The resolvent algebra of non-relativistic Bose fields: a C*-dynamical approach to interacting quantum systems

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AQFT: WHERE OPERATOR ALGEBRA MEETS MICROLOCAL ANALYSIS

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Motivation

Many body theory for Bosons: Bose fields

- annihilation and creation operators, **complex** $f, g \in \mathcal{D}(\mathbb{R}^s)$,

$$[a(f), a^*(g)] = \langle f, g \rangle \, 1 \,, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0 \,.$$

- fields (**real** linear)

$$\phi(f) \doteq a^*(f) + a(f) \,.$$

- Fock space $\mathcal{F}$: generated by fields from vacuum Ω
- dynamics: (pair potentials $V \in C_0(\mathbb{R}^s)$)

$$\boldsymbol{H} = \int d\boldsymbol{x} \, \partial a^*(\boldsymbol{x}) \partial a(\boldsymbol{x}) + \int d\boldsymbol{x} \int d\boldsymbol{y} \; a^*(\boldsymbol{x}) a^*(\boldsymbol{y}) V(\boldsymbol{x} - \boldsymbol{y}) a(\boldsymbol{x}) a(\boldsymbol{y})$$

Standard framework (“bookkeeping”), but …

Longstanding question:

Existence of a "kinematical" C*-algebra $\mathfrak{A}$, encoding the CCRs, such that the solutions of the Heisenberg equation lie in $\mathfrak{A}$, *i.e.*

$$\partial_t A(t) = i\,[\boldsymbol{H}, A(t)]\,,\ A(0) \in \mathfrak{A}_{\text{o}}\ \text{ implies }\ A(t) \in \mathfrak{A}_{\text{o}},\ t \in \mathbb{R}\,?$$

Solutions $A(t) = \alpha(t)(A)$, where $\alpha(t)$ are automorphisms of $\mathfrak{A}$, $t \in \mathbb{R}$.

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Apparent obstructions to this idea: [Bratteli, Robinson]

> First, no global statement of the time development as a group of *-automorphisms of an appropriate C^*-algebra has been obtained and, second, there are plausible physical reasons for believing that such a development, if it existed, would be discontinuous in the norm topology.

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> The real trouble maker is $\rho(x)$ which in some representation like the Fock representation is well defined but in others, like the one based on the tracial state, is truly infinite. Though there is no doubt that for a smooth v H from (1.2) determines a time evolution in the Fock representation, an automorphism of the algebra of observables, which would be valid in any state, cannot be expected in general.

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"Large field problem". **Solution:** resolvent algebra of fields

Outline

- Resolvent algebra
- Gauge transformations
- Structure of observables
- Dynamics of observables and fields
- Applications
- Summary

Resolvent algebra

Resolvent algebra **abstractly** defined by relations between symbols

$$R(\lambda, f) = (i\lambda + \phi(f))^{-1}, \quad \lambda \in \mathbb{R}\backslash\{0\}, \ f \in \mathcal{D}(\mathbb{R}^s)$$

1. $R(\lambda, 0) = (i\lambda)^{-1} 1$
2. $R(\lambda, f)^* = R(-\lambda, f)$
3. $R(\lambda, f) - R(\mu, f) = i(\mu - \lambda) R(\lambda, f) R(\mu, f)$
4. $[R(\lambda, f),\, R(\mu, g)] = i\sigma(f, g)\, R(\lambda, f)\, R(\mu, g)^2 R(\lambda, f)$
5. $\nu\, R(\nu\lambda,\, \nu f) = R(\lambda, f)$
6. $R(\lambda, f) R(\mu, g)$
   $= R(\lambda + \mu,\, f + g)[R(\lambda, f) + R(\mu, g) + i\sigma(f, g) R(\lambda, f)^2 R(\mu, g)]$

Resolvent algebra: unital C*-algebra $\mathfrak{R}$ generated by all sums and products of resolvents; faithfully represented on Fock space $\mathcal{F}$.

Gauge transformations

On $\mathfrak{R}$ acts global gauge group $\Gamma \simeq U(1)$ given by

$$\gamma(u)(R(\lambda, f)) \doteq R(\lambda, e^{iu} f) \overset{\mathcal{F}}{=} e^{iuN} R(\lambda, f) e^{-iuN}, \quad u \in [0, 2\pi],$$

N particle number operator.

