Weak Quasi-Hopf Algebras and Conformal Field Theory

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AQFT and DHR and abstract duality for compact groups

- One of the key ideas in AQFT is that the theory should be formulated only in terms of local observable quantities.
- From the mathematical point of view one starts from a net $\mathcal{A}$ of local observables i.e. a map $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ from the set of doble cones in the four dimensional Minkowski space-time into the set of von Neumann algebras acting on a fixed Hilbert space $\mathcal{H}_0$ (the vacuum Hilbert space) + natural axioms.
- Other mathematical objects such as the the global gauge group or the unobservable charged field operators should be recovered from the representation theory of the net $\mathcal{A}$ encoding the charge structure of the theory.

- How to do this?
- Consider only DHR representations with finite statistical dimension. i.e. representations that look like the vacuum (in the sense of unitary equivalence) in the causal complement $\mathcal{O}'$ of every $\mathcal{O}$ and that admit conjugate representations, and let $\mathrm{Rep}(\mathcal{A})$ be the corresponding representation category.
- The vacuum representation π_0 is the defining representation of $\mathcal{A}$ on $\mathcal{H}_0$. Clearly $\pi_0 \in \mathrm{Rep}(\mathcal{A})$.
- The crucial step in the DHR analysis (1969-1971) is the following. The representations of the form $\pi = \pi_0 \circ \rho$ with ρ a localized and transportable endomorphisms with finite statistical dimension of the quasi-local C*-algebra $(\cup_{\mathcal{O}\subset M}\mathcal{A}(\mathcal{O}))^{-\|\cdot\|}$ define a full subcategory equivalent to $\mathrm{Rep}(\mathcal{A})$.

- The composition of endomorphisms gives rise to a tensor product operation $(\pi_0 \circ \rho) \otimes (\pi_0 \circ \sigma) := \pi_0 \circ \rho\sigma$ which, together with the existence of conjugates, induces on $\mathrm{Rep}(\mathcal{A})$ the structure of rigid semisimple C*-tensor category.
- Moreover, there are natural unitary isomorphisms $c(\pi_1, \pi_2) \in \mathrm{Hom}(\pi_1 \otimes \pi_2, \pi_2 \otimes \pi_1)$ encoding the representations of the permutation groups related to particle statistics. These makes $\mathrm{Rep}(\mathcal{A})$ into a symmetric (up to a $\mathbb{Z}_2$-grading) rigid semisimple C*-tensor category with simple unit . Here the space-time dimension $= 4$ (in fact ≥ 3) is crucial.
- It was already noticed in one of the first DHR papers that when the irreducible DHR endomorphism of $\mathcal{A}$ are all automorphisms then $\mathrm{Rep}(\mathcal{A})$ is tensor equivalent to a ($\mathbb{Z}_2$-graded) category of unitary representations of a compact abelian group G. Moreover, the net $\mathcal{A}$, can be obtained as a fixed-point net $\mathcal{F}^G$ of a graded-local field net $\mathcal{F}$.

- The natural generalization allowing non abelian gauge groups required almost twenty years and was completed by Doplicher and Roberts in a series of three papers (1989-1990).
- They obtained the following remarkable abstract group duality result: Let $\mathcal{C}$ be an essentially small symmetric rigid semisimple C*-tensor category with simple unit then $\mathcal{C} \simeq \mathrm{Rep}(G)$ for a unique (up to isomorphisms) compact group G. A very similar result was obtained independently by Deligne (1990) with rather different methods.
- Now let $\mathcal{A}$ be a net of local observables and let G be the compact group associated with $\mathrm{Rep}(\mathcal{A})$ through the above duality result. Then Doplicher and Roberts also gave a crossed product construction of a canonical field net $\mathcal{F} = \mathcal{A} \rtimes \mathrm{Rep}(\mathcal{A})$ with an action of G such that $\mathcal{A} = \mathcal{F}^G$ and all DHR superselection sectors of $\mathcal{A}$ are realized in the vacuum Hilbert space of $\mathcal{F}$ and labelled by equivalence classes $\xi \in \hat{G}$.

