Umbral Calculus and Approximation Operators

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Three unrelated topics?

N black beads, $N-M$ white beads, sampling with replacement + addition of c beads of the same colour.

X_n is number of black beads after n drawings (Markov-Pólya distribution)

I WLLN (Weak Law of Large Numbers) for X_n: $\frac{X_n}{n} \xrightarrow{P} ?$

II Convergence of Stancu operator: $f \in C[0,1]$,

$$\left\| \frac{1}{1^{[n,-a]}} \sum_{k=0}^{n} \binom{n}{k} x^{[k,-a]} (1-x)^{[n-k,-a]} f\left(\frac{k}{n}\right) - f \right\|_{\infty} \to ?$$

where $x^{[k,-a]} = x(x+a)\ldots(x+(k-1)a)$

III $$\sum_{k=0}^{n} \left(k - n\frac{\alpha}{\alpha+\beta}\right)^2 \frac{\binom{\alpha+k-1}{k}\binom{\beta+n-k-1}{n-k}}{\binom{\alpha+\beta+n-1}{n}} \to ?$$

Outline

- explain relation between WLLN and Popoviciu approximation operators

- use Umbral Calculus to compute moments of probability distributions on $0, 1, \ldots, n$

- extensions to other types of approximation operators

Weierstrass approximation

$f : [0,1] \to \mathbb{R}$ continuous, $\mathcal{P} = \mathbb{R}[x]$

$\exists (p_n)_{n\in\mathbb{N}} \in \mathcal{P}: \ p_n(x) \to f(x)$ uniformly in x

Probabilistic proof (Bernstein, ...): $X_n \sim \mathrm{Bin}(n,x)$

$$p_n(x) := \mathbb{E} f\left(\frac{X_n}{n}\right) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

Chebyshev inequality:

$$\mathbb{P}(|X - \mathbb{E}(X)| > \varepsilon) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2}$$

This implies $\mathbb{E} f\left(\frac{X_n}{n}\right) \to f(x)$ (variant of WLLN)

Convergence is uniform in $x \in [0,1]$ since

$$\mathrm{Var}\left(\frac{X_n}{n}\right) = \frac{x(1-x)}{n} \leq \frac{1}{4n}$$

Popoviciu operators

$$L_n^{D,I}(f) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

is positive, linear operator on $C[0,1]$.

Generalization: Popoviciu (1931) + papers by Manole, Moldovan, Lupaş, Sablonniére, Stancu, ...

$$L_n^{Q,I}(f) := \frac{1}{q_n(1)} \sum_{k=0}^{n} q_k(x) q_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

where $q_n(x) \geq 0$ for all n and

$$\forall n, x, y : \; q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y)$$

$$\mathbb{P}(X_n = k) = \frac{q_k(x)\, q_{n-k}(1-x)}{q_n(1)} \;, k = 0, \ldots, n$$

$Q : \mathcal{P} \to \mathcal{P}$ linear operator with $Qq_k := q_{k-1}$.

$$L_n^{Q,I}(1) = 1;\; L_n^{Q,I}(x) = \mathbb{E}\left(\frac{X_n}{n}\right);\; L_n^{Q,I}(x^2) = \mathbb{E}\left(\frac{X_n^2}{n^2}\right)$$

Examples

- $$q_k(x) = \frac{x^k}{k!} \quad (Q = D)$$

  $$\mathbb{P}\left(X_n^{\alpha,\beta} = k\right) = \binom{n}{k} \left(\frac{\alpha}{\alpha+\beta}\right)^k \left(\frac{\beta}{\alpha+\beta}\right)^{n-k}.$$

  $X_n^{\alpha,\beta} \sim \text{Bin}\left(n, \frac{\alpha}{\alpha+\beta}\right)$.

  Corresponding approximation operator: Bernstein operator

- $$q_k(x) = \frac{x(x+a)\dots(x+(k-1)a)}{k!}$$

  (shifted factorials $Q = \frac{1}{a}(I - E^{-a})$)

  $X_n^{\alpha,\beta}$ has Markov-Pólya distribution.

