SIMPLICIAL COMPLEXES OF SMALL CODIMENSION

MATTEO VARBARO AND RAHIM ZAARE-NAHANDI

Abstract. We show that a Buchsbaum simplicial complex of small codimension must have large depth. More generally, we achieve a similar result for CM\(_t\) simplicial complexes, a notion generalizing Buchsbaum-ness, and we prove more precise results in the codimension 2 case. Along the paper, we show that the CM\(_t\) property is a topological invariant of a simplicial complex.

1. Introduction

In [11], Hartshorne proposed his tantalizing conjecture concerning smooth varieties of small codimension in some projective space. Precisely, if \( R = K[x_1, \ldots, x_n] \) is the polynomial ring in \( n \) variables over a field \( K \), the conjecture declaims:

**Conjecture 1.1.** (Hartshorne) If \( I \subseteq R \) is a homogeneous ideal of height \( h \) less than \((n-1)/3\) such that \( \text{Proj} R/I \) is nonsingular, then \( I \) is a complete intersection.

If \( h = 2 \), then the condition \( h < (n-1)/3 \) is equivalent to \( n > 7 \). In this case, by a result of Evans and Griffith [6, Theorem 3.2], the conjecture is equivalent to:

**Conjecture 1.2.** If \( I \subseteq R \) is a homogeneous ideal of height 2 such that \( \text{Proj} R/I \) is nonsingular, and \( n > 7 \), then \( R/I \) is Cohen-Macaulay.

The present article has no pretension to give new insights on the conjecture of Hartshorne: the only result in this direction is Corollary 3.6, stating that \( R/I \) has depth larger than \( n - 2h \) if furthermore \( I \) admits a square-free initial ideal. Rather, this paper brings the philosophy of the conjecture to the world of combinatorial commutative algebra, as it had already been done, to some extent, in [3].

If \( \Delta \) is a simplicial complex in \( n \) variables, \( \text{Proj} K[\Delta] \) is almost never smooth, so Hartshorne’s conjecture is not interesting when stated for \( \text{Proj} K[\Delta] \). The notion of Cohen-Macaulay-ness in codimension \( t \) was introduced, independently and with the sole difference concerning a purity matter, in [16] and in [9]. In [16] this concept was suggested as the right one to measure the singularities of a simplicial complex: \( \Delta \) is Cohen-Macaulay in codimension \( t \) (according to [9]) if and only if \( \Delta \) is pure of singularity dimension less than \( t - 1 \) (according to [16]). In particular, if \( \Delta \) has negative singularity dimension, it is Buchsbaum. So, somehow Buchsbaum-ness plays the role of ‘smooth-ness’ for simplicial complexes. This way of thinking is also supported from the results in the recent paper [2], which imply that, if the ideal defining a smooth projective variety has a square-free Gröbner degeneration, then

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the associated simplicial complex is Buchsbaum. With this definition in mind, the same philosophy that led Hartshorne to make his conjecture brings one to expect the following: If $\Delta$ is a Buchsbaum simplicial complex with small codimension, then $K[\Delta]$ should have large depth.

In this note, we show that if $\Delta$ is a $(d-1)$-dimensional Buchsbaum simplicial complex on $d+2$ vertices, then depth $K[\Delta] \geq d-1$. Moreover, in this case $K[\Delta]$ is not Cohen-Macaulay in codimension $t$ if and only if $\Delta$ is the Alexander dual of (the clique complex of) the $(d+2)$-cycle (Proposition 4.2). More generally, if $\Delta$ is a $(d-1)$-dimensional Buchsbaum simplicial complex on $n$ vertices, then depth $K[\Delta] \geq 2d-n+1$. Even more generally, if $\Delta$ is Cohen-Macaulay in codimension $t$, then $K[\Delta]$ satisfies the condition of Serre $S_{2d-n-t+2}$ (Corollary 3.5). Along the way, we also prove that being Cohen-Macaulay in codimension $t$ is a topological invariant (Theorem 2.5).