- Here some more details on the DR abstract duality result in view of possible generalizations.
- One can define a dimension function $d(X)$, $X \in \mathcal{C}$ having positive integer values. It satisfies $d(X \otimes Y) = d(X)d(Y)$ and $d(X \oplus Y) = d(X) + d(Y)$.
- One can define a faithful $*$-tensor functor $\mathscr{F} : \mathcal{C} \to \mathrm{Hilb}$ satisfying $\dim(\mathscr{F}(X)) = d(X)$ (here the Cuntz algebra plays a crucial role).
- One can can recover the compact group G as the group $\mathrm{Nat}_{\otimes}(\mathscr{F})$ of monoidal natural unitary transformations $\eta_X : \mathscr{F}(X) \to \mathscr{F}(X)$ (this last step is essentially the classical Tannaka-Krein duality).

2D QFT

- In two-space time dimensions the DR reconstruction does not work in general. This is because the usual permutation symmetry related to particle statistics is weakened to braid group statistics.
- If $\mathcal{A}$ is a net of local observables on a 2D space-time then the category Rep($\mathcal{A}$) of DHR representations of $\mathcal{A}$ with finite statistical dimension is still a rigid semisimple C*-tensor category with simple unit which is in general no longer symmetric but only braided.
- The values of the statistical dimension need not to be integers. For example it can take the values $d(\pi) = 2\cos(\frac{\pi}{n})$, n=3, 4, 5, ...
- In particular Rep($\mathcal{A}$) will be no longer equivalent to a representation category Rep(G) for a compact group G. This fact makes things more complicated but also very exciting. Are there more general symmetry objects that can be used to replace compact groups. Quantum groups?

Hopf algebras and generalizations

- Original motivation for Hopf algebras: algebraic topology (50s)
- Further motivations: duality for locally compact groups (G. Kac 60s); quantum groups (Drinfeld-Jimbo, Woronowicz 80s).
- I will focus on the representation theory aspects.
- A paradigmatic example is the algebra $A := \mathbb{C}G$ of a finite group G. The category $\mathrm{Rep}(A)$ of finite dimensional unital representations of A is equivalent to $\mathrm{Rep}(G)$ as a linear category. On the other hand the tensor structure of $\mathrm{Rep}(G)$ is not directly visible from $\mathrm{Rep}(A)$: $\pi_1, \pi_2 \in \mathrm{Rep}(A) \Rightarrow \pi_1 \otimes \pi_2 \in \mathrm{Rep}(A \otimes A)$.

- Recall that the tensor structure on Rep(G) is obtained from the diagonal embedding $G \ni g \mapsto (g, g) \in G \times G$. This gives rise to a unital homomorphism $\Delta : A = \mathbb{C}G \to A \otimes A \simeq \mathbb{C}(G \times G)$. Δ is called coproduct.
- The tensor product $\underline{\otimes}$ on the objects of Rep(A) is then given by $\pi_1 \underline{\otimes} \pi_2 := \pi_1 \otimes \pi_2 \circ \Delta \in \text{Rep}(A)$.
- In order to get a unit and a rigid structure on Rep(A) one further need a special one-dimensional representation $\varepsilon : A \to \mathbb{C}$, the counit, (this comes from the trivial representation of G) and a suitable antiautomorphism $S : A \to A$, the antipode (this comes from the map $g \mapsto g^{-1}$ in G).
- Some of the properties of the triple $(A, \Delta, \varepsilon, S)$ above can be abstracted to the notion of Hopf algebra. Here I only mention the coassociativity of Δ i.e. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ which makes Rep(A) into a strict tensor category.

