Dual graphs of projective schemes

Matteo Varbaro (University of Genova)

August 26th, Haeundae, Busan, KOREA
Let $X \subseteq \mathbb{P}^n$ be a projective scheme over $K = \overline{K}$.
Motivations

Let $X \subseteq \mathbb{P}^n$ be a projective scheme over $K = \overline{K}$.

The main motivation for this talk comes from the desire of understanding how global properties of $X \subseteq \mathbb{P}^n$ influence the combinatorial configuration of its irreducible components.
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The main motivation for this talk comes from the desire of understanding how global properties of $X \subseteq \mathbb{P}^n$ influence the combinatorial configuration of its irreducible components.

One way to make precise the concept of “combinatorial configuration of its irreducible components” is by meaning of the dual graph of $X$ ....
Given $X \subseteq \mathbb{P}^n$, if $X_1, \ldots, X_s$ are its irreducible components, we form the **dual graph** $G(X)$ as follows:

The vertex set of $G(X)$ is $\{1, \ldots, s\}$.

Two vertices $i \neq j$ are connected by an edge if and only if:

$$\dim(X_i \cap X_j) = \dim(X) - 1.$$ 

From now on we will consider only equidimensional schemes.

**Note:** If $X$ is a projective curve, then $\{i, j\}$ is an edge if and only if $X_i \cap X_j \neq \emptyset$ (the empty set has dimension $-1$).

If $\dim(X) > 1$, by intersecting $X \subseteq \mathbb{P}^n$ with a generic hyperplane, we get a projective scheme in $\mathbb{P}^{n-1}$ of dimension one less, and same dual graph! Iterating this trick we can often reduce questions to curves.
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Hartshorne’s connectedness theorem

Given $X \subseteq \mathbb{P}^n$ and the unique saturated homogeneous ideal $I_X \subseteq S = K[x_0, \ldots, x_n]$ s.t. $X = \text{Proj}(S/I_X)$, let us recall that $X \subseteq \mathbb{P}^n$ is **arithmetically Cohen-Macaulay** (resp. **arithmetically Gorenstein**) if $S/I_X$ is Cohen–Macaulay (resp. Gorenstein).

A classical result by Hartshorne is that aCM schemes are connected in codimension one:

**Hartshorne’s connectedness theorem**

If $X \subseteq \mathbb{P}^n$ is aCM, then $G(X)$ is a connected graph.

On the other hand ......
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Theorem A, Benedetti-Bolognese-V. 2015

For any connected graph $G$, there exists a reduced aCM curve $C \subseteq \mathbb{P}^n$ such that $G(C) = G$.

Furthermore, $\text{reg}(C) = \text{reg}({I}_C) = 3$ and the irreducible components of $C$ are rational normal curves no 3 of which meet at one point.

Moreover:

Benedetti-Bolognese-V. 2015

For a connected graph $G$, the following are equivalent:

1. There is a curve $C \subseteq \mathbb{P}^n$ such that no 3 of its irreducible components meet at one point, $\text{reg}(C) = 2$, and $G(C) = G$.
2. $G$ is a tree.
For any connected graph $G$, there exists a reduced aCM curve $C \subseteq \mathbb{P}^n$ such that $G(C) = G$. Furthermore, $\text{reg}(C) = \text{reg}(I_C) = 3$ and the irreducible components of $C$ are rational normal curves no 3 of which meet at one point.
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From graphs to curves

**Theorem A, Benedetti-Bolognese-V. 2015**

For any connected graph $G$, there exists a reduced ACM curve $C \subseteq \mathbb{P}^n$ such that $G(C) = G$. Furthermore, $\text{reg}(C) = \text{reg}(I_C) = 3$ and the irreducible components of $C$ are rational normal curves no 3 of which meet at one point.

Moreover:

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For a connected graph $G$, the following are equivalent:

- There is a curve $C \subseteq \mathbb{P}^n$ such that no 3 of its irreducible components meet at one point, $\text{reg}(C) = 2$, and $G(C) = G$.
- $G$ is a tree.
From graphs to curves

Notice that not any graph can be obtained as the dual graph of a line arrangement $C = \bigcup_{i=1}^{n} L_i$. For example, one can see that the graph $G$ having:

- $\{1, \ldots, 6\}$ as vertices;
- $\{\{i, j\} : 1 \leq i < j \leq 6\}$ as edges

is not the dual graph of any line arrangement.