The paper is structured as follows: a brief review of some preliminaries and conventions is given in Section 2, where the topological invariance of Cohen-Macaulay-ness in an arbitrary codimension is also proved. Section 3 is devoted to the connection between Cohen-Macaulay-ness of a simplicial complex in some codimension with linearity of the Stanley-Reisner ideal of the Alexander dual of the simplicial complex up to a certain step. This leads to a connection between Cohen-Macaulay-ness in a certain codimension with the $S_r$ condition of Serre. Some corollaries and relevant examples are also given. In Section 4, the case of codimension 2 simplicial complexes is analyzed in more detail, and a combinatorial proof of the main result of Section 3 in the codimension 2 case is provided.

### 2. Preliminaries and Conventions

Let $R = K[x_1, \ldots, x_n]$ be the ring of polynomials over a field $K$, equipped with the standard grading. For integers $p \geq 1$ and $d \geq 2$, we say that a simplicial complex $\Delta$ on $n$ vertices satisfies the Green-Lazarsfeld property $N_{d,p}$ if $I_{\Delta}$ is generated in degree $d$ and the first $p$ steps of the minimal graded free resolution

$$
\cdots \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_1} F_0 \longrightarrow I_{\Delta} \longrightarrow 0
$$

of $I_{\Delta}$ are linear, in the sense that $\varphi_1, \ldots, \varphi_{p-1}$ are represented by matrices of linear forms.

A simplicial complex $\Delta$ is said to satisfy the Serre’s condition $S_r$ if $H_i(\text{link}_\Delta F; K)$ vanishes for all $F \in \Delta$ and for all $i < \min\{r-1, \dim(\text{link}_\Delta F)\}$, where $H_i(\Delta; K)$ is the $i$th reduced homology group of $\Delta$ over the field $K$. This is equivalent to the usual definition of the condition $S_r$ on $K[\Delta]$.

By a CM$_t$ simplicial complex, we mean a pure simplicial complex $\Delta$ which is Cohen-Macaulay in codimension $t$, namely a simplicial complex such that $\text{link}_\Delta F$ is Cohen-Macaulay for all $F \in \Delta$ with $|F| \geq t$.

**Remark 2.1.** Let $\Delta$ be a pure simplicial complex of dimension $d-1$. It follows by the definition that $\Delta$ satisfies the $S_r$ condition $\implies$ $\Delta$ is CM$_{d-r}$. The vice versa is false, just think to a disconnected Buchsbaum simplicial complex $\Delta$ (such a $\Delta$ is CM$_1$ but does not even satisfy $S_2$). On the other hand, we will show in Corollary 3.5 that $\Delta$ is CM$_t$ on $n$ vertices $\implies$ $\Delta$ satisfies the $S_{2d-n-t+2}$ condition.

**Remark 2.2.** The notion of singularity dimension has been considered in [16] as follows: a simplicial complex $\Delta$ has singularity dimension less than $m$ if $\text{link}_\Delta F$ is
Cohen-Macaulay for all $F \subseteq \Delta$ with $\dim F \geq m$ (by convention, $\dim \emptyset = -1$). So a simplicial complex $\Delta$ is CM$_t$ if and only if it is pure and has singularity dimension less than $t - 1$.

**Remark 2.3.** The phrase “Cohen-Macaulay in codimension $t$” in the present paper has a different meaning from the phrase “Cohen-Macaulay in codimension $c$” considered in [16]. In fact, according to [16, Definition 3.6], even if $\Delta$ is a pure simplicial complex of dimension $d - 1$, then in [16] “$\Delta$ Cohen-Macaulay in codimension $c$” means that $\text{link}_\Delta F$ is Cohen-Macaulay for all $F \subseteq \Delta$ with $|F| = d - 1 - c$.

For an $R$-module $M$ we write $\dim M$ for the Krull dimension of $M$; when $M = 0$ we write by convention $\dim M = -\infty$.

