$ defined in (21), (22) only depends on the two parameters $c_1 = R\omega_1$ and $c_2 = R\omega_2$.

**Proof** The characteristic function of the circular corona is the difference of the characteristic functions of the circle of radius $\omega_2$ and the circle of radius $\omega_1$. It follows that

$$H_B(x) = \frac{\omega_2}{2\pi} \frac{J_1(\omega_2 |x|)}{|x|} - \frac{\omega_1}{2\pi} \frac{J_1(\omega_1 |x|)}{|x|}. \quad (33)$$

Therefore, in equation (21), the use of the polar coordinates $x = (\rho \cos \theta, \rho \sin \theta)$, $x' = (\rho' \cos \theta', \rho' \sin \theta')$ and the change of variables $z = \rho / R$, $z' = \rho' / R$ implies that the Slepian operator depends on $c_1$ and $c_2$. The relation between the eigensystem of the Slepian operator and the singular system of $A$ concludes the proof.

For this geometry the far-field pattern corresponding to a perfectly conducting scatterer with Dirichlet boundary conditions is given by [9]

$$u_\infty(\phi; \theta, |\omega|) = -\sqrt{\frac{4}{\pi |\omega|}} e^{-i\pi/4} \sum_{n=-\infty}^{\infty} \frac{\hat{r}^n J_n(|\omega| R/2)}{H_n^{(1)}(|\omega| R/2)} e^{in(\theta - \phi)} e^{-in\pi/2}, \quad (34)$$
where \( J_n(x) \) is the Bessel function of the first kind of order \( n \) and \( H_n^{(1)}(x) \) is the Hankel function of the first kind of order \( n \). In this equation \( \phi \) denotes the observation angle, i.e. \( \hat{x} = (\cos \phi, \sin \phi) \) and \( \theta \) denotes the incident angle, i.e. \( d = (\cos \theta, \sin \theta) \). Furthermore, as defined in the previous section, the spatial frequencies \( \omega \) are related to the wavenumber \( k \) by \( k = |\omega|/2 \). The computation of the far-field pattern by truncating equation (34) up to an appropriate \( n \) leads to the data for the physical optics inverse problem. In Figure 1 we computed the matrix \( h(\omega) \) in (19) in the case \( R = 1 \), for \( 256 \times 256 \) uniformly sampled values of \( \omega \) over a square of side \( 2\omega_{\text{max}} \), \( \omega_{\text{max}} = 40 \), and compared the result with the Fourier transform of the unit circle (the central rows of the two matrices are represented). This figure points out also experimentally that physical optics is a high-frequency approximation. However, in the computations we expect that reliable restorations can be already obtained from frequencies smaller than the ones indicated by condition (20), since the use of an even badly approximated data set for some frequencies is in any case better than the absence of information in the same frequency range.

We applied the projected Landweber algorithm with both positivity and compact support constraints to the data matrix affected by 1% pixelwise added gaussian noise. The support is a disk of radius 2. In order to simulate a physical optics experiment, i.e. to compute \( \hat{g}(\omega) \) according to (24),(19), the matrix has been pixelwise multiplied by a mask which is 1 for \( \omega_1 < |\omega| < \omega_2 \) and zero elsewhere. The regularization parameter which, in the case of the projected Landweber method, is represented by the iteration number, is fixed by minimizing the root mean squared error (\( \text{rmse} \))

\[
\text{rmse} = \frac{\|f_n - f_0\|_F}{\|f_0\|_F},
\]