However, by taking $6$ generic lines $L_i \subseteq \mathbb{P}^2$ and blowing up $\mathbb{P}^2$ along the points $P_{1,2} = L_1 \cap L_2$ and $P_{3,4} = L_3 \cap L_4$, the strict transform of $\bigcup_{i=1}^{n} L_i$ will have $G$ as dual graph!
Notice that not any graph can be obtained as the dual graph of a line arrangement $C$, that is a union of lines $C = \bigcup_{i=1}^{s} L_i$. 

For example, one can see that the graph $G$ having:

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A graph is $d$-connected if it has $>d$ vertices, and the deletion of $<d$ vertices, however chosen, leaves it connected. Menger theorem (Max-flow-min-cut). A graph is $d$-connected iff between any 2 vertices one can find $d$ vertex-disjoint paths.

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**Menger theorem (Max-flow-min-cut).**

A graph is $d$-connected iff between any 2 vertices one can find $d$ vertex-disjoint paths.
Theorem B, Benedetti–Bolognese–V. 2015

Let $X \subseteq \mathbb{P}^n$ be an arithmetically Gorenstein projective scheme such that $\text{reg}(X) = \text{reg}(I_X) = r + 1$. If $\text{reg}(q) \leq \delta$ for all primary components $q$ of $I_X$, then $G(X)$ is $\left\lfloor \frac{(r + \delta - 1)}{\delta} \right\rfloor$-connected.

When $\delta$ can be chosen to be 1, i.e. when $X$ is a (reduced) union of linear spaces (a subspace arrangement), we recover the following:

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The important things to know are that:

1. $G(C)$ is 10-connected.
2. The diameter of $G(C)$ is 2.
3. There is a partition $V_1, \ldots, V_9$ of the nodes of $G(C)$ such that the induced subgraph of $G(C)$ on each $V_i$ is a triangle.
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$$C \subseteq \left( \bigcup_{i=1}^{9} H_i \right) \cap X \quad \text{and} \quad (f, g) \subseteq I_C,$$

where $f$ is the cubic polynomial defining $X$ and $g = \prod_{i=1}^{9} \ell_i$. 

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In particular $C \subseteq \mathbb{P}^3$ is an an arithmetically Gorenstein subspace arrangement of regularity $\deg(f) + \deg(g) - 1 = 3 + 9 - 1 = 11$. Thus our result confirms that $G(C)$ is 10-connected.
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If $I \subseteq S = K[x_0, \ldots, x_n]$ is a height 2 monomial ideal, it is easy to show that, if $S/I$ is Cohen-Macaulay, then $\text{diam}(G(X)) \leq 2$. 
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We know many aCM line arrangements in $\mathbb{P}^3$ not arising like this (e.g. the previous 27 lines), but still their dual graph has diameter $\leq 2$ (many experiments by Michela Di Marca).
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**Question**

Is $\text{diam}(G(C)) \leq 2$ for any aCM line arrangement $C \subseteq \mathbb{P}^3$?
We say that a projective scheme $X \subseteq \mathbb{P}^n$ is Hirsch if

$$\text{diam}(G(X)) \leq \text{codim}_{\mathbb{P}^n} X.$$
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The previous question, thus, can be rephrased as:

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Hirsch embeddings

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Be careful:

- There exist nonreduced complete intersections $C \subseteq \mathbb{P}^3$ such that $C_{\text{red}} \subseteq \mathbb{P}^3$ is a line arrangement and $\text{diam}(G(C))$ is arbitrarily large.

- For large $n$, there are arithmetically Gorenstein line arrangements that are not Hirsch (Santos).
Many projective embeddings, however, are Hirsch:

**Adiprasito–Benedetti 2014**

If $X \subseteq \mathbb{P}^n$ is aCM and $I_X$ is a monomial ideal generated by quadrics, then $X \subseteq \mathbb{P}^n$ is Hirsch.
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If $X$ is an arrangement of lines, no 3 of which meet in the same point, canonically embedded in $\mathbb{P}^n$, then $X \subseteq \mathbb{P}^n$ is Hirsch.
Many projective embeddings, however, are Hirsch:

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**Conjecture: Benedetti–V. 2014**

If $X \subseteq \mathbb{P}^n$ is a (reduced) aCM scheme and $I_X$ is generated by quadrics, then $X \subseteq \mathbb{P}^n$ is Hirsch.
Sketch of the proof of Theorem B

By taking generic hyperplane sections, we can reduce ourselves to consider \( \dim X = 1 \).

If \( I = \cap_{i=1}^s q_i \) is a primary decomposition of a homogeneous ideal \( I \subseteq S = K[ x_0, \ldots, x_n ] \) and \( \text{Proj}( S/I ) \) has dimension 1, then:

\[
\text{reg}( I ) \leq \sum_{i=1}^s \text{reg}( q_i ) .
\]

Let \( I_X = \cap_{i=1}^s q_i \) be the primary decomposition of \( I_X \), choose \( A \subseteq \{1, \ldots, s\} \) of cardinality less than \( \lfloor \left( r + \delta - 1 \right) / \delta \rfloor \) and let \( B = \{1, \ldots, s\} \setminus A \).