  Corresponding approximation operator: Stancu operator

Characterization of X_n

$$\forall n, x, y: \; q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y)$$

implies

$$\sum_{n=0}^{\infty} q_n(x)\, z^n = e^{x g(z)}$$

Define independent random variables X, Y:

$$P(X=k) = \theta^k\, q_k(\alpha)\, e^{-\alpha g(\theta)}$$

$$P(Y=h) = \theta^h\, q_h(\beta)\, e^{-\beta g(\theta)}.$$

Then

$$P(X=k \mid X+Y=n) = \frac{q_k(\alpha)\, q_{n-k}(\beta)}{q_n(\alpha+\beta)}.$$

Popoviciu-Sheffer operators

Crăciun (2001) studied generalization of Popoviciu operators:

$$L_n^{Q,S}(f) := \frac{1}{s_n(1)} \sum_{k=0}^{n} q_k(x) s_{n-k}(1-x)\, f\left(\frac{k}{n}\right)$$

where $q_k(x) \geq 0$, $s_k(x) \geq 0$

$$s_n(x+y) = \sum_{k=0}^{n} q_k(x)\, s_{n-k}(y)$$

cf. $q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y)$

$Q : \mathcal{P} \to \mathcal{P}$ linear operator with $Qq_k := q_{k-1}$. $S : \mathcal{P} \to \mathcal{P}$ invertible linear operator with $Ss_k := q_k$.

$(s_k)_{k\in\mathbb{N}}$ are called Sheffer sequences

Examples (continued)

- $$q_k(x) = \frac{x\,(x+ak)^{k-1}}{k!}$$

  (Abel polynomials with delta operator DE^{-a})

  $X_n^{\alpha,\beta}$ has quasi-binomial distribution.

  Corresponding approximation operator: "second operator" introduced by Cheney and Sharma (1964)

- $$s_k(x) = \frac{(x+ka)^k}{k!}$$

  (Sheffer sequence for $Q = DE^{-a}$ and $S = E^a Q' = I - aD$)

  Corresponding approximation operator: "first operator" introduced by Cheney and Sharma (1964)

Korovkin theorems

Necessary conditions for $(L_n f)(x) \to f(x)$ uniformly in x:

$$||L_n\, x - x||_\infty \to 0 \text{ and } ||L_n\, x^2 - x^2||_\infty \to 0.$$

Special case of one of the Korovkin theorems in approximation theory (cf. version of WLLN presented earlier).

Main problem : compute $L_n^{Q,S}\, x^\ell = \mathbb{E}\left(\frac{X_n^\ell}{n^\ell}\right)$

$$\mathbb{P}(X_n = k) = \frac{q_k(\alpha)\, s_{n-k}(\beta)}{s_n(\alpha+\beta)},\ \ k = 0, \ldots, n$$

Approach (first $S = I$) : define

$$V_\ell\, q_n = \sum_{k=0}^{n} k^{[\ell]}\, q_k(\alpha)\, q_{n-k} \Rightarrow \frac{V_\ell\, q_n(\beta)}{q_n(\alpha+\beta)} = \mathbb{E}\left(\frac{X_n^{[\ell]}}{n^\ell}\right)$$

$k^{[\ell]} = k(k-1)\ldots(k-\ell+1)$ (factorial moments)

Expansions: Umbral Calculus

Use "Taylor" expansion for V_ℓ in terms of Q

$$q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y)$$

$Q : \mathcal{P} \to \mathcal{P}$ linear operator with $Q q_n := q_{n-1}$.

$$\sum_{n=0}^{\infty} q_n(x)\, z^n = e^{x\, g(z)}$$

Operator Expansion Theorem

$$T = \sum_{k=0}^{\infty} [T\, q_k]_{x=0}\, Q^k$$

Apply to $T = D$:

$$D = \sum_{k=0}^{\infty} (D q_k)(0)\, Q^k = g(Q) \Leftrightarrow Q = g^{-1}(D)$$

Apply to $T = E^\alpha$ (shift-operator):

$$E^a = \sum_{k=0}^{\infty} (E^\alpha q_k)(0)\, Q^k = e^{\alpha g(Q)} = e^{\alpha D}$$

Moments: $S = I$, $s_n = q_n$

Apply expansion formulas to V_ℓ (cf. Manole 1987):

$$\begin{aligned} V_\ell &= \sum_{k=0}^{\infty} (V_\ell q_k)(0)\, Q^k \\ &= \sum_{k=0}^{\infty} k^{[\ell]}\, q_k(\alpha)\, Q^k \\ &= \sum_{k=\ell}^{\infty} k^{[\ell]}\, q_k(\alpha)\, Q^k \\ &= Q^\ell \sum_{k=0}^{\infty} k^{[\ell]}\, q_k(\alpha)\, Q^k \\ &= Q^\ell \frac{d^\ell}{dQ^\ell} \left(\sum_{k=0}^{\infty} q_k(\alpha)\, Q^k \right) \\ &= Q^\ell \frac{d^\ell}{dQ^\ell} e^{\alpha\, g(Q)} \end{aligned}$$

