Cohomological and projective dimensions
Let $R$ be a ring, $I \subset R$ an ideal and $M$ an $R$-module. By $H^i_I(M)$ we mean the $i$th local cohomology module of $M$ with support in $I$. One way to think at it is by the following isomorphism:

$$H^i_I(M) \cong \lim_{\rightarrow} \text{Ext}^i_R(R/I_n, M)$$

where $(I_n)_{n \in \mathbb{N}}$ is an inverse system of ideals cofinal with $(I^n)_{n \in \mathbb{N}}$:

$$\forall \ n \in \mathbb{N} \quad I_{n+1} \subset I_n \quad \text{and} \quad \exists \ k, m \in \mathbb{N} : \ I_k \subset I^n \quad \text{and} \quad I^m \subset I_n$$
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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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It is not difficult to prove that:

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The proof is easy; for all $e \in \mathbb{N}$:

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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On the other hand \( \text{pd}(R/I) = (r - t + 1)(s - t + 1) \), so:

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\text{cd}(R, I) > \text{pd}(R/I) \text{ (a part from trivial cases).}
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As one can check, for all \( p \leq n - 4 \), this provides examples of graded ideals \( I \subset R \) for which \( \text{cd}(R, I) > \text{pd}(R/I) = p \).

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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COHOMOLOGICAL AND PROJECTIVE DIMENSIONS

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**Hartshorne:** $H^i_{DR}(X) \cong H^i(X_h, \mathbb{C})$ (singular cohomology).

$\text{depth}(R/I) \geq 3 \implies X$ is connected. Moreover it is well known:

$$X \text{ connected (Zariski)} \iff X_h \text{ connected (euclidean)}$$

Thus $H^0_{DR}(X) \cong H^0(X_h, \mathbb{C}) \cong \mathbb{C}$ ((ii) $\checkmark$).

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So it remains to show (**iii**), i.e. $H^1(X_h, \mathbb{C}) = 0$. 

The proof of (iii) relies on the celebrated exponential sequence:

$$0 \rightarrow \mathbb{Z}_{X_h} \cdot 2\pi i \rightarrow \mathcal{O}_{X_h} \xrightarrow{\exp_{X_h}} \mathcal{O}_{X_h}^* \rightarrow 0.$$ 

This is well known, but references can be found only if $X$ is reduced (I radical), so I would like to explain it in the general case.
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This is well known, but references can be found only if $X$ is reduced ($I$ radical), so I would like to explain it in the general case.
First of all the problem is local. Therefore we can assume that $X \subset \mathbb{A}^n$ is affine. So we have the maps:

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}^n} & \xrightarrow{\exp_{\mathbb{C}^n}} & \mathcal{O}_{\mathbb{C}^n}^* \\
\downarrow & & \downarrow \\
\mathcal{O}_{X_h} & \xrightarrow{\exp_{X_h}} & \mathcal{O}_{X_h}^*
\end{array}
$$

where the vertical maps are the natural projections. Notice that all the above maps are surjective! We want to show that:

$$
\mathcal{O}_{X_h} \xrightarrow{\exp_{X_h}} \mathcal{O}_{X_h}^*,
$$

where $\exp_{X_h}(f) = \exp_{\mathbb{C}^n}(f)$, is a well-defined map.
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$$O_{\mathbb{C}^n} \xrightarrow{\operatorname{exp}_{\mathbb{C}^n}} O_{\mathbb{C}^n}^*$$

$$\downarrow \quad \quad \quad \downarrow$$

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$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}^n} & \overset{\exp_{\mathbb{C}^n}}{\longrightarrow} & \mathcal{O}^*_{\mathbb{C}^n} \\
\downarrow & & \downarrow \\
\mathcal{O}_{X_h} & \rightarrow & \mathcal{O}^*_{X_h}
\end{array}
$$

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where $\exp_{X_h}(\bar{f}) = \exp_{\mathbb{C}^n}(\bar{f})$, is a well-defined map.
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\[
\begin{array}{c}
\mathcal{O}_{\mathbb{C}^n} \xrightarrow{\exp_{\mathbb{C}^n}} \mathcal{O}_{\mathbb{C}^n}^* \\
\downarrow \quad \downarrow \\
\mathcal{O}_X^h \quad \mathcal{O}_X^h^*
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\]

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$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}^n} & \xrightarrow{\exp_{\mathbb{C}^n}} & \mathcal{O}^*_{\mathbb{C}^n} \\
\downarrow & & \downarrow \\
\mathcal{O}_X^h & & \mathcal{O}^*_X^h
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$$