**Remark 2.4.** Notice that $\Delta$ is a pure $(d - 1)$-dimensional simplicial complex if and only if

$$\dim \text{Ext}^{n-i}_R(K[\Delta], R) < 0 \quad \forall \ i < d.$$  

On the other hand, it has been proved in [16, Corollary 7.4] that $\Delta$ has singularity dimension $< m$ if and only if

$$\dim \text{Ext}^{n-i}_R(K[\Delta], R) \leq m \quad \forall \ i < d.$$  

So, if $\Delta$ has singularity dimension $< m$ and $\text{depth}K[\Delta] > m$, then $\Delta$ is pure. In particular, since $\text{depth}K[\Delta] > 0$ for any simplicial complex $\Delta$, the following are equivalent:

1. $\Delta$ is Buchsbaum.
2. $\Delta$ has singularity dimension $< 0$.
3. $\Delta$ is CM$_1$.

A property of a simplicial complex $\Delta$ is a topological invariant of $\Delta$ if it holds for any simplicial complex whose geometric realization is homeomorphic to the one of $\Delta$. Next we prove that the properties of satisfying $S_r$, being CM$_1$, and having singularity dimension $< m$ are topological invariants. This fact has essentially been proved by Yanagawa in [22]. We report his result in our context for the convenience of the reader. We keep the same notations used in [22].

**Theorem 2.5.** Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex on $n$ vertices. Then, for all $i \in \mathbb{N}$,

$$\dim \text{Ext}^{n-i}_R(K[\Delta], R)$$

is a topological invariant of $\Delta$. In particular, satisfying $S_r$, being CM$_1$, and having singularity dimension $< m$ are topological invariants.

**Proof.** Let $X$ be a topological realization of $\Delta$. If $\dim \text{Ext}^{n-i}_R(K[\Delta], R) \leq 0$, then $\dim \text{Ext}^{n-i}_R(K[\Delta], R) = 0$ if and only if $\text{Ext}^{n-i}_R(K[\Delta], R) = 0$ if and only if $H^{i-1}(X; K) = 0$, so we can assume that $\dim \text{Ext}^{n-i}_R(K[\Delta], R) > 0$.

Notice that $\text{Ext}^{n-i}_R(K[\Delta], R) = 0$ for $i > d$ or $i \leq 0$, and that $\text{Ext}^{n-d}_R(K[\Delta], R)$ is always $d$-dimensional. Therefore we will assume that $0 < i < d$. In this situation, [22, Theorem 4.1] yields that $\dim \text{Ext}^{n-i}_R(K[\Delta], R) - 1$ is equal to the dimension of the support of the sheaf $K^{-i-1}(\mathcal{D}_X^*)$ on $X$, where $\mathcal{D}_X^*$ is the Verdier dualizing complex of $X$ with coefficients in $K$. So we have that $\dim \text{Ext}^{n-i}_R(K[\Delta], R)$ is a topological invariant of $\Delta$.

For the last part, notice that being pure is obviously a topological invariant and:

1. $\Delta$ satisfies $S_r$ (for $r \geq 2$) $\iff \dim \text{Ext}^{n-i}_R(K[\Delta], R) < i - r \quad \forall \ i < d$. 


(2) \( \Delta \) has singularity dimension < \( m \) \( \iff \) \( \dim \text{Ext}^n_R(\mathbb{K}[\Delta], R) \leq m \) \( \forall i < d \).

(3) \( \Delta \) is CM\(_t\) \( \iff \) \( \Delta \) is pure and \( \dim \text{Ext}^n_R(\mathbb{K}[\Delta], R) < t \) \( \forall i < d \).

\[ \square \]

For further concepts and notations on simplicial complexes and combinatorial commutative algebra we refer to the standard books [19], [12] and [17].

3. The CM\(_t\) Property of Simplicial Complexes versus the Serre Condition \( S_r \)

In this section, for a simplicial complex \( \Delta \) of dimension \( d - 1 \) on \( n \) vertices, applying a subadditivity result of Herzog and Srinivasan to the Betti diagram of the Stanley-Reisner ideal of \( \Delta \), it is shown that if \( \Delta \) satisfies CM\(_t\) for some \( t \geq 0 \), then \( \Delta^\vee \) satisfies the \( N_{n-d,2d-n-t+2} \) condition. In other words, the minimal graded free resolution of \( I_{\Delta^\vee} \) is linear on the first \( 2d - n - t + 2 \) steps. This leads to the implication that if \( \Delta \) is CM\(_t\) for some \( t \geq 0 \), then the Stanley-Reisner ring of \( \Delta \) satisfies the \( S_{2d-n-t+2} \) condition of Serre.