Fact: action of gauge transformations discontinuous in C*-sense, but one has

Lemma

Let $R \in \mathfrak{R}$. Its Fourier components are elements of $\mathfrak{R}$, i.e.

$$R_m \overset{\mathcal{F}}{=} (1/2\pi)^{-1} \int_0^{2\pi} du \, e^{imu} \, \gamma(u)(R) \in \mathfrak{R}, \quad m \in \mathbb{Z}.$$

Note: integral is defined in the strong operator topology on $\mathcal{F}$.

Observable algebra: $\mathfrak{A} = \mathfrak{R}^\Gamma \subset \mathfrak{R}$ (gauge invariant elements).

Outline of proof:

Given $f \in \mathcal{D}(\mathbb{R}^s)$, put (i) $L \doteq \mathbb{C}\, f$, (ii) $\mathcal{F}(L) \subset \mathcal{F}$ Fock space over L, (iii) $\mathfrak{R}(L) \subset \mathfrak{R}$ resolvent algebra generated by $R(\lambda, g)$, where $g \in L$, $\lambda \in \mathbb{R}\backslash\{0\}$. Consider

$$u, v \mapsto e^{im(u-v)}\gamma(v)(R(\lambda, f))^* \, \gamma(u)(R(\lambda, f)) = e^{im(u-v)} R(-\lambda, e^{iv} f)\, R(\lambda, e^{iu} f) \in \mathfrak{R}(L)\,.$$

Commutator of underlying field operators:

$$[\phi(e^{iu} f), \phi(e^{iv} f)] = (e^{i(v-u)} - e^{-i(v-u)})\, \langle f, f\rangle \neq 0 \quad \text{if} \quad (u-v) \neq \pi\mathbb{Z}\,.$$

So the operator is compact on $\mathcal{F}(L)$ if $(u-v) \neq \pi\mathbb{Z}$. Hence, the resulting function has values in the *compact ideal* $\mathfrak{C}(L) \subset \mathfrak{R}(L)$ for almost all $(u, v) \in \mathbb{R}^2$ and it is bounded. This implies (s.o. topology)

$$\int_0^{2\pi} dv \int_0^{2\pi} du\, e^{im(u-v)}\gamma(v)(R(\lambda, f))^* \, \gamma(u)(R(\lambda, f)) = \Big| \int_0^{2\pi} du\, e^{imu}\gamma(u)(R(\lambda, f))\Big|^2 \in \mathfrak{C}(L)\,.$$

Polar decomposition: $\int_0^{2\pi} du\, e^{imu}\gamma(u)(R(\lambda, f)) \in \mathfrak{C}(L) \subset \mathfrak{R}$.

Structure of observables

Facts:

- $\mathfrak{A}$ acts faithfully on Fock space $\mathcal{F} = \oplus_n \mathcal{F}_n$ (since $\mathfrak{R}$ does)
- $\rho_n(\mathfrak{A}) \doteq \mathfrak{A} \upharpoonright \mathcal{F}_n$ disjoint, non-faithful representations of $\mathfrak{A}$, $n \in \mathbb{N}_0$

Strategy of analysis:

- clarify structure of each $\rho_n(\mathfrak{A})$
- understand relation between different algebras $\rho_n(\mathfrak{A})$, $n \in \mathbb{N}_0$

Definition: $\mathfrak{C}_m$ compact operators on $\mathcal{F}_m$. Natural embedding into $\mathcal{F}_n$

$$\mathfrak{C}_m \mapsto \mathfrak{C}_{m\,n} \doteq \mathfrak{C}_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m}, \quad 0 \le m \le n$$

$\mathfrak{K}_n \doteq$ linear span of $\mathfrak{C}_{m\,n}$, $0 \le m \le n$ (AF algebra).

Proposition

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Definition: $\{\mathfrak{K}_n, \epsilon_n\}_{n\in\mathbb{N}_0}$ where $\epsilon_n(\mathfrak{K}_n) \doteq \mathfrak{K}_n \otimes_s 1 \subset \mathfrak{K}_{n+1}$ (directed system)

Relation between algebras $\mathfrak{K}_n = \rho_n(\mathfrak{A})$, $n \in \mathbb{N}_0$? Clustering properties!

$$\boldsymbol{\Phi}^n(\boldsymbol{x}) \doteq \underbrace{\Phi_1 \otimes_s \cdots \otimes_s \Phi_{n-1}}_{\boldsymbol{\Phi}^{n-1}} \otimes_s \Phi_n(\boldsymbol{x}) \in \mathcal{F}_n\,, \quad \Phi_1, \dots, \Phi_n \in \mathcal{F}_1\,,\ \boldsymbol{x} \in \mathbb{R}^s\,.$$

Proposition

Let $n \in \mathbb{N}$, $A \in \mathfrak{A}$.