where \( f_0 \) is the characteristic function of the unit disk and \( \| \cdot \|_F \) is the Frobenius norm. The results of this numerical experiment are given in Table 1, for different values of the pair \([\omega_1, \omega_2]\) measuring the size of the band-pass filtering corona. These values have been chosen by observing that, for this geometry, condition (20) becomes \( |\omega| > 2\pi \) and that for each fixed value of \( \omega_2 \) the smallest \( \text{rmse} \) error is obtained with \( \omega \) equal to a fraction of \( 2\pi \) of the order of \( 1/4 \). The results in Table 1 clearly show that the restoration error slowly decreases with increasing \( \omega_2 \) but that unacceptably high errors are obtained for a frequency range completely out of the physical optics regime. In this table, the effect of the use of projections is also shown. Indeed Tikhonov method systematically provides worse reconstructions with respect to the projected Landweber scheme (we notice that the restoration errors given by the Landweber method without projection are essentially the same as for Tikhonov). Finally in Table 2, the same experiment is performed, this time with data produced by an 'inverse crime' procedure, i.e. by computing the Fourier transform of the unit disk and by multiplying it by the usual mask. The restoration errors are everywhere of the same order of magnitude except for the case \([0, 6]\), where the use of the wrong approximation is particularly influent.
Table 1. Restoration of the unit disk by means of the projected Landweber method for different sizes of the band-pass filtering corona. The data have been computed by means of equations (19), (24) and (34). 'p-Land' stands for projected Landweber method and 'Tik' stands for Tikhonov method.

<table>
<thead>
<tr>
<th>( \omega_1, \omega_2 )</th>
<th>rmse (p-Land)</th>
<th>rmse (Tik)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,6)</td>
<td>( \approx 1. )</td>
<td>( \approx 1. )</td>
</tr>
<tr>
<td>(1.5,12)</td>
<td>0.24</td>
<td>0.70</td>
</tr>
<tr>
<td>(1.5,18)</td>
<td>0.21</td>
<td>0.69</td>
</tr>
<tr>
<td>(1.5,24)</td>
<td>0.18</td>
<td>0.68</td>
</tr>
<tr>
<td>(1.5,30)</td>
<td>0.16</td>
<td>0.68</td>
</tr>
</tbody>
</table>

Table 2. Restoration of the unit disk by means of the projected Landweber method for different sizes of the band-pass filtering corona. The data have been computed by means of an 'inverse crime' procedure, i.e. by computing the Fourier transform of the unit disk. 'p-Land' stands for projected Landweber method.

<table>
<thead>
<tr>
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<tr>
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</tr>
<tr>
<td>(1.5,12)</td>
<td>0.18</td>
</tr>
<tr>
<td>(1.5,18)</td>
<td>0.14</td>
</tr>
<tr>
<td>(1.5,24)</td>
<td>0.10</td>
</tr>
<tr>
<td>(1.5,30)</td>
<td>0.06</td>
</tr>
</tbody>
</table>

5. Comparison with the linear sampling method

The linear sampling method is a linear method for the solution of inverse scattering problems in the resonance region. The method was formulated for the first time in [10] and rigorously proved in [5]. Reviews of the method in the case of acoustic and electromagnetic problems are in [6] and [8] respectively. At the basis of the method there is the far-field equation

\[
\int_{\Omega} u_{\infty}(\hat{x}; d, k) g_z(d) ds(d) = \Phi_{\infty,z}(\hat{x}) ,
\]

(36)

where \( \Omega = \{ d \in \mathbb{R}^2, |d| = 1 \} \) represents the set of all the incident directions, \( z \) is a point in \( \mathbb{R}^2 \) and \( \Phi_{\infty,z}(\hat{x}) \) is the far-field pattern of the fundamental solution of the Helmholtz equation. The solution of (36) does not exist in general, although an approximate solution exists which grows up for \( z \) approaching the boundary of the scatterer from inside and stays large outside. From a computational point of view, in [12] and [24]
it is shown how classical regularization schemes can be applied in order to produce stable approximate solutions of (36). The visualization algorithm based on the linear sampling method can be formulated as follows [12]: for each point of a grid containing the scatterer: 1) construct a one-parameter family of regularized solutions of (36); 2) select an optimal regularized solution by fixing the regularization parameter through some optimality criterion; 3) plot the norm of the regularized solution: the boundary of the scatterer will correspond to the points of the computational grid where this norm is large. Numerical and experimental validations of this method in the case of 2D and 3D electromagnetic scattering are given in [7, 24, 8]. In particular, these papers point out that the linear sampling method provides reliable reconstructions in a computationally efficient way but with coarse spatial resolution. The deterioration of the reconstructions due to the use of limited aperture data are described in [11, 8].