First we recall a generalization of the Eagon-Reiner’s theorem given in [8].

**Theorem 3.1.** [8, Theorem 3.1] Let \( \Delta \) be a simplicial complex on \( n \) vertices, \( \Delta^\vee \) its Alexander dual and \( I_\Delta \subset R \) the Stanley-Reisner ideal of \( \Delta \). Then the following are equivalent:

(i) \( \Delta^\vee \) is a CM\(_t\) simplicial complex of dimension \( d - 1 \).

(ii) \( \beta_{0,j}(I_\Delta) = 0 \) \( \forall j > n - d \) and \( \beta_{i,i+j}(I_\Delta) = 0 \) \( \forall j > n - d \) and \( i + j \leq n - t \).

I.e., the Betti diagram \( \beta_{i,i+j}(I_\Delta) \) looks like in Figure 1.

\[
\begin{array}{cccccccc}
\hline
i & 0 & 1 & \cdots & i & \cdots & d-t-1 & d-t & \cdots & \text{projdim} \\
\hline
n-d & \ast & \ast & \cdots & \ast & \cdots & \ast & \ast & \cdots & \ast \\
n-d+1 & 0 & 0 & \cdots & 0 & \cdots & 0 & \ast & \cdots & \ast \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
j & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n-t-1 & 0 & 0 & \ast & \cdots & \ast & \ast & \ast & \ast & \ast \\
n-t & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{regularity} & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\hline
\end{array}
\]

*Figure 1. The shape of the Betti diagram of \( I_\Delta \) when \( \Delta^\vee \) is CM\(_t\).*

On the other hand, Herzog and Srinivasan [13] proved the following “subadditivity” result on the Betti numbers of monomial ideals.
Theorem 3.2. [13, Corollary 4]. Let $I = (u_1, \ldots, u_m)$ be a monomial ideal of $R$, and let $e = \max \{ \deg(u) \}$. Then for all $j_0 \in \mathbb{Z}$:

$$\beta_{i,j}(I) = 0 \quad \forall \ j > j_0 \implies \beta_{i+1,j}(I) = 0 \quad \forall \ j > j_0 + e.$$  

(3.1)

Now we prove the main result of the paper.

Theorem 3.3. Let $\Delta$ be a $(d-1)$-dimensional $CM_t$ simplicial complex on $n$ vertices. Then $\Delta^\vee$ satisfies the $N_{n-d,2d-n-t+2}$ condition.

Proof. Notice that $I_{\Delta^\vee}$ is generated in degree $n-d$. Hence the assertion is trivially valid for $2d-n-t+2 \leq 1$. Therefore, we may assume that $2d-n-t \geq 0$. Then, (3.1) gives us

$$\beta_{i,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > j_0 \implies \beta_{i+1,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > j_0 + n-d.$$  

By Theorem 3.1, we know that, for all $i \in \mathbb{N}$,

$$\beta_{i,j}(I_{\Delta^\vee}) = 0 \quad \forall \ i+n-d < j \leq n-t,$$  

(3.2) and

$$\beta_{0,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > n-d.$$  

(3.3)

Now, suppose that $1 \leq i \leq 2d-n-t+1$, and assume we have already proved that

$$\beta_{i-1,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > i-1+n-d.$$  

(3.4)

By (3.4) together with (3.1) we have $\beta_{i,j}(I_{\Delta^\vee}) = 0$ for all $j > i-1+2n-2d$. In particular, we have $\beta_{i,j}(I_{\Delta^\vee}) = 0$ for $i = 2d-n-t+1$, $j > (2d-n-t+1)-1+2n-2d = n-t$. On the other hand (3.2) guarantees us that $\beta_{i,j}(I_{\Delta^\vee}) = 0$ for all $i+n-d < j \leq n-t$. Putting all together we get

$$\beta_{i,j}(I_{\Delta^\vee}) = 0 \quad \forall \ j > i+n-d.$$  

□

In [20] and, independently, in [23], the following refinement of the result of Herzog and Srinivasan is proved:

Theorem 3.4. [20, Theorem 6.2, the $\mathbb{Z}$-graded part]. With the notation of Theorem 3.3, one has:

$$\beta_{i,k}(I) = 0, \forall k = j_0, \ldots, j_0 + e - 1 \implies \beta_{i+1,j_0+e}(I) = 0.$$  

This result can be applied to study the Betti numbers of $\Delta^\vee$ (inferring analog results to Theorem 3.3) when $\Delta$ has singularity dimension less than $m$.

For $r \geq 2$, by a result of Yanagawa [21, Corollary 3.7], for a simplicial complex $\Delta$ of codimension $c$, $K[\Delta]$ satisfies the $S_r$ condition of Serre if and only if $I_{\Delta^\vee}$ satisfies the $N_{c,r}$ condition. Therefore, an interesting consequence of Theorem 3.3 is the following:

Corollary 3.5. Let $\Delta$ be a simplicial complex of dimension $d-1$ on $n$ vertices. Assume that $\Delta$ is CM$_t$ for some $t \geq 0$. Then $\Delta$ satisfies the $S_{2d-n-t+2}$ condition. In particular, if $\Delta$ is Buchsbaum, then $\text{depth}K[\Delta] \geq 2d-n+1$.  

In particular, if $\Delta$ is Buchsbaum, then $\text{depth}K[\Delta] \geq 2d-n+1$.  

□
The following corollary is in the spirit of Hartshorne’s conjecture and goes in the direction of a question raised in [2, Question 4.2].

**Corollary 3.6.** Let $I \subseteq R$ be a homogeneous ideal of height $h$ such that $\text{Proj} R/I$ is nonsingular. If $I$ has a square-free initial ideal with respect to some term order, then $\text{depth} R/I > n - 2h$.

**Proof.** Let $J$ be a square-free initial ideal of $I$. Since $R/I$ is generalized Cohen-Macaulay, $R/J$ is Buchsbaum by [2, Corollary 2.11]. By Corollary 3.5, then, $\text{depth} R/J \geq n - 2h + 1$. We conclude since the depth cannot go up by taking the initial ideal. \qed

Another consequence, interestingly related to the result of Brehm and Kühnel [1, Theorem B], is the following:

**Corollary 3.7.** Let $\Delta$ be a $(d - 1)$-dimensional Buchsbaum simplicial complex on $n$ vertices such that $H_i(\Delta; K) \neq 0$ for some $i \geq 1$. Then $n \geq 2d - i$.

**Remark 3.8.** Being the combinatorial manifolds a very special case of Buchsbaum simplicial complexes, even if the conclusion of Corollary 3.7 is slightly weaker than the one in [1, Theorem B], it applies to a much larger class of simplicial complexes.

**Example 3.9.** Since Theorem 3.3 and Corollary 3.5 are trivial for $t \geq 2d - n + 1$, it is natural to ask for examples of $\text{CM}_t$ simplicial complexes that are not $\text{CM}_{t-1}$ for $1 \leq t \leq 2d - n$. Murai and Terai [18, Example 3.5] considered the following simplicial complex:

$$\Delta = \langle \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 3, 4, 5\}, \{1, 3, 4, 6\}, \{1, 3, 5, 6\}, \{2, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\} \rangle,$$

where $\Delta$ satisfies $S_3$ but is not Cohen-Macaulay. Thus $\Delta$ is Buchsbaum and the condition $1 \leq t \leq 2d - n$ is satisfied. Now if $v$ is a new vertex, by [10, Theorem 3.1 (ii)], the cone on $\Delta$ with vertex $v$ is $\text{CM}_2$ but not Buchsbaum, and again we have $1 \leq t \leq 2d - n$. Taking further cones, one gets a family of $\text{CM}_t$ simplicial complexes which are not $\text{CM}_{t-1}$ and we have $1 \leq t \leq 2d - n$.