(i) $\lim_{\boldsymbol{x}\to\infty} \langle \boldsymbol{\Psi}^n(\boldsymbol{x}), \rho_n(A)\, \boldsymbol{\Phi}^n(\boldsymbol{x})\rangle = n^{-1} \langle \boldsymbol{\Psi}^{n-1}, \rho_{n-1}(A)\, \boldsymbol{\Phi}^{n-1}\rangle \langle \Psi_n, \Phi_n \rangle$

(ii) $\rho_n(A) = \sum_{m=0}^n C_{m\,n}$ *implies* $\rho_{n-1}(A) = \sum_{m=0}^n \frac{n-m}{n}\, C_{m\, n-1}$

Recall notation: $C_{k\,l} = C_k \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{l-k} \in \mathfrak{C}_{k\,l} \subset \mathfrak{K}_l$.

Definition: $\{\mathfrak{K}_n, \kappa_n\}_{n\in\mathbb{N}_0}$ where $\kappa_n : \mathfrak{K}_n \to \mathfrak{K}_{n-1}$ homomorphism given by $\kappa_n(\sum_{m=0}^n C_{m\,n}) = \sum_{m=0}^n \frac{n-m}{n}\, C_{m\, n-1}$ (inverse system)

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Remark: C*-algebra, algebraic operations component-wise defined. Preceding result implies $\mathfrak{A} \subset \mathfrak{K}$. Extend $\mathfrak{A}$ in order to obtain equality!

Definition: $\overline{\mathfrak{A}}$ is the C*-algebra of bounded operators on $\mathcal{F}$ satisfying $A \upharpoonright \oplus_{m=0}^{n} \mathcal{F}_m \in \mathfrak{A} \upharpoonright \oplus_{m=0}^{n} \mathcal{F}_m$, $n \in \mathbb{N}_0$.

Remark: $\mathfrak{A}$ dense in $\overline{\mathfrak{A}}$ with regard to topology induced by seminorms $\| \cdot \|_n$, $n \in \mathbb{N}_0$. Differences between algebras only visible in states containing an infinity of particles.

Theorem

Map $A \mapsto \{A \upharpoonright \mathcal{F}_n \in \mathfrak{K}_n\}_{n\in\mathbb{N}_0}$ defines isomorphism between $\overline{\mathfrak{A}}$ and $\mathfrak{K}$.

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Dynamics of observables and fields

Strategy:

- establish stability of $\mathfrak{K}_n$ under action of dynamics, $n \in \mathbb{N}_0$
- check consistency of dynamics with structure of inverse limit

Analysis:

- Consider restrictions $\boldsymbol{H} \upharpoonright \mathcal{F}_n = H_n, \ n \in \mathbb{N}_0,$ where

$$H_n = \sum_i \boldsymbol{P}_i^2 + \sum_{j \neq k} V(\boldsymbol{Q}_j - \boldsymbol{Q}_k)\,, \quad i, j, k \in \{1, \dots, n\}.$$

Define automorphic action of dynamics on $\mathcal{B}(\mathcal{F}_n)$

$$\alpha_n(t) \doteq \mathrm{Ad}\, e^{itH_n}\,, \quad t \in \mathbb{R}\,.$$

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Let $n \in \mathbb{N}_0$, then

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Note: $\mathfrak{K}_n$ has ideals; result not true for simple subalgebras of $\mathcal{B}(\mathcal{F}_n)$.

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Proposition

$\kappa_n \circ \alpha_n(t) = \alpha_{n-1}(t) \circ \kappa_n$ *on* $\mathfrak{K}_n$, $n \in \mathbb{N}_0$.

Consequence: $\{K_n\}_{n \in \mathbb{N}_0} \in \mathfrak{K}$ implies $\{\alpha_n(t)(K_n)\}_{n \in \mathbb{N}_0} \in \mathfrak{K}$, $t \in \mathbb{R}$.