- A Hopf algebra is a quadruple $(A, \Delta, \varepsilon, S)$ of the type described above an can be considered as a generalization of the notion of group.
- By relaxing coassociativity one obtain the notion of quasi-Hopf algebra first introduced by Drinfeld. These allows more flexibility in dealing with non strict tensor categories: non-trivial associators $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$. This is done trough a suitable element $\Phi \in A \otimes A \otimes A$ satisfying a 3-cocycle condition related to the pentagon equation. Accordingly the data of a quasi-Hopf algebra is given by a quintuple $(A, \Delta, \varepsilon, S, \Phi)$
- Quasi-Hopf algebras are not sufficiently general to describe many interesting tensor categories related to QFT. This is because, when A is semisimple, the function D on the fusion ring $\mathrm{Gr}(\mathrm{Rep}(A))$ defined by $D([\pi]) := \dim(V_\pi)$, where V_π is the representation space of π, is a positive integral valued dimension function and there are many fusion categories do not admitting such a function by the uniqueness of the Frobenius-Perron dimension.

- In the early 90s Mack and Schoumerus suggested the following solution to the above problem: give up to the request that Δ is unital so that a wak quasi-Hopf algebra is again a quintuple $(A, \Delta, \varepsilon, S, \Phi)$ with a possibly non-unital coproduct.
- In this way $\Delta(I)$ is an idempotent in $A \otimes A$ commuting with $\Delta(A)$ but typically different from $I \otimes I$.
- The tensor product $\pi_1 \underline{\otimes} \pi_2$ in Rep(A) is now defined by the restriction of $\pi_1 \otimes \pi_2 \circ \Delta$ to $\pi_1 \otimes \pi_2 \circ \Delta(I) V_{\pi_1} \otimes V_{\pi_2}$.
- Now, for a semisimple A, the additive function $D : \mathrm{Gr}(\mathrm{Rep}(A)) \to \mathbb{Z}_{>0}$ defined by $D([\pi]) := \dim(V_\pi)$ is only a weak dimension function i.e. it satisfies $D([\pi_1 \underline{\otimes} \pi_2]) \leq D([\pi_1])D([\pi_2])$, $D([\iota]) = 1$ and $D(\overline{\pi}) = D(\pi)$ and this gives no important restrictions.

Tannakian results

- Let $\mathcal{C}, \mathcal{C}'$ be tensor categories. A linear functor $\mathscr{F} : \mathcal{C} \to C'$ together with natural transformations $F_{X,Y} : \mathscr{F}(X) \otimes \mathscr{F}(Y) \to \mathscr{F}(X \otimes Y)$ and $G_{X,Y} : \mathscr{F}(X \otimes Y) \to \mathscr{F}(X) \otimes \mathscr{F}(X)$ satisfying

$$F_{\iota,X} = F_{X,\iota} = 1_{\mathscr{F}(X)},\ G_{\iota,X} = G_{X,\iota} = 1_{\mathscr{F}(X)},$$

$$F_{X,Y} \circ G_{X,Y} = 1_{\mathscr{F}(X \otimes Y)}$$

  is called a weak quasi-tensor functor.
- Although many results hold with more generality I will focus on fusion categories (rigid, semisimple, tensor categories with simple unit and finitely many equivalence classes of simple objects).

- The following result are due to Häring-Oldenburg (late 90s).
- Let $\mathcal{C}$ be a fusion category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Vect}$ be a faithful weak quasi-tensor functor. Then $A := \mathsf{Nat}(\mathscr{F})$ is a semisimple algebra admitting a structure of weak-quasi Hopf algebra such that $\mathsf{Rep}(A)$ is tensor equivalent to $\mathcal{C}$.
- Let $\mathcal{C}$ be a fusion category and $D : \mathrm{Gr}(\mathcal{C}) \to \mathbb{Z}_{\geq 0}$ be an integral weak dimension then there exists a faithful weak quasi-tensor functor $\mathscr{F} : \mathcal{C} \to \mathrm{Vect}$ such that $D([X]) = \dim(\mathscr{F}(X))$ for all $X \in \mathcal{C}$.
- Every fusion category admit infinitely many integral weak dimension functions.
- Extra structure on $\mathcal{C}$ gives extra structure on A: brading $\leftrightarrow$ R-matrix ; C*-tensor structure on $\mathcal{C} \leftrightarrow \Omega$ - involutive structure on A (in particular A is a C*-algebra).