**Remark 3.10.** Often, the minimal number of vertices necessary for triangulating a given $(d - 1)$-dimensional combinatorial manifolds is more than $2d$. An exception is an 8-dimensional combinatorial manifold, the so-called “Brehm and Kühnel manifold”, which has 6 combinatorially different triangulations on 15 vertices (see [1], [15, Proposition 48] and [14]).

4. **The CM$_t$ property and minimal chord-less cycles of graphs**

In this section, we focus on pure $(d - 1)$-dimensional simplicial complexes on $d + 2$ vertices, i.e., pure codimension two simplicial complexes. If $\Delta$ is such a simplicial complex, then its Alexander dual is flag, i.e., $\Delta^\vee$ is the clique complex of a graph $G$. In general, the clique complex and the independence complex of a graph $H$ will be denoted by $\Delta(H)$ and $\Delta_H$, respectively. Also, by $\overline{H}$ we will denote the complementary graph of $H$.

**Theorem 4.1.** Let $\Delta$ be a pure $(d - 1)$-dimensional codimension two simplicial complex. Then the following are equivalent:

(i) $\Delta$ is $\text{CM}_t$,

(ii) $\Delta^\vee$ satisfies the $N_{2,d-t}$ condition,
(iii) $\Delta$ satisfies the $S_{d-t}$ condition,
(iv) Every cycle of the 1-skeleton $G$ of $\Delta'$ of length at most $d - t + 2$ has a chord.

Proof. The equivalence of (i), (ii) and (iii) is simply an application of Theorem 3.3, Corollary 3.5 in the case $n = d + 2$, and Remark 2.1. The equivalence of (ii) and (iv) follows by [5, Theorem 2.1].

Proposition 4.2. If $\Delta$ is a codimension two Buchsbaum simplicial complex, then
$$\text{depth} K[\Delta] \geq \dim \Delta.$$ Furthermore, $K[\Delta]$ is Cohen-Macaulay if and only if the 1-skeleton $G$ of $\Delta'$ is not the $(d + 2)$-cycle.

Proof. Notice that $\Delta$ being Buchsbaum is equivalent to $\Delta$ being CM$_1$. So the first part of the statement follows by Theorem 4.1. If $K[\Delta]$ is not CM$_0$, again Theorem 4.1 implies that $G$ has an induced chord-less $(d + 2)$-cycle (in those notations, so $d = \dim \Delta + 1$). Since the number of vertices is $d + 2$, $G$ is actually the $(d + 2)$-cycle.

Remark 4.3. In particular, if $\Delta$ is a codimension two Buchsbaum simplicial complex which is not Cohen-Macaulay, then
$$\text{projdim} K[\Delta] = 3.$$
One might expect that, in general, if $\Delta$ is a codimension 2 simplicial complex which is CM$_1$ but not CM$_{1-1}$, then
$$\text{projdim} K[\Delta] = t + 2.$$ This is false: a simple example is the Alexander dual of $\Delta = \langle \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}, \{5, 6\} \rangle$ which has dimension $d - 1$ where $d = 4 = 6 - 2 = n - 2$. Then $\Delta$ is CM$_2$ but not CM$_1$. Nevertheless, the projective dimension of the Stanley-Reisner ring of $\Delta$ is 3.

For the sake of documenting a different method, we give an alternative proof, more combinatorial, for the equivalence of (i) and (iv) in Theorem 4.1.

Theorem 4.4. Let $G$ be a simple graph on $[n] = \{1, \ldots, n\}$ with no isolated vertices. Let $\Delta = \Delta(G)$ be the clique complex of $G$. Let $r \geq 3$ be an integer. Then $\Delta'$ is CM$_{n-r}$ if and only if every cycle of $G$ of length at most $r$ has a chord.

Proof. The “if” direction follows by [5, Theorem 2.1], [21, Corollary 3.7] and Remark 2.1, so we focus on the “only if” part.

