Square-free Gröbner degenerations

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Basic notions in Commutative Algebra

Let K be a field, $S = K[X_1, \dots, X_n]$ be the polynomial ring in n variables over K equipped with a positive graded structure (namely $\deg(X_i)$ is a positive integer for any $i = 1, \dots, n$).

Given a homgeneous ideal $I \subset S$, the *Krull dimension* of $R = S/I$ ($\dim R$) is equal to the minimum $d \in \mathbb{N}$ such that $R/(f_1, \dots, f_d)$ is a finite dimensional K-vector space for some homogeneous elements $f_1, \dots, f_d \in R$.

Homogeneous elements $f_1, \dots, f_d \in R$ satisfying the property above are called a *homogeneous system of parameters* for R.

Basic notions in Commutative Algebra

Homogeneous elements of positive degree $f_1, \dots, f_k \in R = S/I$ form an *R-regular sequence* if $\overline{f_i}$ is a nonzero divisor of $R/(f_1, \dots, f_{i-1})$ for any $i = 1, \dots, k$. The *depth* of R ($\operatorname{depth} R$) is the maximum k such that there is a R-regular sequence of length k. It is easy to see that

$$\operatorname{depth} R \leq \dim R.$$

If equality above holds R is said *Cohen-Macaulay*.

Proposition

The following are equivalent:

- R is Cohen-Macaulay.
- Every homogeneous system of parameters for R is an R-regular sequence.

Monomial orders

Recall that $S = K[X_1, \dots, X_n]$: let Mon(S) be the set of monomials of S:

$$\mathrm{Mon}(S) = \{X_1^{u_1} \cdots X_n^{u_n} : (u_1, \dots, u_n) \in \mathbb{N}^n\}.$$

Definition

A *monomial order* on S is a total order $<$ on Mon(S) such that:

(i) $1 \leq \mu$ for every $\mu \in \mathrm{Mon}(S)$;

(ii) If $\mu_1, \mu_2, \nu \in \mathrm{Mon}(S)$ such that $\mu_1 \leq \mu_2$, then $\mu_1\nu \leq \mu_2\nu$.

Note that, if $<$ is a monomial order on S and μ, ν are monomials such that $\mu|\nu$, then $\mu \leq \nu$: indeed $1 \leq \nu/\mu$, so

$$\mu = 1 \cdot \mu \leq (\nu/\mu) \cdot \mu = \nu.$$

Monomial orders

Remark

The fact that a monomial order refines the divisibility partial order on $\mathrm{Mon}(S)$, together with the Hilbert basis theorem, makes a monomial order on S a well-order on $\mathrm{Mon}(S)$. This is the starting point for the theory of *Gröbner bases*.

Note that, fixed a monomial order $<$ on S, every nonzero polynomial $f \in S$ can be written uniquely as

$$f = a_1\mu_1 + \ldots + a_k\mu_k$$

with $a_i \in K \setminus \{0\}$, $\mu_i \in \mathrm{Mon}(S)$ and $\mu_1 > \mu_2 > \ldots > \mu_k$.

Definition

The *initial monomial* of f (w.r.t. $<$) is $\mathrm{in}(f) = \mu_1$.

Definition

If I is an ideal of $S = K[X_1, \dots, X_n]$, then the ideal $\text{in}(I) = (\{\text{in}(f) : f \in I\}) \subset S$ is called the *initial ideal* of I (w.r.t. $<$).

Definition

Polynomials $f_1, \dots, f_m$ of an ideal $I \subset S$ are a *Gröbner basis* of I (w.r.t. $<$) if $\text{in}(I) = (\text{in}(f_1), \dots, \text{in}(f_m))$.

Example

Consider the ideal $I = (f_1 = X_1^2 - X_2^2, f_2 = X_1X_3 - X_2^2)$ of $K[X_1, X_2, X_3]$. For the lexicographic monomial order the polynomials f_1, f_2 are not a Gröbner basis of I, indeed $X_1X_2^2 = \text{in}(X_3f_1 - X_1f_2)$ is a monomial of $\text{in}(I)$ which is not in the ideal $(\text{in}(f_1) = X_1^2, \text{in}(f_2) = X_1X_3)$.

Gröbner bases

Remark

As we saw, a system of generators of I may not be a Gröbner basis of I. On the contrary, a Gröbner basis of I is always a system of generators of I. Once again, the Hilbert basis theorem implies that, at least, any ideal of S has a finite Gröbner basis.

There is a way to compute a Gröbner basis of an ideal I (and so $\text{in}(I)$) starting from a system of generators of I, namely the *Buchsberger's algorithm*. Since monomial ideals are much simpler to study than polynomial ideals, it is important to relate properties of the ideal I with properties of the monomial ideal $\text{in}(I)$.

Remark

For any monomial order on S and ideal $I \subset S$ it is easy to check that $\dim S/I = \dim S/\text{in}(I)$.

To describe further relations, it is useful to see the passage from I to $\text{in}(I)$ as a deformation, interpretation given in the 80s by many people. From this interpretation it easily follows that

$$\operatorname{depth} S/\text{in}(I) \leq \operatorname{depth} S/I$$

for any homogeneous ideal $I \subset S$. In particular, given a homogeneous ideal $I \subset S$ and a monomial order on S, we have:

Proposition

$S/\text{in}(I)$ is Cohen-Macaulay $\implies$ S/I is Cohen-Macaulay.

It is easy to find examples where the converse to the above implication fails...