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\mathcal{O}_{X_h} & & \mathcal{O}_{X_h}^*
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where $\exp_{X_h}(\bar{f}) = \exp_{\mathbb{C}^n}(\bar{f})$, is a well-defined map.
To show that $\exp_{X_h}$ is well-defined we can argue on the stalks. Let $P$ be a point of $X_h$. We can assume that $P = 0$. So let $a \subset A = \mathbb{C}\{x_1, \ldots, x_n\}$ be so that $\mathcal{O}_{\mathbb{C}^n,0} \cong A$ and $\mathcal{O}_{X_h,0} \cong A/a$. Let $f \in a$:

$$\exp_{\mathcal{O}_{\mathbb{C}^n,0}}(f) - 1 = \sum_{m \geq 1} \frac{f^m}{m!} \in a.$$ 

So it makes sense to write the commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}^n} & \xrightarrow{\exp} & \mathcal{O}_{\mathbb{C}^n} \\
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\mathcal{O}_{X_h} & \xrightarrow{\exp} & \mathcal{O}_{X_h}
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Notice that $\exp_{X_h}$ is surjective.
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\downarrow & & \downarrow \\
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\end{array}$$

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So it makes sense to write the commutative diagram:

\[ \begin{array}{ccc}
\mathcal{O}_{\mathbb{C}^n} & \xrightarrow{\exp_{\mathbb{C}^n}} & \mathcal{O}_{\mathbb{C}^n}^* \\
\downarrow & & \downarrow \\
\mathcal{O}_{X_h} & \xrightarrow{\exp_{X_h}} & \mathcal{O}_{X_h}^* 
\end{array} \]

Notice that $\exp_{X_h}$ is surjective.
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\downarrow & & \downarrow \\
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$$\exp_{\mathbb{C}^n,0}(f) - 1 = \sum_{m \geq 1} \frac{f^m}{m!} \in a.$$ 

So it makes sense to write the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}^n} & \xrightarrow{\exp_{\mathbb{C}^n}} & \mathcal{O}^*_{\mathbb{C}^n} \\
\downarrow & & \downarrow \\
\mathcal{O}_{X_h} & \xrightarrow{\exp_{X_h}} & \mathcal{O}^*_h
\end{array}
\]

Notice that $\exp_{X_h}$ is surjective.
Now we want to show that:

\[ 0 \to \mathbb{Z}X_h \cdot 2\pi i \to \mathcal{O}_X^* \to \mathcal{O}_h \to 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{C_n,0}(f) - 1 \in \mathfrak{a} \). Then \( \exp_{C_n,0}(f) - 1 \in \sqrt{\mathfrak{a}} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{\mathfrak{a}} \). But

\[ \exp_{C_n,0}(f') - 1 = \sum_{m \geq 1} f'^m/m! = f'^{-1} \cdot (1 + \sum_{m \geq 1} f'^m/(m + 1)!)) = 1. \]

The element \( g = \sum_{m \geq 1} f'^m/(m + 1)! \in \sqrt{\mathfrak{a}} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( \mathfrak{a} \).
Now we want to show that:

\[ 0 \to \mathbb{Z} X_h \cdot 2\pi i \to \mathcal{O} X_h \xrightarrow{\exp_{X_h}} \mathcal{O}^*_X \to 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in a \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{a} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{a} \). But

\[
\exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = i^k \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)! ) \in a.
\]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{a} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( a \).
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is exact. For the discussion above we have just to show exactness in the middle. Let $f \in A$ such that $\exp_{n,0}(f) - 1 \in \alpha$. Then $\exp_{n,0}(f) - 1 \in \sqrt{\alpha}$. Since the above sequence is exact if $X$ is reduced, there exist $k \in \mathbb{Z}$ such that $f' = f - 2\pi ik \in \sqrt{\alpha}$. But

$$\exp_{n,0}(f') - 1 = \sum_{m \geq 1} f'^m/m! = ik \cdot (1 + \sum_{m \geq 1} f'^m/(m+1)! ) \in \alpha.$$ 

The element $g = \sum_{m \geq 1} f'^m/(m+1)! \in \sqrt{\alpha}$. This means that $1 + g$ is invertible in $A$, so $f'$ is actually an element of $\alpha$. 


