The universal C*-algebra of the electromagnetic field: spacelike linearity and topological charges

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Algebraic Quantum Field Theory:
Where Operator Algebra meets Microlocal Analysis

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Joint project with D. Buchholz, G. Ruzzi and E. Vasselli

- The C*-algebra of the e.m. field, defined by the e.m. potential on the 4-dim Minkowski spacetime [Buchholz ESI lectures 2012]
- Also based on a former key result by J. Roberts [Roberts 77]:

  the commutator of the e.m. field F with its Hodge dual field $\star F$ supported respectively on two surfaces whose boundaries are causally disjoint but linked together, is not vanishing

  Remind: $\star$ interchange the electric and magnetic parts of F

  [BCRV16] The universal C*-algebra of the electromagnetic field. *Lett. Math. Phys.*, **106**, 269–285, (2016). `arXiv:1506.06603`

  [BCRV17] The universal C*-algebra of the electromagnetic field II. Topological charges and spacelike linear fields. *Lett. Math. Phys.*, **107**, 201–222, (2017). `arXiv:1610.03302`

- A related project

  The net of causal loops and connection representations

  for abelian gauge theories on a 4-dim globally hyperbolic spacetime using a net of local C*-algebras with loops supported observables joint work with G. Ruzzi and E. Vasselli [CRV12], [CRV15]

An AQFT roadmap for QED by structural algebraic properties

- Define the Universal C*-algebras of e.m. (observable) field $\mathfrak{V}$ that will appear in any theory incorporating electromagnetism (vacuum or non-trivial current) e.g. QED
- $\mathfrak{V}$ defines a net on a poset of regions of $\mathbb{R}^4$, with covariance and causality but may not contain the dynamic of the theory
- Fix an interesting (pure, vacuum, ...) state ω. The GNS representation $(\pi_\omega, \mathscr{H}_\omega, \Omega)$ on the algebras $\mathfrak{V} \setminus \ker \pi_\omega$ gives the dynamical information of the net and distinguishes different theories
- We obtain any e.m. theory that satisfy the Haag-Kastler axioms: the physical content of a theory is encoded in its observable net

Linearity on test functions

- Models in Haag-Kastler axioms do not require an *a priori* unrestrained condition of linearity of the field over the set of test functions
- This is just a matter of convenience, e.g. symplectic spaces, Weyl algebras, Wightman framework
- Other examples of non linearity also appear in other contests of AQFT: definition of CFT and perturbative AQFT

Topological charges and spacelike linearity for the e.m. field

- Following [Roberts 77], Topological charges result from commutators of the intrinsic (gauge invariant) vector potential A in spacelike separated, topologically non-trivial regions
- These commutators are in the center of the algebra $\mathfrak{V}$: existence in [BCRV16], non trivial examples in [BCRV17]
- The vector potential A is well defined in all regular, pure states of $\mathfrak{V}$ but topological charges vanish if A is unrestrained linear on the test functions: hence no topological charges in the Wightman framework
- Nevertheless, we exhibit regular vacuum states with non-trivial topological charges: the vector potential is homogeneous and spacelike linear on test functions
- Such states also exist in the presence of non-trivial electric currents
- We also exhibit topological charges for theories with several e.m. potentials depending linearly on the test functions, e.g. scaling (short distance) limit of non-abelian gauge fields with asymptotic freedom

Outline

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Notations

Minkowski spacetime $\mathbb{R}^4$, signature $(+,-,-,-)$ and causal disjointness $\perp$

