Primes in arithmetic progressions to large moduli

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How many primes are less than $x$ and congruent to $a \pmod{q}$? 

Theorem (Siegel-Walfisz)

If $q \leq (\log x)^A$ and $\gcd(a, q) = 1$ then 

$$\pi(x; q, a) = (1 + o(1)) \pi(x) \phi(q).$$

Theorem (GRH Bound)

Assume GRH. If $q \leq x^{1/2 - \epsilon}$ and $\gcd(a, q) = 1$ then 

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Introduction

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**Theorem (Bombieri-Vinogradov)**

Let $Q < x^{1/2-\epsilon}$. Then for any $A$

$$
\sum_{q \sim Q, (a,q)=1} \sup_{(a,q)=1} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}
$$

**Corollary**

For most $q \leq x^{1/2-\epsilon}$, we have

$$
\pi(x; q, a) = (1 + o(1)) \frac{\pi(x)}{\phi(q)}
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for every $a$ with $\gcd(a, q) = 1$. 

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Beyond GRH

Pioneering work by Bombieri, Fouvry, Friedlander, Iwaniec went beyond the $x^{1/2}$ barrier in special circumstances.

\[ \sum_{q \sim x^\theta} (q, a) = \frac{\pi(x; q, a) - \pi(x) \phi(q)}{\ll a (\theta - 1/2)^2 x (\log \log x)} + \frac{x \log 3}{\log x}. \]

This is non-trivial when $\theta$ is very close to $1/2$.

\[ \sum_{q \sim x^{4/7 - \epsilon}} (q, a) = \frac{\pi(x; q, a) - \pi(x) \phi(q)}{\ll a A x \log A x}. \]

This is often an adequate substitute for BV with exponent $4/7$.
Beyond GRH

Pioneering work by Bombieri, Fouvry, Friedlander, Iwaniec went beyond the $x^{1/2}$ barrier in special circumstances.

**Theorem (BFI1)**

Fix $a$. Then we have (uniformly in $\theta$)

$$\sum_{q \sim x^\theta \atop (q,a)=1} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll a (\theta - 1/2)^2 \frac{x(\log \log x)^O(1)}{\log x} + \frac{x}{\log^3 x}.$$

This is non-trivial when $\theta$ is very close to $1/2$.

**Theorem (BFI2)**

Fix $a$. Let $\lambda(q)$ be ‘well-factorable’. Then we have

$$\sum_{q \sim x^{4/7-\epsilon} \atop (q,a)=1} \lambda(q) \left( \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right) \ll_{a, A} \frac{x}{\log^A x}.$$

This is often an adequate substitute for BV with exponent $4/7$!
More recently, Zhang went beyond $x^{1/2}$ for smooth/friable moduli.

**Theorem (Zhang, Polymath)**

\[
\sum_{q \leq x^{1/2+7/300-\epsilon}} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}
\]

The implied constant is independent of $a$. 
New results

**Theorem (M.)**

Let $\delta < 1/42$ and $Q_\delta := \{ q \sim x^{1/2+\delta} : \exists d|q \text{ s.t. } x^{2\delta+\epsilon} < d < x^{1/14-\delta}\}$.

$$\sum_{\substack{q\in Q_\delta \atop (q,a)=1}} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x(\log \log x)^O(1)}{\log^5 x}. $$
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$$\sum_{q \in Q_{\delta}} \left| \pi(x ; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_a \frac{x(\log \log x)^{O(1)}}{\log^5 x}.$$ 

Corollary

Let $\delta < 1/42$. For $(100 - O(\delta))\%$ of $q \sim x^{1/2+\delta}$ we have

$$\pi(x ; q, a) = (1 + o(1)) \frac{\pi(x)}{\phi(q)}$$

Corollary

$$\sum_{q_1 \sim x^{1/21}} \sum_{q_2 \sim x^{10/21-\epsilon}} \left| \pi(x ; q_1 q_2, a) - \frac{\pi(x)}{\phi(q_1 q_2)} \right| \ll_a \frac{x(\log \log x)^{O(1)}}{\log^5 x}.$$
Theorem (M.)

Let $\lambda(q)$ be ‘very well factorable’. Then we have

$$\sum_{\substack{q \leq x^{3/5-\epsilon} \\ (q,a)=1}} \lambda(q) \left( \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right) \ll_{a,A} \frac{x}{(\log x)^A}.$$ 

The $\beta$-sieve weights are ‘very well factorable’ for $\beta \geq 2$. 

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$$\sum_{q \leq x^{3/5 - \epsilon} \atop (q,a) = 1} \lambda(q) \left( \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right) \ll_{a,A} x \left( \frac{\log x}{A} \right)^{6}.$$

The $\beta$-sieve weights are ‘very well factorable’ for $\beta \geq 2$.

