Square-free Gröbner degenerations

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Commutative algebra and its applications

Basic notions in Commutative Algebra

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Remark

For any monomial order on S and ideal $I \subset S$, one has $\dim S/I = \dim S/\mathrm{in}(I)$. If furthermore I is homogeneous, $\dim_K(S/I)_j = \dim_K(S/\mathrm{in}(I))_j$ for any $j \in \mathbb{Z}$.

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Proposition

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It is easy to find examples where the converse to the above implication fails...

Motivations

An important instance in which the previous implication can be reversed is:

Theorem (Bayer-Stillman, 1987)

For a degree reverse lexicographic monomial order, if the coordinates are generic (with respect to I), then

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On a different perspective, *Algebras with Straightening Law (ASL)* were introduced in the 80s by De Concini, Eisenbud and Procesi. This notion arose as an axiomatization of the underlying combinatorial structure observed by many authors in classical algebras like coordinate rings of flag varieties, their Schubert subvarieties and various kinds of rings defined by determinantal equations.

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Conjecture (Herzog)

Let $I \subset S$ be a homogeneous ideal such that $\operatorname{in}(I)$ is square-free for some monomial order. Then

$$S/\operatorname{in}(I) \text{ is Cohen-Macaulay} \iff S/I \text{ is Cohen-Macaulay.}$$

Attempts

A first approach that one could try to prove Herzog's conjecture is to exploit Bayer and Stilman result and to show that:

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Let $I \subset S$ be a homogeneous ideal such that $\mathrm{in}(I)$ is a square-free monomial ideal. Then, $\mathrm{gin}(\mathrm{in}(I)) = \mathrm{gin}(I)$.

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Unfortunately it was exhibited a counterexample to the above statement by Conca in 2007.

Attempts

Theorem (_ , 2009)

Let $I \subset S$ be a homogeneous ideal such that S/I is Cohen-Macaulay. Then the simplicial complex associated to $\sqrt{\text{in}(I)}$ via the Stanley-Reisner correspondence, $\Delta(\sqrt{\text{in}(I)})$, is strongly connected.

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Since a Cohen-Macaulay simplicial complex is strongly connected, the above statement goes in direction of Herzog's conjecture (in(I) is square-free iff $\sqrt{\text{in}(I)} = \text{in}(I)$).

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Since a Cohen-Macaulay simplicial complex is strongly connected, the above statement goes in direction of Herzog's conjecture ($\text{in}(I)$ is square-free iff $\sqrt{\text{in}(I)} = \text{in}(I)$). However, it is not difficult to produce examples of homogeneous ideals $I \subset S$ such that S/I is Cohen-Macaulay but $S/\sqrt{\text{in}(I)}$ is not (for a monomial ideal $J \subset S$, it is always true that depth $S/J \leq$ depth $S/\sqrt{J}$, so there are more chances of being Cohen-Macaulay for $S/\sqrt{\text{in}(I)}$ than for $S/\text{in}(I)$).

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As a consequence, we get the following:

Corollary

For any ASL A, we have that A is Cohen-Macaulay if and only if A_D is Cohen-Macaulay.

In view of the above result it is natural to inquire in more details homogeneous ideals admitting a square-free initial ideal.

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Conjecture (Constantinescu, De Negri, _ , 2019)

Let $I \subset S$ be a homogeneous prime ideal defining a smooth variety. If $\mathrm{in}(I)$ is square-free for some monomial order, then $\Delta(\mathrm{in}(I))$ is contractible.

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This conjecture it is mostly open; at least we know that $\Delta(\text{in}(I))$ has to be connected (indeed, strongly connected) by a result of Kalkbrener and Sturmfels. Some evidence for the conjecture:

Gröbner degenerations of smooth projective varieties

Theorem (Constantinescu, De Negri, _ , 2019)

Let $I \subset S$ be a homogeneous prime ideal defining a smooth variety. If $\text{in}(I)$ is square-free for a degrevlex monomial order, then $\Delta(\text{in}(I))$ is contractible.

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If $\dim S/I = 2$, that is $C = \text{Proj}\, S/I$ is a projective curve, we can say something also for monomial orders different from degrevlex:

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THANKS FOR YOUR ATTENTION!