Assume that $\Delta'$ is CM$_{n-r}$. We prove by induction on $r$ that every cycle of $G$ of length at most $r$ has a chord. The first case $r = 3$ is trivial. Assume that, on the contrary, $G$ has a chord-less $r$-cycle $C$. Let $V(C) = \{v_1, \ldots, v_r\}$ and $E(C) = \{\{v_1, v_2\}, \ldots, \{v_{r-1}, v_r\}, \{v_r, v_1\}\}$ be the vertex set and the edge set of $C$, respectively. Then the induced subgraph $\overline{C}$ on $V(C)$ is the graph $K_r \setminus E(C)$, where $K_r$ is the complete graph on $V(C)$. Clearly, $K_r \setminus E(C)$ has $(\binom{r}{2}) - r = r(r-3)/2$ edges. Let $F$ be the simplex on $V(G) \setminus V(C)$. Then, $|F| = n - r$ and $F$ is a face of $\Delta'$ because $V(C) \notin \Delta$. Thus $\Gamma = \text{link}_{\Delta'} F$ should be Cohen-Macaulay. We prove that this is not the case. Observe that the only facets of $\Delta'$ which contain $F$ are $F \cup (V(C) \setminus \{v_i, v_j\})$ for some $\{v_i, v_j\} \in \overline{C}$. Therefore,
$$\Gamma = \text{link}_{\Delta'} F = \langle V(C) \setminus \{v_i, v_j\} : \{v_i, v_j\} \in \overline{C} \rangle.$$ In particular, $\dim \Gamma = r - 3$. We determine $h_{r-2}$ by computing the $f$-vector of $\Gamma$: to this purpose, notice that every subset of the vertex set of $\Gamma$ of cardinality $\leq r - 3$
is also a face of $\Gamma$. To see this, let $E = V(C) \setminus \{v_i, v_j, v_k\}$ be a subset of the vertex set of $\Gamma$ of cardinality $r - 3$. Choose $1 \leq l \leq r$ such that $l \notin \{i, j, k\}$. Then at least one of the pairs $(i, l)$, $(j, l)$ and $(k, l)$ will be a non-consecutive pair modulo $r$. Let $(i, l)$ be such a pair. Then, $\{i, l\} \in \mathcal{G}$, and hence, $E \subset V(C) \setminus \{v_i, v_j\}$, i.e., $E$ is a face of $\Gamma$. Therefore we got:

$$f_{-1} = 1, f_i = \binom{r}{i+1}, i = 0, \ldots, r - 4 \text{ and } f_{r-3} = r(r - 3)/2.$$ 
Consequently,

$$h_{r-2} = \sum_{i=0}^{r-2} (-1)^{r-2-i} f_{i-1} = \left( \sum_{i=0}^{r-3} (-1)^{r-i} \binom{r}{i} \right) + r(r - 3)/2 =$$

$$(1 - 1)^r + \binom{r}{r - 1} - \binom{r}{r - 2} - 1 + r(r - 3)/2 = -1.$$ 
Hence $\Gamma$ is not Cohen-Macaulay. This completes the proof. □

**Corollary 4.5.** With the assumptions of Theorem 4.4, assume that $G$ is $r$-chordal, i.e., it has no chordless cycles of length greater than $r$. Then $\Delta^v$ is CM$_{n-r}$ if and only if $I_{\Delta} = I(\mathcal{G})$ has a linear resolution.

**Proof.** The assertion follows by Theorems 4.1 and 4.4 and Fröberg’s result that $I_{\Delta} = I(\mathcal{G})$ has a linear resolution if and only if $G$ is chordal [7]. □

**Remark 4.6.** It is easy to see that if $G$ is a bipartite graph or a chordal graph, then $\mathcal{G}$ can only have chordless four cycles (e.g., see [8, Lemma 4.2 and Lemma 4.6]). Assume that $G$ is a graph on $n$ vertices which is either bipartite or chordal.

If the Alexander dual of $\Delta(\mathcal{G}) = \Delta_G$ is CM$_{n-4}$, then by Corollary 4.5, $I(G)$ has a linear resolution.

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