- use $\frac{V_\ell\, q_n(\beta)}{q_n(\alpha+\beta)} = \mathbb{E}\left(\frac{X_n^{[\ell]}}{n^\ell}\right)$
- use $g'(Q) = \left(\left(g^{-1}(D)\right)'\right)^{-1} = (Q')^{-1}$

Special case $\ell = 1$

$$
\begin{aligned}
V_1 q_n(\beta) &= \alpha g'(Q)\, Q\, E^{\alpha} q_n(\beta) \\
&= \alpha \left(Q'\right)^{-1} Q E^{\alpha} q_n(\beta) \\
&= \alpha E^{\alpha} \left(\left(Q'\right)^{-1} q_{n-1} \right) (\beta) \\
&= \alpha E^{\alpha} n \frac{q_n(x)}{x} \,|_{x=\beta} \\
&= n\alpha \,\frac{q_n(\alpha+\beta)}{\alpha+\beta}
\end{aligned}
$$

N.B. Rodrigues formula: $n\, q_n = (x(Q')^{-1})\, q_{n-1}$
(equivalent to Lagrange inversion)

$$
\mathbb{E}\left(\frac{X_n}{n}\right) = \frac{V_1\, q_n(\beta)}{q_n(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}
$$

Similarly:

$$
V_2\, q_n(\beta) = \alpha \left[E^{\alpha} \left\{ g''(Q) + \alpha \left(g'\left(Q\right)\right)^2 \right\} q_{n-2} \right] (\beta)
$$

Moment formulas: general S

Same ideas as before, but now use $S\,s_n = q_n$ + commutativity of operators

$$\mathbb{E}\left(X_n^{[\ell]}\right) = \sum_{k=0}^{n} k^{[\ell]} \frac{q_k(\alpha)\, s_{n-k}(\beta)}{s_{n-k}(\alpha+\beta)}$$

$$= \frac{V_\ell\, s_n(\beta)}{s_n(\alpha+\beta)}$$

$$= \frac{Q^\ell \frac{d^\ell}{dQ^\ell}\, e^{\alpha\, g(Q)} s_n(\beta)}{s_n(\alpha+\beta)}$$

Special case $\ell = 1$:

$$\mathbb{E}\left(X_n\right) = \frac{\alpha\left[E^\alpha \left(Q'\right)^{-1} s_{n-1}\right](\beta)}{s_n(\alpha+\beta)}$$

$$= \frac{\alpha\left[E^\alpha S^{-1}\left(\frac{nq_n(x)}{x}\right)\right]_{x=\beta}}{s_n(\alpha+\beta)}$$

Extensions: generalized differentiation

$$D_c\, x^k = \frac{c_k}{c_{k-1}}\, x^{k-1}$$

shift operator $E_c^y = \Phi(yD_c)$ with $\Phi(t) = \sum_{n=0}^{\infty} \frac{t^n}{c_n}$

$$E_c^y q_n(x) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y)$$

$$\sum_{n=0}^{\infty} q_n\,(x)\, t^n = \Phi(xg(t)) \text{ and } Q = g^{-1}\,(D_c)$$

Choices for c_k:

- $c_k = k!$: ordinary differentiation
- $c_k = (1-q)\ldots(1-q^k)/(1-q)^k$: q-differentiation
  $D_q p(x) = (p(x) - p(qx))/(x - qx)$

  Special case $q = 0$: divided differences

Divided differences

Case $c_k = 1$: $D_c\, x^k = x^{k-1}$

$$E_c^y p\,(x) = \frac{xp(x) - yp(y)}{x - y}$$

$$\mathbb{P}(X_n^{\alpha,\beta} = k) = \frac{(\alpha - \beta)\, q_k\,(\alpha)\, q_{n-k}\,(\beta)}{\alpha q_k\,(\alpha) - \beta q_k\,(\beta)}$$

$$\mathbb{E}\left(\frac{X_n^{[\ell]}}{n^\ell}\right) = \frac{(\alpha - \beta)\, Q^\ell\, \frac{d^\ell}{dQ^\ell}\,\left(\frac{1}{1-\alpha g(Q)}\right) q_n(\beta)}{\alpha q_k\,(\alpha) - \beta q_k\,(\beta)}$$

Open problems:

- Probability distributions of this form??

- Interpretation in terms of conditional distributions??

Conclusions

- we have seen link between WLLN and approximation theory
- Umbral Calculus yields tools to compute moments of certain prob. distributions
- extensions need further investigation (interpretation?)
- ...

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