Now we want to show that:

\[ 0 \rightarrow \mathbb{Z}X_h \xrightarrow{2\pi i} \mathcal{O}_{X_h} \xrightarrow{\exp_{X_h}} \mathcal{O}_{X_h}^* \rightarrow 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^*,0}(f) - 1 \in a \). Then \( \exp_{\mathbb{C}^*,0}(f) - 1 \in \sqrt{a} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{a} \). But

\[ \exp_{\mathbb{C}^*,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = i^k \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)!) = 0. \]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{a} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( a \).
Now we want to show that:

\[ 0 \to \mathbb{Z}X_h \xrightarrow{2\pi i} \mathcal{O}X_h \xrightarrow{\exp X_h} \mathcal{O}^*_X \to 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \mathfrak{a} \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{\mathfrak{a}} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{\mathfrak{a}} \). But

\[
\exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = f' \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)!) = \mathfrak{a}
\]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{\mathfrak{a}} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( \mathfrak{a} \).
Now we want to show that:

\[ 0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{C^n,0}(f) - 1 \in \mathfrak{a} \). Then \( \exp_{C^n,0}(f) - 1 \in \sqrt{\mathfrak{a}} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{\mathfrak{a}} \). But

\[ \exp_{C^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = f' \cdot (1 + \sum_{m \geq 1} f'^m / (m+1)!) = 1. \]

The element \( g = \sum_{m \geq 1} f'^m / (m+1)! \in \sqrt{\mathfrak{a}} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( \mathfrak{a} \).
Now we want to show that:

\[ 0 \to \mathbb{Z} \chi_h \cdot 2\pi i \to \mathcal{O} \chi_h \xrightarrow{\exp_{\chi_h}} \mathcal{O}^*_\chi_h \to 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \mathfrak{a} \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{\mathfrak{a}} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{\mathfrak{a}} \). But

\[
\exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} \frac{f'^m}{m!} = f' \cdot (1 + \sum_{m \geq 1} \frac{f'^m}{(m+1)!}) \in \mathfrak{a}.
\]

The element \( g = \sum_{m \geq 1} \frac{f'^m}{(m+1)!} \in \sqrt{\mathfrak{a}} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( \mathfrak{a} \).
Now we want to show that:

\[ 0 \to \mathbb{Z} \times_h \cdot 2\pi i \to \mathcal{O}_X \times_h \exp_{X_h} \to \mathcal{O}_{X_h}^* \to 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in a \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{a} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{a} \). But

\[
\exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = f' \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)!) \in a.
\]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{a} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( a \).
Now we want to show that:

\[ 0 \rightarrow \mathbb{Z}X_h \xrightarrow{\cdot 2\pi i} \mathcal{O}X_h \xrightarrow{\exp X_h} \mathcal{O}^*_X \rightarrow 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \mathfrak{a} \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{\mathfrak{a}} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi i k \in \sqrt{\mathfrak{a}} \). But

\[ \exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = f' \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)!) \in \mathfrak{a}. \]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{\mathfrak{a}} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( \mathfrak{a} \).
Now we want to show that:

\[ 0 \rightarrow \mathbb{Z} \chi_{X_{h}} \xrightarrow{2\pi i} \mathcal{O}X_{h} \xrightarrow{\exp_{X_{h}}} \mathcal{O}^{*}_{X_{h}} \rightarrow 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{C_{n},0}(f) - 1 \in \alpha \). Then \( \exp_{C_{n},0}(f) - 1 \in \sqrt{\alpha} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi i k \in \sqrt{\alpha} \). But

\[
\exp_{C_{n},0}(f') - 1 = \sum_{m \geq 1} f'^{m}/m! = f' \cdot (1 + \sum_{m \geq 1} f'^{m}/(m+1)!) \in \alpha.
\]

The element \( g = \sum_{m \geq 1} f'^{m}/(m+1)! \in \sqrt{\alpha} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( \alpha \).
Now we want to show that:

\[ 0 \rightarrow \mathbb{Z} \cdot X_h \xrightarrow{2\pi i} \mathcal{O}_{X_h} \xrightarrow{\exp X_h} \mathcal{O}_{X_h}^* \rightarrow 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \mathfrak{a} \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{\mathfrak{a}} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{\mathfrak{a}} \). But

\[ \exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = f' \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)!) \in \mathfrak{a}. \]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{\mathfrak{a}} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( \mathfrak{a} \).
Now we want to show that:

\[ 0 \rightarrow \mathbb{Z}X_h \xrightarrow{\cdot 2\pi i} \mathcal{O}X_h \xrightarrow{\exp X_h} \mathcal{O}^*_X \rightarrow 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \mathfrak{a} \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{\mathfrak{a}} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{\mathfrak{a}} \). But