Motivations

An important instance in which the above implication can be reversed is:

Theorem (Bayer-Stillman, 1987)

For a degree reverse lexicographic monomial order, if the coordinates are generic (with respect to I), then

$$S/\operatorname{in}(I) \text{ is Cohen-Macaulay} \iff S/I \text{ is Cohen-Macaulay.}$$

On a different perspective, *Algebras with Straightening Law (ASL)* were introduced in the 80s by De Concini, Eisenbud and Procesi. This notion arose as an axiomatization of the underlying combinatorial structure observed by many authors in classical algebras like coordinate rings of flag varieties, their Schubert subvarieties and various kinds of rings defined by determinantal equations.

Motivations

Any ASL A has a discrete counterpart A_D that is defined by square-free monomials of degree 2, and it was proved by DEP that A is Cohen-Macaulay whenever A_D is so. This can also be seen because A can be realized as S/I in such a way that $A_D \cong S/\operatorname{in}(I)$ with respect to some monomial order. In this case, in all the known examples also the converse implication was true. This lead Herzog to conjecture the following:

Conjecture (Herzog)

Let $I \subset S$ be a homogeneous ideal such that $\operatorname{in}(I)$ **is square-free** for some monomial order. Then

$$S/\operatorname{in}(I) \text{ is Cohen-Macaulay} \iff S/I \text{ is Cohen-Macaulay.}$$

Stanley-Reisner correspondence

One can attach to a square-free monomial ideal $J \subset S$ a simplicial complex on n vertices $\Delta(J)$ with set of faces

$$\{\{i_1, \dots, i_k\} : X_{i_1} \cdots X_{i_k} \notin J\}.$$

Vice versa, a simplicial complex on n vertices Γ yields a square-free monomial ideal of S

$$I_\Gamma = (X_{i_1} \cdots X_{i_k} : \{i_1, \dots, i_k\} \notin \Gamma).$$

Clearly $\Delta(I_\Gamma) = \Gamma$ and $I_{\Delta(J)} = J$, and this is the so-called **Stanley-Reisner correspondence**. It links aspects from commutative algebra, from topology and from combinatorics.

Conjecture (Herzog)

Let $I \subset S$ be a homogeneous ideal such that $\mathrm{in}(I)$ **is square-free** for some monomial order. Then

$$S/\,\mathrm{in}(I) \text{ is Cohen-Macaulay} \iff S/I \text{ is Cohen-Macaulay}.$$

Attempts

A first approach that one could try to prove Herzog's conjecture is to exploit Bayer and Stilman result and to show that:

???

Let $I \subset S$ be a homogeneous ideal such that $\mathrm{in}(I)$ is a square-free monomial ideal. Then, $\mathrm{gin}(\mathrm{in}(I)) = \mathrm{gin}(I)$.

Unfortunately it was exhibited a counterexample to the above statement by Conca in 2007.

Attempts

Theorem (_ , 2009)

Let $I \subset S$ be a homogeneous ideal such that S/I is Cohen-Macaulay. Then the simplicial complex associated to $\sqrt{\text{in}(I)}$ via the Stanley-Reisner correspondence, $\Delta(\sqrt{\text{in}(I)})$, is strongly connected.

Since a Cohen-Macaulay simplicial complex is strongly connected, the above statement goes in direction of Herzog's conjecture ($\text{in}(I)$ is square-free iff $\sqrt{\text{in}(I)} = \text{in}(I)$). However, it is not difficult to produce examples of homogeneous ideals $I \subset S$ such that S/I is Cohen-Macaulay but $S/\sqrt{\text{in}(I)}$ is not (for a monomial ideal $J \subset S$, it is always true that $\text{depth}\, S/J \leq \text{depth}\, S/\sqrt{J}$, so there are more chances of being Cohen-Macaulay for $S/\sqrt{\text{in}(I)}$ than for $S/\text{in}(I)$).

The main result

Actually the property described just above has been very important to prove the following:

Theorem (Conca-_ , 2018)

Let $I \subset S$ be a homogeneous ideal such that $\text{in}(I)$ is square-free for some monomial order. Then $S/\text{in}(I)$ is Cohen-Macaulay $\iff$ S/I is Cohen-Macaulay. More generally,

$$\dim_K H^i_{\mathfrak{m}}(S/I)_j = \dim_K H^i_{\mathfrak{m}}(S/\text{in}(I))_j \quad \forall\ i,j \in \mathbb{Z}.$$

As a consequence, we get the following:

Corollary

For any ASL A, we have that A is Cohen-Macaulay if and only if A_D is Cohen-Macaulay.

In view of the above result it is natural to inquire in more details homogeneous ideals admitting a square-free initial ideal.

Conjecture (Constantinescu, De Negri, _ , 2019)

Let $I \subset S$ be a homogeneous prime ideal defining a smooth variety. If $\mathrm{in}(I)$ is square-free for some monomial order, then $\Delta(\mathrm{in}(I))$ is contractible.

Though all known ideals defining smooth projective varieties and admitting a square-free initial ideal (Grassmannians, Veronese embeddings and Segre products of projective spaces...) satisfy the above conjecture, it is mostly open; we could prove it in a few cases:

Gröbner degenerations of smooth projective varieties

Theorem (Constantinescu, De Negri, _ , 2019)

Let $I \subset S$ be a homogeneous prime ideal defining a smooth variety. If $\text{in}(I)$ is square-free for a degrevlex monomial order, then $\Delta(\text{in}(I))$ is contractible.

If $\dim S/I = 2$, that is $C = \text{Proj}\, S/I$ is a projective curve, we can say something also for monomial orders different from degrevlex:

Theorem (Constantinescu, De Negri, _ , 2019)

Let $I \subset S$ be a homogeneous prime ideal defining a smooth curve and assume that $\text{in}(I)$ is square-free for some monomial order. Then:

- $\Delta(\text{in}(I))$ has at least one leaf.
- If $K = \mathbb{Q}$, $\Delta(\text{in}(I))$ has no cycles or more than one cicles.

THANK YOU FOR YOUR ATTENTION !