- The weak quasi-Hopf algebra associated to a fusion category $\mathcal{C}$ is highly non-unique: it depends on the choice of D and, once D is fixed, on the weak quasi-tensor structure on the functor $\mathscr{F} : \mathcal{C} \to \mathrm{Vect}$. In the latter case however they are unique up to a "twist".
- It seems that until a recent work by Ciamprone and Pinzari (2017), where examples from quantum groups at roots of unity are considered, they have almost been forgotten.
- A different Hopf algebraic object, the weak Hopf algebras intruduced by Böhm and Szlachanyi (middle 90s) received much more attention an found important applications.

- For example in their recent beautiful book on tensor categories Etingof, Gelaki, Nikshych and Ostrik write "This structure is called a weak quasi-Hopf algebra, and in principle it allows one to speak about any finite tensor category in explicit linear-algebraic terms. However, this structure is so cumbersome that it seems better not to consider it, and instead to use the language of tensor categories, which is the point of view of this book."
- In the remaining part of this talk I would like to try to convince you that, despite their problems, they can be useful and natural in the investigation of conformal field theory (CFT).

Conformal nets and vertex operator algebras

- Conformal nets and vertex operator algebras (VOAs) gives two mathematically rigorous frameworks for chiral conformal quantum field theories (chiral CFTs) i.e. CFTs on S^1.
- Conformal nets are the chiral CFT version of AQFT and are given by nets of von Neumann algebras $I \subset \mathcal{A}(I)$ over the intervals $I \subset S^1$ acting on given Hilbert space $\mathcal{H}$ + axioms.
- A VOA) is a vector space V together with a linear map (the state-field correspondence)

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathrm{End}(V)$$

from V into the set of operator valued formal distributions acting on V + axioms.

- Conformal nets and VOAs have very interesting representation theories.
- A representation π of a conformal net $\mathcal{A}$ is a family $\pi_I, I \subset S^1$ where each π_I is a representation of $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$. The family is assumed to be compatible with the net structure.
- A VOA-module for the VOA V is a vector space M together with a linear map

$$a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a^M_{(n)} z^{-n-1}, \quad a^M_{(n)} \in \mathrm{End}(M)$$

which is compatible with the vertex algebra structure of V.

- Under suitable rationality conditions (that I will resume with the term completely rational) conformal nets and VOAs gives rise to very interesting examples of modular tensor categories (in particular fusion categories).
- If $\mathcal{A}$ is a completely rational conformal net then $\mathrm{Rep}(\mathcal{A})$ is a modular C*-tensor category (Kawahigashi, Longo, Müger 2001).
- If V is a completely rational VOA then $\mathrm{Rep}(V)$ is a modular tensor category (Huang 2008).

From VOAs to conformal nets

- A general connection between VOAs and conformal nets has been recently considered by Carpi, Kawahigashi, Longo and Weiner.
- One first need a suitable definition of unitary VOA (Dong, Lin and CKLW).
- For sufficiently nice (simple) unitary VOAs called strongly local one can define a map $V \mapsto \mathcal{A}_V$ into the class of conformal nets.
- Conjecture 1: The map $V \mapsto \mathcal{A}_V$ gives a one-to-one correspondence between the class of simple unitary VOAs and the class of conformal nets.
- Conjecture 2: The map $V \mapsto \mathcal{A}_V$ gives gives a one-to-one correspondence between the class of completely rational unitary VOAs and the class of completely rational conformal nets. Moreover, if V is completely rational we have a tensor equivalence $\mathrm{Rep}(V) \simeq \mathrm{Rep}(\mathcal{A}_V)$.

- Recently it has been suggested by Carpi, Weiner and Xu (in preparation) to consider a strong integrability condition on unitary VOA-modules of a strongly local V which allows to define a map $M \mapsto \pi_M$ from V-modules to representations of $\mathcal{A}_V$. In certain cases this gives an isomorphism of linear C*-categories $\mathscr{F} : \mathrm{Rep}^u(V) \to \mathrm{Rep}(\mathcal{A}_V)$ where $\mathrm{Rep}^u(V)$ is the category of unitary V-modules. Further examples have been recently given by Gui (2017).
- Conjecture 3: Assume that V is completely rational and strongly local. Then $\mathrm{Rep}^u(V)$ admits a structure of modular tensor category such that the forgetful functor : $\mathrm{Rep}^u(V) \to \mathrm{Rep}(V)$ is a braided tensor equivalence. Moreover, the functor $\mathscr{F} : \mathrm{Rep}^u(V) \to \mathrm{Rep}(\mathcal{A}_V)$ discussed above admits a tensor structure.