\[ \exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = f' \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)!) \in \mathfrak{a}. \]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{\mathfrak{a}} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( \mathfrak{a} \).
Now we want to show that:

\[
0 \rightarrow \mathbb{Z} X_h \overset{2\pi i}{\longrightarrow} \mathcal{O} X_h \overset{\exp X_h}{\longrightarrow} \mathcal{O}^*_X \rightarrow 0
\]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in a \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{a} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{a} \). But

\[
\exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = f' \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)!) \in a.
\]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{a} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( a \).
Now we want to show that:

\[ 0 \rightarrow \mathbb{Z}X_h \cdot 2\pi i \rightarrow \mathcal{O}X_h \xrightarrow{\exp_{X_h}} \mathcal{O}_X^* \rightarrow 0 \]

is exact. For the discussion above we have just to show exactness in the middle. Let \( f \in A \) such that \( \exp_{\mathbb{C}^n,0}(f) - 1 \in a \). Then \( \exp_{\mathbb{C}^n,0}(f) - 1 \in \sqrt{a} \). Since the above sequence is exact if \( X \) is reduced, there exist \( k \in \mathbb{Z} \) such that \( f' = f - 2\pi ik \in \sqrt{a} \). But

\[
\exp_{\mathbb{C}^n,0}(f') - 1 = \sum_{m \geq 1} f'^m / m! = f' \cdot (1 + \sum_{m \geq 1} f'^m / (m + 1)! ) \in a.
\]

The element \( g = \sum_{m \geq 1} f'^m / (m + 1)! \in \sqrt{a} \). This means that \( 1 + g \) is invertible in \( A \), so \( f' \) is actually an element of \( a \).
The exponential sequence yields a long exact sequence of abelian groups:

\[ 0 \to H^0(X_h, \mathbb{Z}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}^*) \to H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}^*) \to \ldots \]

By GAGA $H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X)$, and the latter is an artinian $\mathbb{C}$-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So $H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}^*)$ is surjective, and thus

\[ H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \]

is injective.
The exponential sequence yields a long exact sequence of abelian groups:

\[ 0 \to H^0(X_h, \mathbb{Z}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}^*_{X_h}) \to \]
\[ H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}^*_{X_h}) \to \ldots \]

By GAGA, \( H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X) \), and the latter is an artinian \( \mathbb{C} \)-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So \( H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}^*_{X_h}) \) is surjective, and thus

\[ H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \]

is injective.
The exponential sequence yields a long exact sequence of abelian groups:

\[ 0 \rightarrow H^0(X_h, \mathbb{Z}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}^*) \rightarrow \]
\[ H^1(X_h, \mathbb{Z}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h}^*) \rightarrow \ldots \]

By GAGA \( H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X) \), and the latter is an artinian \( \mathbb{C} \)-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So \( H^0(X_h, \mathcal{O}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}^*) \) is surjective, and thus

\[ H^1(X_h, \mathbb{Z}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h}) \]

is injective.
The conclusion

The exponential sequence yields a long exact sequence of abelian groups:

\[ 0 \to H^0(X_h, \mathbb{Z}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}^*_{X_h}) \to \]
\[ H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}^*_{X_h}) \to \ldots \]

By GAGA \( H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X) \), and the latter is an artinian \( \mathbb{C} \)-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So \( H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}^*_{X_h}) \) is surjective, and thus

\[ H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \]

is injective.
The exponential sequence yields a long exact sequence of abelian groups:

\[ 0 \rightarrow H^0(X_h, \mathbb{Z}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}^*) \rightarrow H^1(X_h, \mathbb{Z}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h}^*) \rightarrow \ldots \]

By GAGA \( H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X) \), and the latter is an artinian \( \mathbb{C} \)-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So \( H^0(X_h, \mathcal{O}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}^*) \) is surjective, and thus

\[ H^1(X_h, \mathbb{Z}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h}) \]

is injective.
The exponential sequence yields a long exact sequence of abelian groups:

\[ 0 \to H^0(X_h, \mathbb{Z}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}^*) \to \]
\[ H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}^*) \to \ldots \]

By GAGA, \( H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X) \), and the latter is an artinian \( \mathbb{C} \)-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So \( H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}^*) \) is surjective, and thus

\[ H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \]

is injective.
The exponential sequence yields a long exact sequence of abelian groups:

$$0 \rightarrow H^0(X_h, \mathbb{Z}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}^*) \rightarrow H^1(X_h, \mathbb{Z}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h}^*) \rightarrow \ldots$$