In the present section we want to compare the properties of this algorithm with respect to the projected Landweber algorithm with both positivity and compact support constraints applied to equation (23), (21), (22). Before describing some numerical experiments the following two remarks point out what we mean with the term 'comparison' in this context.

**Remark 5.1.** A scattering experiment is defined by choosing the incident direction and the wavenumber of the incident wave: given a pair \((k, d) \in \mathbb{R}_+ \times \mathbb{R}^2\), the far-field pattern is a function of the observation direction \(\hat{x} \in \Omega\). The physical optics approach (i.e., the regularized solution of the Bojarski identity by applying, for example, the projected Landweber method) and the linear sampling method use different kinds of data under different frequency regimes. In order to realize the physical optics conditions, one has to perform all the scattering experiments corresponding to all the incident directions and all the values of the wavenumbers where the approximation is reasonable. However, the corresponding far-field patterns are measured only in the back scattering direction. On the other hand, in the linear sampling method the far-field patterns corresponding to all the incident directions and only one wavenumber are measured everywhere. Table 3 summarizes the experimental conditions corresponding to the two approaches followed in this paper. In [4] the weak scattering approach, with distributions as unknowns, and the linear sampling method are applied to the same sets of data and the corresponding reconstructions systematically compared.

**Remark 5.2.** Both approaches require the solution of a linear Fredholm integral equation of the first kind. However the solution of the Bojarski's identity directly leads to the reconstruction of the obstacle boundary while, in the case of the linear
sampling method, the far-field equation must be solved many times, for all the points of the computational grid, in order to obtain a visualization of the scatterer.

Therefore these two approaches cannot be directly compared. However they can be effectively used under complementary experimental conditions, and, which is more important, they can be combined to improve the restoration accuracy.

**Table 3.** Values of $d$, $k$, and $\hat{x}$ describing the experimental conditions where the physical optics and the linear sampling method approaches are applicable.

<table>
<thead>
<tr>
<th>Method</th>
<th>$d$</th>
<th>$k$</th>
<th>$\hat{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>physical optics</td>
<td>$\Omega$</td>
<td>$[k_1, k_2] \in \mathbb{R}$</td>
<td>$-d$</td>
</tr>
<tr>
<td>linear sampling</td>
<td>$\Omega$</td>
<td>$k = \bar{k}$</td>
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Our first numerical experiment concerns the reconstruction of an ellipse with semiaxiss $R_1 = 1$ and $R_2 = 2$ respectively. The scattering data have been computed by means of a code implementing the Nyström method [9] and are affected by 1% gaussian noise. For the physical optics approach we considered a grid of 512x512 uniformly sampled spatial frequencies on a squared grid with side $2\omega_{max}$, $\omega_{max} = 80$ (the Nyström data are computed on a disk in the frequency domain and interpolated to the squared grid by means of cubic splines). For the linear sampling method approach we used the Tikhonov algorithm to solve the far-field equation on a computational grid of 61x61 uniformly sampled points; the far-field pattern is computed for 32 observation angles and 32 incident angles, uniformly spaced over $[0, 2\pi]$. The results of this experiment are in Figure 2: Figure 2(a) contains the original profile to be restored; Figure 2(b) and Figure 2(c) show the reconstruction provided by the physical optics approach when the coronal mask is characterized by radii $\omega_1 = 0$, $\omega_1 = 6$ and $\omega_1 = 1.5$, $\omega_2 = 30$ respectively; the support constraint is given by a disk of radius 3. Finally, Figure 2(d) shows the reconstruction provided by the linear sampling method for $k = 0.45$, i.e. $|\omega| = 0.9$. The solid curve corresponds to the profile selected by means of an edge detection algorithm (Canny algorithm, [22]). These restorations clearly show that the effectiveness of the linear sampling method in the case of regimes out of the physical optics conditions is enhanced by the use of the edge-detection post-processing step (although, even without post-processing, the linear sampling method is able to coarsely visualize the shape of the scatterer within a computational time of a few second). The out-of-band extrapolation properties of the projected Landweber method when the physical optics conditions are fulfilled are also pointed out.
The physical condition of high-frequency scattering is much more influential than the geometrical condition which consists in assuming that the scatterer is a convex object. To show this we consider the case of the kite profile described by the parametrized equation