From VOAs to C*-tensor categories

- The following result has been obtained using weak quasi-Hopf algebra techniques. It seems to be me that this is a good argument for the claim that weak quasi-Hopf algebra are useful.
- Theorem (Carpi, Ciamprone, Pinzari): Let V be a completely rational VOA. Assume that every V-module is unitarizable and that $\mathrm{Rep}(V)$ is tensor equivalent to a C*-tensor category. Then, $\mathrm{Rep}^u(V)$ admit the structure of a braided C*-tensor category, unique up to unitary equivalence, such that the forgetful functor $: \mathrm{Rep}^u(V) \to \mathrm{Rep}(V)$ is a tensor equivalence.

- Let $\mathfrak{g}$ be a complex simple Lie algebra, let k be a positive integer and let $V_{\mathfrak{g}_k}$ be the corresponding simple level k affine VOA. It is known that $V_{\mathfrak{g}_k}$ is a unitary completely rational VOA and that every $V_{\mathfrak{g}_k}$-module is unitarizable.
- By a result of Finkelberg (1996) based on the work Kazhdan and Lusztig we know that $\mathrm{Rep}(V_{\mathfrak{g}_k})$ is tensor equivalent to the "semisimplified" category $\widetilde{\mathrm{Rep}(G_q)}$ associated to the representations of the quantum group G_q, with G the simply connected Lie group associated to $\mathfrak{g}$ and $q = e^{\frac{i\pi}{d(k+h^\vee)}}$, $h^\vee$ = dual Coxeter number, $d = 1$ if $\mathfrak{g}$ is ADE, $d = 2$ if $\mathfrak{g}$ is BCF and $d = 3$ if $\mathfrak{g}$ is G_2.

- It was shown by Wenzl and Xu (1998) that $\widetilde{\text{Rep}}(G_q)$ is tensor equivalent to a C*-tensor category.
- As a consequence we get that $\text{Rep}^u(V_{\mathfrak{g}_k})$ is a C*-tensor category.
- The same result has been proved by Gui in a series of two paper (ArXiv 2017)in the special cases $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 2$ and $\mathfrak{g} = \mathfrak{so}_{2n}$, $n \geq 3$, by a completely different method based on Connes fusions for bimodules and a deep analysis of the analytic properties of the smeared intertwiners operators for VOA modules.
- Our method works also other VOAs like e.g. lattice VOAs and certain holomorphic orbifolds.

The Zhu algebra as a weak quasi-Hopf algebra

- Let V be completely rational. In 1998 Zhu introduced a finite-dimensional semisimple algebra $A(V)$ and a functor $\mathscr{F}_V : \mathrm{Rep}(V) \to \mathrm{Vect}$ with the following properties: for each $M \in \mathrm{Rep}(V)$, $A(V)$ acts on $\mathscr{F}_V(M)$ and in this way $\mathscr{F}_V$ gives rise to a linear equivalence from $\mathrm{Rep}(V)$ to $\mathrm{Rep}(A(V))$. Moreover $\mathscr{F}_V(V) = \mathbb{C}\Omega$.
- It follows that we can identify $A(V)$ with $\mathrm{Nat}(\mathscr{F}_V)$. Hence, if $D_V([M]) := \dim(\mathscr{F}_V(M))$ defines a weak dimension function then the Zhu algebra $A(V)$ admit the structure of a weak quasi-Hopf algebra such that there is a tensor equivalence $\mathrm{Rep}(V) \simeq \mathit{Rep}(A(V))$.
- D_V is not always a weak dimension function. A counterexample is given e.g. by the Ising. However D_V is a weak dimension function in many interesting cases e.g. if V is a unitary affine VOA.

THANK YOU VERY MUCH!