By GAGA $H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X)$, and the latter is an artinian $\mathbb{C}$-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So $H^0(X_h, \mathcal{O}_{X_h}) \rightarrow H^0(X_h, \mathcal{O}_{X_h}^*)$ is surjective, and thus $H^1(X_h, \mathbb{Z}_{X_h}) \rightarrow H^1(X_h, \mathcal{O}_{X_h})$ is injective.
The exponential sequence yields a long exact sequence of abelian groups:

\[ 0 \rightarrow H^0(X_h, \mathbb{Z}_X) \rightarrow H^0(X_h, \mathcal{O}_X) \rightarrow H^0(X_h, \mathcal{O}_{X_h}^*) \rightarrow \]
\[ H^1(X_h, \mathbb{Z}_X) \rightarrow H^1(X_h, \mathcal{O}_X) \rightarrow H^1(X_h, \mathcal{O}_{X_h}^*) \rightarrow \ldots \]

By GAGA $H^0(X_h, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X)$, and the latter is an artinian $\mathbb{C}$-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So $H^0(X_h, \mathcal{O}_X) \rightarrow H^0(X_h, \mathcal{O}_{X_h}^*)$ is surjective, and thus

\[ H^1(X_h, \mathbb{Z}_X) \rightarrow H^1(X_h, \mathcal{O}_X) \]

is injective.
The exponential sequence yields a long exact sequence of abelian groups:

\[
0 \to H^0(X_h, \mathbb{Z}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}^*) \to \]
\[
H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}^*) \to \ldots
\]

By \textit{GAGA} \( H^0(X_h, \mathcal{O}_{X_h}) \cong H^0(X, \mathcal{O}_X) \), and the latter is an artinian \( \mathbb{C} \)-algebra. For such an algebra, it is easy to show that the exponential map from the additive group to its multiplicative group of units is surjective. So \( H^0(X_h, \mathcal{O}_{X_h}) \to H^0(X_h, \mathcal{O}_{X_h}^*) \) is surjective, and thus

\[
H^1(X_h, \mathbb{Z}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h})
\]

is injective.
Using again GAGA, $H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X)$. Moreover

$$H^1(X, \mathcal{O}_X) \cong H^2_m(R/I)_0$$

where $R = \mathbb{C}[x_1, \ldots, x_n]$, $I \subset R$ is such that $X \cong \text{Proj}(R/I)$ and $m$ is the maximal irrelevant. Our assumption was depth$(R/I) \geq 3$. In particular $H^2_m(R/I) = 0$, so $H^1(X_h, \mathcal{O}_{X_h}) = 0$. Eventually, by the injection $H^1(X_h, \mathbb{Z}_{X_h}) \hookrightarrow H^1(X_h, \mathcal{O}_{X_h})$ we deduce $H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h}) = 0$. By the universal coefficient theorem

$$H^1(X_h, \mathbb{C}) = 0,$$

and this was the missing piece (iii) to infer $\text{cd}(R, I) \leq n - 3$. □
Using again GAGA, $H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X)$. Moreover

$$H^1(X, \mathcal{O}_X) \cong H^2_{m}(R/I)_0$$

where $R = \mathbb{C}[x_1, \ldots, x_n]$, $I \subset R$ is such that $X \cong \text{Proj}(R/I)$ and $m$ is the maximal irrelevant. Our assumption was $\text{depth}(R/I) \geq 3$. In particular $H^2_{m}(R/I) = 0$, so $H^1(X_h, \mathcal{O}_{X_h}) = 0$. Eventually, by the injection $H^1(X_h, \mathbb{Z}_{X_h}) \hookrightarrow H^1(X_h, \mathcal{O}_{X_h})$ we deduce $H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h}) = 0$. By the universal coefficient theorem

$$H^1(X_h, \mathbb{C}) = 0,$$

and this was the missing piece (iii) to infer $\text{cd}(R, I) \leq n - 3$. $\square$
Using again GAGA, $H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X)$. Moreover