\[ x(t) = (a_1 \sin(t), \cos(t) + a_2 \cos(2t) - a_2) \quad t \in [0, 2\pi] \]  

The two parameters \( a_1 \) and \( a_2 \) can tune the convexity of the kite. For example, for \( a_1 = 1.5 \) and \( a_2 = 0.65 \) the shape of the scatterer is in Figure 3(a), i.e. the kite is a concave object. To reconstruct this kite, the projected Landweber method has been applied to the physical optics data in the cases \( \omega_1 = 0, \omega_2 = 6 \) and \( \omega_1 = 1.5, \omega_2 = 30 \) with the results in Figure 3(b) and Figure 3(c) respectively (the support is the disk of radius 3); in the high-frequency range, concavity results in some more intense ringing in the bottom (concave) region of the object. Figure 3(d) shows the performance of the linear sampling method for \( k = 0.85 \), i.e. again out of the physical optics regimes. Again edge-detection has been used to select an optimal profile.

Under physical optics conditions the projected Landweber algorithm and the linear sampling method can be used together. More precisely, the linear sampling method can be used to infer information about the support function of the scatterer. The procedure could be the following one:

(i) a fixed-frequency scattering experiment is performed where the far-field pattern is measured for all incident and observation angles and only one wave number;

(ii) the linear sampling method is applied and an assessment of the support function of the scatterer is made;

(iii) a new scattering experiment is performed, where all the incident directions and all the wavenumbers for which the physical optics approximation is valid are used, and where the far-field patterns are measured only in the back-scattering direction;

(iv) finally the projected Landweber method is applied, where the support constraint is based on the reconstruction provided by the linear sampling method in step (ii). That is, at each iteration, all pixels outside the profile selected by means of the linear sampling method are set to zero while the other pixels are kept untouched. The positivity constraint is implemented analogously.

The result of this procedure is shown in Figure 4 and Figure 5. In Figure 4 a convex version of the kite in equation (37), with \( a_1 = 1.5 \) and \( a_2 = 0.2 \) (Figure 4(a)) is reconstructed first by using the linear sampling method in the case \( k = 0.5 \), with 32 incident waves and 32 observation angles and 1% gaussian noise added to the far-field matrix entries. The result is given in Figure 4(b), where the usual edge detection procedure provides an estimate of the contour of the kite. Then we perform a physical
optics experiment in the range of frequencies \( \omega_1 = 1.5, \omega_2 = 30 \). The computational grid in the frequency domain is a square with side equal to 80 and 512x512 sampled points. The profile provided in Figure 4(b) is used (after some interpolation) as a support constraint for the projected Landweber method applied to the physical optics data. The restoration error obtained in this way is 0.22.

This combined use of the two methods is particularly effective in the restoration of two objects, when super-resolution effects can remove artifacts due to inner diffraction between the scatterers. In Figure 5 we restore the two convex kites in Figure 5(a). The procedure is exactly the same as in Figure 4. In Figure 5(b) the linear sampling method plus edge detection (this time with \( k = 1 \)) provides the support constraint for the application of the projected Landweber algorithm to physical optics data. The result in Figure 5(c) is rather satisfactory, particularly if compared to the reconstruction in Figure 5(d) which has been obtained by using the Tikhonov method, i.e. with no constraint applied in the regularization procedure. The restoration errors are 0.26 in the case of the projected Landweber method and 0.82 in the case of the Tikhonov method. The restoration with the combined use of the two approaches allowed us to reduce the phantoms between the objects obtained with both the linear sampling method and the Tikhonov method. Of course this procedure is the more efficient the more precise the support description given by the linear sampling method is. However, in the numerical cases we addressed, the linear sampling method together with the edge detection procedure always provided an over-estimate of the support, so that the objects are rather precisely reconstructed, although small parts of the artifacts are still present in the visualization.