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Let $\alpha(t)$, $t \in \mathbb{R}$, *be the group of automorphisms of* $\mathcal{B}(\mathcal{F})$ *fixed by a Hamiltonian* $\boldsymbol{H}$ *with pair potential* $V \in C_0(\mathbb{R}^s)$.

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(ii) *There is a dense (in l.c.t.) subalgebra* $\overline{\mathfrak{A}}_\alpha \subset \overline{\mathfrak{A}}$ *on which action is pointwise norm continuous, i.e.* $(\overline{\mathfrak{A}}_\alpha, \alpha)$ *is a C*-dynamical system.*

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Action of dynamics on non-gauge invariant operators (fields)

Definition: $V_f \doteq a^*(f)(1 + a^*(f)a(f))^{-1/2}$, $f \in \mathcal{D}(\mathbb{R}^s)$ normalized

Facts:

- $V_f^* V_f = 1$, $V_f V_f^* = E_f$ (projection onto $ker\, a(f)^\perp$)
- $\sigma_f(\,\cdot\,) \doteq V_f \cdot V_f^*$ morphism of $\overline{\mathfrak{A}}$ (non-unital)
- $V_f^* V_g,\ V_g V_f^* \in \overline{\mathfrak{A}}$ (transportability of morphisms).

Action of dynamics:

- $\alpha(t)(V_f) = \big(\alpha(t)(V_f) V_f^*\big) V_f, \quad \alpha(t)(V_f^*) = V_f^* \big(V_f \alpha(t)(V_f^*)\big),$

Proposition

Let $\alpha(t)$, $t \in \mathbb{R}$, be defined as above and pick normalized $f \in \mathcal{D}(\mathbb{R}^s)$.

- *Cocycles $\alpha(t)(V_f)V_f^*,\ V_f\,\alpha(t)(V_f^*) \in \overline{\mathfrak{A}},\ t \in \mathbb{R}$.*
- *C^*-algebra generated by $\overline{\mathfrak{A}}$ and V_f, V_f^* on $\mathcal{F}$ is stable under the automorphic action of $\alpha(t)$, $t \in \mathbb{R}$.*

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Applications

1. Quasi local structure of observables
2. (Approximate) ground states and condensates
3. Particle properties and collision theory
4. Equilibrium states

(1) Quasi local structure of observables

Consider spatial translations, fixed by $\alpha_{\boldsymbol{x}}(R(\lambda, f)) = R(\lambda, f_{\boldsymbol{x}})$, and put $\boldsymbol{\alpha}(t, \boldsymbol{x}) \doteq \boldsymbol{\alpha}(t) \circ \boldsymbol{\alpha}(\boldsymbol{x}) = \boldsymbol{\alpha}(\boldsymbol{x}) \circ \boldsymbol{\alpha}(t)$ for $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^s$

Corollary

Let $A, B \in \overline{\mathfrak{A}}$ *and* $t \in \mathbb{R}$. *Then*

$$\lim_{\boldsymbol{x} \to \infty} \| [\alpha(t, \boldsymbol{x})(A), B] \|_n = 0 \, , \quad n \in \mathbb{N}_0 \, .$$

Question: Do there hold more specific bounds for given potential V?

(2) (Approximate) ground states and condensates

(a) Ground state: Ω for (renormalized) Hamiltonian $\boldsymbol{H}_r = \boldsymbol{H} + E(N)$.

(b) Approximate (non-Fock) ground states and condensates:

$$\boldsymbol{\Psi}_{L,n} = (n!)^{-1/2}\, \Phi_L \otimes_s \cdots \otimes_s \Phi_L \in \mathcal{F}_n\,, \quad n \in \mathbb{N}$$

where $\boldsymbol{x} \mapsto \Phi_L(\boldsymbol{x}) \doteq L^{-s/2}\, \Phi(\boldsymbol{x}/L) \in \mathcal{F}_1$ is normalized.

Let $V \geq 0$, $\widehat{\Omega}_n$ outgoing Møller operator, $\widehat{\boldsymbol{\Psi}}_{L,n} \doteq \widehat{\Omega}_n \boldsymbol{\Psi}_{L,n}$, then

$$0 \leq \langle \widehat{\boldsymbol{\Psi}}_{L,n}, H_n \widehat{\boldsymbol{\Psi}}_{L,n} \rangle = \langle \boldsymbol{\Psi}_{L,n}, H_{0,n} \boldsymbol{\Psi}_{L,n} \rangle = nL^{-2} \int d\boldsymbol{x}\, |\partial \Phi(\boldsymbol{x})|^2$$

Consider states $\omega_{n,L}(\,\cdot\,) \doteq \langle \widehat{\boldsymbol{\Psi}}_{L,n},\, \cdot\, \widehat{\boldsymbol{\Psi}}_{L,n} \rangle$ for $n \to \infty$, $nL^{-2} = c$.