Let \( I_A = \cap_{i \in A} q_i \), \( I_B = \cap_{i \in B} q_i \) and \( X_A = \text{Proj}( S/I_A ) \), \( X_B = \text{Proj}( S/I_B ) \).
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Let $I_A = \bigcap_{i \in A} q_i$, $I_B = \bigcap_{i \in B} q_i$ and $X_A = \Proj(S/I_A)$, $X_B = \Proj(S/I_B)$.
By taking generic hyperplane sections, we can reduce ourselves to consider $\dim X = 1$.

Caviglia 2007

If $I = \bigcap_{i=1}^{s} q_i$ is a primary decomposition of a homogeneous ideal $I \subseteq S = K[x_0, \ldots, x_n]$ and $\operatorname{Proj}(S/I)$ has dimension 1, then:

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By taking generic hyperplane sections, we can reduce ourselves to consider \( \dim X = 1 \).

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If \( I = \cap_{i=1}^{s} q_i \) is a primary decomposition of a homogeneous ideal \( I \subseteq S = K[x_0, \ldots, x_n] \) and \( \text{Proj}(S/I) \) has dimension 1, then:

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$X_A$ and $X_B$ are geometrically linked by $X$ which is a Gorenstein; so by a result of Hartshorne and Schenzel, we have a graded isomorphism

$$H^1_m(S/I_B) \cong H^1_m(S/I_A)^\vee(2 - r).$$
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By Caviglia’s result, $\text{reg}(I_A) \leq |A|\delta \leq r - 1$. 
1. $X_A$ and $X_B$ are geometrically linked by $X$ which is a Gorenstein; so by a result of Hartshorne and Schenzel, we have a graded isomorphism

$$H_1^m(S/I_B) \cong H_1^m(S/I_A)^\vee(2 - r).$$

2. By Caviglia’s result, $\text{reg}(I_A) \leq |A|\delta \leq r - 1$.

3. So $\text{reg}(S/I_A) \leq r - 2$, which implies that $H_1^m(S/I_A)^{r-2} = 0$. 

But then the dual graph of $X_B$, which is the same as the dual graph of $X$ with the vertices of $A$ removed, is connected.
1. $X_A$ and $X_B$ are geometrically linked by $X$ which is Gorenstein; so by a result of Hartshorne and Schenzel, we have a graded isomorphism

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4. So $H_m^1(S/I_B)_0 = H_m^1(S/I_A)_{r-2} = 0$, that is $H^0(X_B, \mathcal{O}_{X_B}) \cong \mathbb{K}$, which implies that $X_B$ is a connected curve.
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5. But then the dual graph of \( X_B \), which is the same as the dual graph of \( X \) with the vertices of \( A \) removed, is connected.
An 'Eisenbud-Goto style' question

Let $X \subseteq \mathbb{P}^n$ be a nondegenerate reduced projective scheme with connected dual graph. Then $\text{reg}(X) \leq \deg(X) - \text{codim}\mathbb{P}^n_X + 1$.

The conjecture is known to be true in its full generality in dimension 1 by Gruson-Lazarsfeld-Peskine and Giaimo; in dimension 2, it is true for smooth surfaces by Lazarsfeld; for smooth threefolds and fourfolds, it is 'almost' true by Kwak.

By the subadditivity result of Caviglia, the EG for curves yields:

**Theorem**

Let $X \subseteq \mathbb{P}^n$ be an equidimensional reduced projective curve. Then $\text{reg}(X) \leq \deg(X)$.
Eisenbud-Goto conjecture (1984)

Let $X \subseteq \mathbb{P}^n$ be a nondegenerate reduced projective scheme with connected dual graph. Then

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Question

Let $X \subseteq \mathbb{P}^n$ be an equidimensional reduced projective scheme. Is it true that:

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If $\text{dim}(X) = 2$, the subadditivity result of Caviglia is not true. However, it is still true that, if $X_1$ and $X_2$ are projective schemes intersecting in dimension 0, then

$$\text{reg}(X_1 \cap X_2) \leq \text{reg}(X_1) + \text{reg}(X_2).$$

This implies that the question above would admit a positive answer in dimension 2 if the EG conjecture was true in dimension 2 in its full generality (not only for irreducible surfaces).
Let $X \subseteq \mathbb{P}^n$ be an equidimensional reduced projective scheme. Is it true that:

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