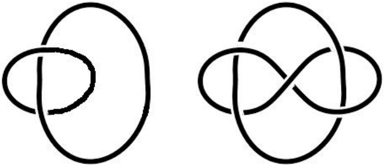
- $\mathscr{D}_n := \mathscr{D}_n(\mathbb{R}^4)$ real smooth compactly supported n-forms on $\mathbb{R}^4$
  in particular $\mathscr{D}_2$ are valued in the antisymmetric tensors of rank two
- $d : \mathscr{D}_n \to \mathscr{D}_{n+1}$ s.t. $dd = 0$ differential
- $d\mathscr{D}_n = \{f \in \mathscr{D}_{n+1} \ : \ df = 0\}$ i.e. closed form in $\mathscr{D}_{n+1}$ Poincaré Lemma
- $\star : \mathscr{D}_n \to \mathscr{D}_{4-n}$ s.t. $\star\star = (-1)^{n+1}$ Hodge dual operator
- $\delta := -\star d \star$ s.t. $\delta : \mathscr{D}_n \to \mathscr{D}_{n-1}$ s.t. $\delta\delta = 0$ co-differential
- $\mathscr{C}_n := \mathscr{C}_n(\mathbb{R}^4) = \{f \in \mathscr{D}_n \ : \ \delta f = 0\}$ s.t. $\delta\mathscr{D}_n \subset \mathscr{C}_{n-1}$ co-closed n-forms
  in particular $\mathscr{C}_1 = \{f \in \mathscr{D}_1 : \mathrm{div}\, f = 0\}$ divergence free 1-forms
- $\delta\mathscr{D}_n = \mathscr{C}_{n-1}$ dual version of the Poincaré Lemma
- a primitive of $f \in \mathscr{D}_n$ is $g \in \mathscr{D}_{n-1}$ s.t. $dg = f$
  a co-primitive of $f \in \mathscr{D}_n$ is $h \in \mathscr{D}_{n+1}$ s.t. $\delta h = f$

Moreover

- $\mathrm{supp}(df), \mathrm{supp}(\star f), \mathrm{supp}(\delta f) \subseteq \mathrm{supp}(f)$ Local action of d, $\star$ and δ
- $\mathscr{P}_+^\uparrow \times \mathscr{D}_n \to \mathscr{D}_n$ s.t. $(P, f) \mapsto f_P := f \circ P^{-1}$ action of Poincaré group
  leaves the space $\mathscr{C}_1$ of divergence-free 1-forms globally invariant

Topological non-trivial regions and Loop functions

Loop-shaped region $\mathcal{L}$: open, bounded and contains some spacelike (pointwise, hence simple) loop $[0,1] \ni t \mapsto \gamma(t)$ i.e. γ is a deformation retract of $\mathcal{L}$ and therefore is homotopy equivalent (i.e. homotopic) to $\mathcal{L}$



Linked loop-shaped regions: Hopf link and Whitehead link

Loop function: for any $\mathcal{L}$ with γ we may choose $\mathcal{O}_0$ a small neighbourhood of the origin such that $(\mathcal{O}_0 + \gamma) \subset \mathcal{L}$ and $s \in \mathscr{D}_0$ a real scalar function with $\mathrm{supp}(s) \subseteq \mathcal{O}_0$ and define

$$x \mapsto l_{s,\gamma}(x) \doteq \int_0^1 dt\, s(x - \gamma(t))\, \dot{\gamma}(t)$$

Then $l_{s,\gamma} \in \mathscr{C}_1$ and $\mathrm{supp}(l_{s,\gamma}) \subseteq (\mathcal{O}_0 + \gamma) \subset \mathcal{L}$.

If $\int dx\, s(x) \neq 0$ there is no $f \in \mathscr{D}_2$ with support in $\mathcal{L}$ s.t. $l_{s,\gamma} = \delta f$, i.e. $l_{s,\gamma}$ is co-closed but not co-exact in this region

The e.m. field F and the intrinsic gauge-invariant e.m. potential A

- The e.m. quantum field is a linear map $F : \mathscr{D}_2 \ni h \mapsto F(h)$
- The e.m. intrinsic vector potential is a map $A : \mathscr{C}_1 \ni f \mapsto A(f)$ s.t.

$$F(h) \doteq A(\delta h), \quad h \in \mathscr{D}_2$$

- A conserved current is a linear map $j : \mathscr{D}_1 \ni g \mapsto j(g)$ s.t.