$$H^1(X, \mathcal{O}_X) \cong H^2_m(R/I)_0$$

where $R = \mathbb{C}[x_1, \ldots, x_n]$, $I \subset R$ is such that $X \cong \text{Proj}(R/I)$ and $m$ is the maximal irrelevant. Our assumption was $\text{depth}(R/I) \geq 3$. In particular $H^2_m(R/I) = 0$, so $H^1(X_h, \mathcal{O}_{X_h}) = 0$. Eventually, by the injection $H^1(X_h, \mathbb{Z}_{X_h}) \hookrightarrow H^1(X_h, \mathcal{O}_{X_h})$ we deduce $H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h}) = 0$. By the universal coefficient theorem

$$H^1(X_h, \mathbb{C}) = 0,$$

and this was the missing piece (iii) to infer $\text{cd}(R, I) \leq n - 3$. \qed
Using again GAGA, $H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X)$. Moreover

$$H^1(X, \mathcal{O}_X) \cong H^2_m(R/I)_0$$

where $R = \mathbb{C}[x_1, \ldots, x_n]$, $I \subset R$ is such that $X \cong \text{Proj}(R/I)$ and $m$ is the maximal irrelevant. Our assumption was $\text{depth}(R/I) \geq 3$. In particular $H^2_m(R/I) = 0$, so $H^1(X_h, \mathcal{O}_{X_h}) = 0$. Eventually, by the injection $H^1(X_h, \mathbb{Z}_{X_h}) \hookrightarrow H^1(X_h, \mathcal{O}_{X_h})$ we deduce $H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h}) = 0$. By the universal coefficient theorem

$$H^1(X_h, \mathbb{C}) = 0,$$

and this was the missing piece (iii) to infer $\text{cd}(R, I) \leq n - 3$. □
Using again GAGA, \( H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X) \). Moreover

\[
H^1(X, \mathcal{O}_X) \cong H^2_m(R/I_0)
\]

where \( R = \mathbb{C}[x_1, \ldots, x_n] \), \( I \subset R \) is such that \( X \cong \text{Proj}(R/I) \) and \( m \) is the maximal irrelevant. Our assumption was \( \text{depth}(R/I) \geq 3 \). In particular \( H^2_m(R/I) = 0 \), so \( H^1(X_h, \mathcal{O}_{X_h}) = 0 \). Eventually, by the injection \( H^1(X_h, \mathbb{Z}_{X_h}) \hookrightarrow H^1(X_h, \mathcal{O}_{X_h}) \) we deduce \( H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h}) = 0 \). By the universal coefficient theorem

\[
H^1(X_h, \mathbb{C}) = 0,
\]

and this was the missing piece (iii) to infer \( \text{cd}(R, I) \leq n - 3 \). \( \square \)
Using again GAGA, $H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X)$. Moreover

$$H^1(X, \mathcal{O}_X) \cong H^2_m(R/I)_0$$

where $R = \mathbb{C}[x_1, \ldots, x_n]$, $I \subset R$ is such that $X \cong \text{Proj}(R/I)$ and $m$ is the maximal irrelevant. Our assumption was $\text{depth}(R/I) \geq 3$. In particular $H^2_m(R/I) = 0$, so $H^1(X_h, \mathcal{O}_{X_h}) = 0$. Eventually, by the injection $H^1(X_h, \mathbb{Z}_{X_h}) \hookrightarrow H^1(X_h, \mathcal{O}_{X_h})$ we deduce $H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h}) = 0$. By the universal coefficient theorem

$$H^1(X_h, \mathbb{C}) = 0,$$

and this was the missing piece (iii) to infer $\text{cd}(R, I) \leq n - 3$. □
Using again GAGA, $H^1(X_h, \mathcal{O}_{X_h}) \cong H^1(X, \mathcal{O}_X)$. Moreover

$$H^1(X, \mathcal{O}_X) \cong H^2_m(R/I)_0$$

where $R = \mathbb{C}[x_1, \ldots, x_n]$, $I \subset R$ is such that $X \cong \text{Proj}(R/I)$ and $m$ is the maximal irrelevant. Our assumption was $\text{depth}(R/I) \geq 3$. In particular $H^2_m(R/I) = 0$, so $H^1(X_h, \mathcal{O}_{X_h}) = 0$. Eventually, by the injection $H^1(X_h, \mathbb{Z}_{X_h}) \hookrightarrow H^1(X_h, \mathcal{O}_{X_h})$ we deduce $H^1(X_h, \mathbb{Z}) \cong H^1(X_h, \mathbb{Z}_{X_h}) = 0$. By the universal coefficient theorem

$$H^1(X_h, \mathbb{C}) = 0,$$

and this was the missing piece (iii) to infer $\text{cd}(R, I) \leq n - 3$. □
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