Corollary

Let $\lambda^+(d)$ be sieve weights for the linear sieve. Then

$$\sum_{q \leq x^{7/12 - \epsilon} \atop (q,a) = 1} \lambda^+(q) \left( \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right) \ll \frac{x}{(\log x)^{A}}.$$
## Comparison

<table>
<thead>
<tr>
<th>Result</th>
<th>Size of $q$</th>
<th>Type of $q$</th>
<th>Proportion of $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFI1</td>
<td>$x^{1/2+o(1)}$</td>
<td>All</td>
<td>$(100 - \delta)%$</td>
</tr>
<tr>
<td>BFI2</td>
<td>$x^{4/7-\epsilon}$</td>
<td>Factorable</td>
<td>$\delta%$</td>
</tr>
<tr>
<td>Zhang</td>
<td>$x^{1/2+7/300-\epsilon}$</td>
<td>Factorable</td>
<td>$\delta%$</td>
</tr>
<tr>
<td>M1</td>
<td>$x^{11/21-\epsilon}$</td>
<td>Partially Factorable</td>
<td>$(100 - \delta)%$</td>
</tr>
<tr>
<td>M2</td>
<td>$x^{3/5-\epsilon}$</td>
<td>Factorable</td>
<td>$\delta%$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Result</th>
<th>Coefficients</th>
<th>Residue class</th>
<th>Cancellation</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFI1</td>
<td>Absolute values</td>
<td>Fixed</td>
<td>$o(1)$</td>
</tr>
<tr>
<td>BFI2</td>
<td>Factorable weights</td>
<td>Fixed</td>
<td>$\log^A x$</td>
</tr>
<tr>
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<td>Absolute values</td>
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</table>

Note that $3/5 > 4/7 > 11/21 > 1/2 + 7/300$. 
The overall proof follows the same lines as previous approaches:

1. Apply a combinatorial decomposition to $\Lambda(n)$

2. Reduce the problem to estimating exponential sums of convolutions.

3. Apply different techniques in different ranges to estimate exponential sums.
   - Bounds from the spectral theory of automorphic forms (Kuznetsov Trace Formula)
   - Bounds from Algebraic Geometry (Weil bound/Deligne bounds)

4. Ensure that (essentially) all ranges are covered.

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Proof overview

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Let us recall the situation when $q \sim x^{1/2+\delta}$ where $\delta > 0$ is fixed but small. Using BFI proof ideas:

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2. Working through the BFI argument their proof can essentially handle all such numbers except for
   - Products $p_1p_2p_3p_4p_5$ of 5 primes with $p_i = x^{1/5+O(\delta)}$
   - Products $p_1p_2p_3p_4$ of 4 primes with $p_i = x^{1/4+O(\delta)}$
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BFI result follows on noting that these terms are only a \( O(\delta) \) proportion of the terms.

We can concentrate on these ‘bad products’.
Consider terms $p_1 p_2 p_3 p_4 p_5$ with $p_i \in [x^{1/5-\delta}, x^{1/5+\delta}]$

- Zhang-style estimates can handle all terms when the modulus is smooth, but are least efficient for products of 5 primes, so don’t help.
Products of 5 Primes

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- Instead we refine some of the estimates for exponential sums coming from Kuznetsov/Kloostermana.

I still can’t handle these terms, but they now contribute $O((\log \log x)^{O(1)} / \log 4^x)$ proportion for a wide range of $q$.

(This is why I only save $4-\epsilon \log x$ factors.)

Algebraic Geometry doesn’t help much, but we can refine Kuznetsov-based estimates to handle these terms.
Consider terms $p_1 p_2 p_3 p_4 p_5$ with $p_i \in [x^{1/5-\delta}, x^{1/5+\delta}]$

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- Refinement of BFI can handle $p_1 p_2 p_3 p_4 p_5$ with $q < x^{4/7-\varepsilon}$ when $p_i \approx x^{1/5}$ except when $p_i \in [x^{1/5 \log^{-A} x}, x^{1/5 \log^A x}]$
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- Provided $q$ has a suitable factor close to $x^{1/2}$, we can handle these terms using the Weil bound.

The technical parts which spectral theory estimates can’t handle are precisely parts that the algebraic geometry estimates are best at *when there is a suitable factor*
As stated these ideas combine to give a result for \( q \sim x^{1/2+\delta} \) for some small \( \delta > 0 \).

To get good numerics, need to refine estimates for other parts of prime decomposition

- Generalize ideas based on Deligne’s work (Fouvry, Kowalski, Michel) to handle products of 3 primes when the modulus has a convenient small factor.
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- Generalize ideas of Fouvry for products of 7 primes when the modulus has a convenient small factor.
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- Generalize ideas of Fouvry for products of 7 primes when the modulus has a convenient small factor.
- Auxilliary estimate when there is a very small factor

Together these improve all terms in the decomposition, with a reasonable range of \( q \)!
Overview

Spectral Theory
- Fouvry-Kowalski-Michel style
- Bombieri-Friedlander-Iwaniec style
- Fouvry style
- Zhang style

Algebraic Geometry
- Products of 3 Primes
- Products of 5 Primes
- Product of 7 Primes
- Combinatorial Decomposition
- Primes in APs

Figure: Outline of steps to prove primes in arithmetic progressions
Thank you for listening.