6. Concluding remarks

The physical optics approach presents the following advantages with respect to the linear sampling method:

- it is a reconstruction method, in the sense that it succeeds in estimating the characteristic function of the object. On the contrary, the linear sampling method is only able to show the size and shape of the object and in this sense it is essentially a visualization method;

- prior knowledges on the scatterer can be exploited in the reconstruction procedure to obtain super-resolution effects. In the linear sampling method the solution of the far-field equation is a function in \( L^2(\Omega) \), neither positive nor compactly supported, and therefore convex constraints cannot be used in the regularization of the integral equation.

On the other hand, the linear sampling method presents the following advantages with respect to the physical optics approach:
it can be applied under more general conditions, even under resonance regimes. On the contrary, physical optics is exclusively a high-frequency method;

- it does not require any information on the geometrical or physical properties of the scatterer, such as whether the object is convex or which boundary condition the total field satisfies on the boundary. On the contrary, physical optics concerns convex sound-soft obstacles;

- it is computationally effective, while the use of convex constraints in the iterative scheme of the physical optics approach typically increases (even notably) the numerical heaviness of the inversion procedure.

As suggested at the end of the previous section, a combined use of the two approaches seems to be particularly effective, when the physical optics conditions are satisfied: the application of the linear sampling method fast provides information of the support of the scatterer, which can be used to obtain super-resolution effects in the out-of-band extrapolation procedure addressed by means of the iterative scheme.

References


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Figure 1. Scattering data for the conducting unit disk. Comparison between the central rows of the Fourier transform of the scatterer (solid) and of $h(\omega)$ in equation (19) (dashed). The Fourier transform has been computed on a 256x256 grid in the frequency range $[-40, 40]$. 
Figure 2. Reconstruction of the conducting ellipse (a); application of the projected Landweber method to the linearized inverse problem when the physical optics condition is not fulfilled (frequency range $\omega_1 = 0, \omega_2 = 6$) (b) and under physical optics conditions (frequency range $\omega_1 = 1.5, \omega_2 = 30$) (c); application of the linear sampling method ($k = 0.45$) with an edge-detection algorithm for selecting an optimal profile (d). In both (b) and (c) the support constraint is a disk of radius 3.
Figure 3. Reconstruction of the conducting kite (a); application of the projected Landweber method to the linearized inverse problem when the physical optics condition is not fulfilled (frequency range $\omega_1 = 0$, $\omega_2 = 6$) (b) and under physical optics conditions (frequency range $\omega_1 = 1.5$, $\omega_2 = 30$) (c); application of the linear sampling method ($k = 0.85$) with and edge-detection algorithm for selecting an optimal profile (d). In both (b) and (c) the support constraint is a disk of radius $3$. 
Figure 4. Combined use of the linear sampling method and the projected Landweber algorithm in the case of physical optics scattering. The convex kite in (a) is reconstructed in (b) by using the linear sampling method \( k = 0.5 \) and in (c) by constraining the iterative scheme with the profile obtained in (b) (frequency range \( \omega_1 = 1.5, \omega_2 = 30 \)).
Figure 5. Combined use of the linear sampling method and the projected Landweber algorithm in the case of physical optics scattering. The two convex kites in (a) are reconstructed in (b) by using the linear sampling method \((k = 1)\) and in (c) by constraining the iterative scheme with the profile obtained in (b) (frequency range \(\omega_1 = 1.5, \omega_2 = 30\)). In (d) the same objects are reconstructed by using the Tikhonov method, i.e. with no convex projection in the regularization procedure.