Singularities, Serre conditions and h-vectors

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Notation

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where $h(t) = h_0 + h_1 t + h_2 t^2 + \ldots + h_s t^s \in \mathbb{Z}[t]$ is the *h-polynomial* of R.

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where $h(t) = h_0 + h_1 t + h_2 t^2 + \ldots + h_s t^s \in \mathbb{Z}[t]$ is the *h-polynomial* of R. We will name the coefficients vector $(h_0, h_1, h_2, \ldots, h_s)$ the *h-vector* of R.

Motivations

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If R is Cohen-Macaulay, then it is easy to see that $h_i \geq 0$ for all $i \geq 0$, however without the CM assumption things get complicated. In this talk I want to discuss conditions on R and/or on X which ensure at least the nonnegativity of the first h_i's and that the degree of $X \subset \mathbb{P}^n$ is bounded below by their sum.

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Theorem

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Theorem

Let K be algebraically closed, $R = \mathrm{Sym}(R_1)/I$ where I has height c and assume that X is connected in codimension 1. If R is reduced, then the minimal quadratic generators of I are $\leq \binom{c+1}{2}$ and R has multiplicity $\geq c+1$; if equality holds, then R is Cohen-Macaulay.

Serre conditions

For $r \in \mathbb{N}$, we say that R *satisfies the Serre condition* (S_r) if:

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{\dim R_{\mathfrak{p}}, r\} \quad \forall\ \mathfrak{p} \in \operatorname{Spec} R.$$

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It turns out that this is equivalent to $\operatorname{depth} R \geq \min\{\dim R, r\}$ and

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In particular, if X is nonsingular, R satisfies the Serre condition (S_r) if and only if $\operatorname{depth} R \geq \min\{\dim R, r\}$.

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Question

If R satisfies (S_r), is it true that $h_i \geq 0$ for all $i = 0, \ldots, r$?

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The above question is known to have a positive answer for Stanley-Reisner rings R (i.e. if R is defined by squarefree monomial ideals) by a result of Murai and Terai ...

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$$I = (x_1, \ldots, x_{r+1})^2 + (x_1 y_1 + \ldots + x_{r+1} y_{r+1}).$$

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Moreover, such an R has Castelnuovo-Mumford regularity $\operatorname{reg} R = 1$ and is Buchsbaum. So the question must be adjusted:

Question

If R satisfies (S_r) and X has nice singularities, is it true that $h_i \geq 0$ for all $i = 0, \dots, r$?

The main result

Theorem (Dao-Ma-_)

Let R satisfy Serre condition (S_r). Suppose either

- K has characteristic 0 and X is Du Bois, or
- K has positive characteristic and X is globally F-split.

Then $h_i \geq 0$ for all $i = 0, \ldots, r$ and the degree of $X \subset \mathbb{P}^n$ is at least $h_0 + h_1 + \ldots + h_{r-1}$.

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Then $h_i \geq 0$ for all $i = 0, \ldots, r$ and the degree of $X \subset \mathbb{P}^n$ is at least $h_0 + h_1 + \ldots + h_{r-1}$. Furthermore, if $\operatorname{reg} R < r$, or if $h_i = 0$ for some $1 \leq i \leq r$, then R is Cohen-Macaulay.

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- If $\mathrm{char}(K) = 0$, X nonsingular $\Rightarrow$ X Du Bois.
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We say that X is *locally Stanley-Reisner* if $\widehat{\mathcal{O}_{X,x}} \cong K[[\Delta_x]]$ for all $x \in X$ (for some simplicial complex Δ_x).

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- R Stanley-Reisner ring $\Rightarrow$ X locally Stanley-Reisner.
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The condition MT_r

Let M be a finitely generated graded S-module, where $S = K[X_0, \ldots, X_n]$. We say that M satisfies the condition MT_r if

$$\operatorname{reg} \operatorname{Ext}_S^{n+1-i}(M, \omega_S) \leq i - r \quad \forall\, i = 0, \ldots, \dim M - 1.$$

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This notion is good for several reasons:

- The condition MT_r does not depend on S.
- The condition MT_r is preserved by taking general hyperplane sections.
- The condition MT_r is preserved by saturating.

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Let M be a finitely generated graded S-module generated in degree ≥ 0 with h-vector $(h_0, \ldots, h_s)$ satisfying MT_r.

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- $h_r + h_{r+1} + \ldots + h_s \geq 0$, or equivalently the multiplicity of M is at least $h_0 + h_1 + \ldots + h_{r-1}$.

Furthermore, if $\operatorname{reg} M < r$ or M is generated in degree 0 and $h_i = 0$ for some $i \leq r$, then M is Cohen-Macaulay.

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1. R satisfies Serre condition (S_r).
2. $\dim \operatorname{Ext}_S^{n+1-i}(R, \omega_S) \leq i - r \quad \forall\ i \in \mathbb{N}$.

So, under our assumptions on X, in order to prove that R satisfies MT_r provided it satisfies (S_r), it is enough to show that

$$\operatorname{reg}\operatorname{Ext}_S^{n+1-i}(R,\omega_S) \leq \dim \operatorname{Ext}_S^{n+1-i}(R,\omega_S) \quad \forall\, i = 0, \ldots, \dim R - 1.$$

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We show more:

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Let $\mathfrak{m}$ be the homogeneous maximal ideal of S and assume either

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So, under our assumptions on X, in order to prove that R satisfies MT_r provided it satisfies (S_r), it is enough to show that

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Then $H^j_{\mathfrak{m}}(\operatorname{Ext}^i_S(R,\omega_S))_{>0} = 0$ for all $i,j \in \mathbb{N}$. In particular, $\operatorname{reg} \operatorname{Ext}^i_S(R,\omega_S) \leq \dim \operatorname{Ext}^i_S(R,\omega_S)$ for all $i \in \mathbb{N}$.

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If $r = 2$, the above corollary is true just assuming that X is geometically reduced...

Questions:

Is it true that $h_i \geq 0$ for all $i \leq r$ provided R satisfies (S_r) and either

1. K has characteristic 0 and X is Du Bois in codimension $r-2$, or
2. K has positive characteristic and X has F-injective singularities ???

THANK YOU !

Some references:

1. H. Dao, L. Ma, M. Varbaro *Regularity, singularities and h-vector of graded algebras*, arXiv:1901.01116.
2. S. Murai, N. Terai, *h-vectors of simplicial complexes with Serre's conditions*, Math. Res. Lett. 16 (2009), no. 6, 1015-1028.