Proposition

All limit points lead to positive energy representations of $(\overline{\mathfrak{A}}_\alpha, \alpha)$.

(2) (Approximate) ground states and condensates

(a) Ground state: Ω for (renormalized) Hamiltonian $\boldsymbol{H}_r = \boldsymbol{H} + E(N)$.

(b) Approximate (non-Fock) ground states and condensates:

$$\boldsymbol{\Psi}_{L,n} = (n!)^{-1/2}\, \Phi_L \otimes_s \cdots \otimes_s \Phi_L \in \mathcal{F}_n\,, \quad n \in \mathbb{N}$$

where $\boldsymbol{x} \mapsto \Phi_L(\boldsymbol{x}) \doteq L^{-s/2}\,\Phi(\boldsymbol{x}/L) \in \mathcal{F}_1$ is normalized.

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Proposition

All limit points lead to positive energy representations of $(\overline{\mathfrak{A}}_\alpha, \alpha)$.

(3) Particle properties and collision theory

"Particle observables" are uncovered at asymptotic times.

Lemma

Let $V \geq 0$ *be short ranged, and let* $A \in \overline{\mathfrak{A}}$ *be localized. Then (weakly)*

$$\lim_{t\to\infty} \alpha(t)(A) = \langle \Omega, A\Omega \rangle \, 1$$

$$\lim_{t\to\infty} \int d\boldsymbol{x}\, h(\boldsymbol{x}/t)\, \alpha(t,\boldsymbol{x})(A_0) = c_s \int d\boldsymbol{p}\, h(2\boldsymbol{p}) \langle \boldsymbol{p}, A_0\, \boldsymbol{p} \rangle\, \widehat{a}^*(\boldsymbol{p})\widehat{a}(\boldsymbol{p}).$$

Here $A_0 \doteq (A - \langle \Omega, A\Omega \rangle 1)$ *and* $\widehat{\ }$ *indicates "outgoing" operators.*

Similarly for "incoming", collision cross sections etc

Collision theory for observables works [Araki, Haag]

(4) Equilibrium states

Theory defined on $\mathbb{R}^s$ (no boxes). Introduce trapping forces, $L > 0$,

$$\boldsymbol{H}_L \doteq \boldsymbol{H} + \int d\boldsymbol{x}\,(\boldsymbol{x}^2/L^4)\; a^*(\boldsymbol{x})a(\boldsymbol{x})\,.$$

Automorphic action $\alpha_L(t) \doteq \mathrm{Ad}\, e^{itH_L}$ on $\mathcal{B}(\mathcal{F})$, $t \in \mathbb{R}$.

Lemma

$\overline{\mathfrak{A}}$ stable under action of $\alpha_L(t)$, and one has pointwise (in l.c.t.)

$$lim_{L\to\infty}\alpha_L(t) = \alpha(t) \quad , t \in \mathbb{R}.$$

V of positive type: $\mathrm{Tr}_{\mathcal{F}}\, e^{-\beta(H_L - \mu N)} < \infty$ for $\beta > 0$, $\mu \leq -V(0)$.

$$\omega_{\beta,\mu,L}(\,\cdot\,) = \mathrm{Tr}\,(e^{-\beta(H_L-\mu N)}\,\cdot\,)/\mathrm{Tr}\,e^{-\beta(H_L-\mu N)}$$

KMS-state with regard to $\alpha_L(t)$, $t \in \mathbb{R}$. Limit states exist (Alaoglu).

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Summary

Results:

- resolvent algebra $\mathfrak{R}$ cures large field problems
- observable algebra $\overline{\mathfrak{A}}$ composed of AF algebras
- automorphic action of interacting dynamics established
- quasi local structure of $\overline{\mathfrak{A}}$ survives under time evolution
- formalism useful for analysis of finite and infinite bosonic systems
- analogous results hold for fermionic systems [Bratteli]

Challenges: treatment of

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- relativistic integrable theories in $d = 2$

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THANK YOU
for your
ATTENTION!