$$\delta j(s) = j(ds) = 0, \quad s \in \mathscr{D}_0$$

1^{st} Maxwell equation

(using e.m. field F)

$$dF(\tau) := F(\delta\tau) = 0$$

(using e.m. vector potential A)

$$F(\delta\tau) = A(\delta^2\tau) = 0, \quad \tau \in \mathscr{D}_3$$

by the Local Poincaré Lemma and independence from co-primitives

2^{nd} Maxwell equation

$$j(g) = F(dg)$$

$$j(g) = A(\delta dg), \quad g \in \mathscr{D}_1$$

Current conservation

$$\delta j(s) = F(d^2 s) = 0$$

$$\delta j(s) = A(\delta d^2 s) = 0, \quad s \in \mathscr{D}_0$$

Locality for F *vs* locality for A

- For the e.m. field F

  Given h_1 and h_2 in $\mathscr{D}_2$ with spacelike separated supports

$$\mathrm{supp}(h_1) \perp \mathrm{supp}(h_2) \quad \Rightarrow \quad [F(h_1), F(h_2)] = 0 \ ,$$

- For the intrinsic e.m. vector potential A

  Given f_1 and f_2 in $\mathscr{C}_1$ such that $\mathrm{supp}(f_1) \times \mathrm{supp}(f_2)$ i.e.

  separated by double cones (or by opposite characteristic wedges)

  the independence from the co-primitive allows to choose two spacelike separated co-primitives:

$$\mathrm{supp}(f_1) \times \mathrm{supp}(f_2) \quad \Rightarrow \quad \exists\, h_1, h_2 \in \mathscr{D}_2 \ , \ \delta h_1 = f_1 \ , \ \delta h_2 = f_2 \quad \text{s.t.} \quad h_1 \perp h_2$$

  obtaining a stronger form of Locality for A

$$\mathrm{supp}(f_1) \times \mathrm{supp}(f_2) \quad \Rightarrow \quad [A(f_1), A(f_2)] = [F(h_1), A(h_2)] = 0$$

  For f_1 and f_2 in $\mathscr{C}_1$ with $\mathrm{supp}(f_1) \perp \mathrm{supp}(f_2)$ but not $\mathrm{supp}(f_1) \times \mathrm{supp}(f_2)$

$$\Rightarrow \quad [A(f_1), A(f_2)] \neq 0$$

The C*-algebras of the e.m. field: definition

Take the *-algebra generated by the formal unitary operators $V(a,g) \doteq e^{iaA(g)}$ with $a \in \mathbb{R}$, $g \in \mathscr{C}_1$ and quotient w.r.t. the ideal given by the relations

$$V(a_1,g)V(a_2,g) = V(a_1+a_2,g),\ V(a,g)^* = V(-a,g),\ V(0,g) = 1 \quad (1)$$

$$V(a_1,g_1)V(a_2,g_2) = V(1,a_1g_1+a_2g_2) \ \text{ if } \ \mathrm{supp}(g_1) \times \mathrm{supp}(g_2) \quad (2)$$

$$\lfloor V(a,g), \lfloor V(a_1,g_1), V(a_2,g_2)\rfloor\rfloor = 1 \ \text{ for any } g \text{ if } \ \mathrm{supp}(g_1) \perp \mathrm{supp}(g_2) \quad (3)$$

(1): algebraic properties of unitary one-parameter groups $a \mapsto V(a,g)$

(2): spacelike linearity and locality properties of A

(3): the symbol $\lfloor\cdot,\cdot\rfloor$ is the group commutator so $\lfloor V(a_1,g_1), V(a_2,g_2)\rfloor$ are central element of topological nature we call topological charges

- The *-algebra generated by the $V(a,g)$ and relations (1) to (3) has a C*-norm induced by all of its GNS reps; its completion w.r.t. this norm, is the universal C*-algebra of the electromagnetic field denoted by $\mathfrak{V}$
- Strongly regular states: $(a_1,\ldots,a_n) \mapsto \omega(V(a_1,g_1)\cdots V(a_n,g_n))$ smooth Then $\pi(V(a,g)) = e^{iaA_\pi(g)}$ are unitaries and $A_\pi(g)$ are self-adjoint on a stable common core, including the GNS vector Ω
- From (2), $A_\pi(g)$ are spacelike linear, i.e. it is true on the common core $a_1A_\pi(g_1) + a_2A_\pi(g_2) = A_\pi(a_1g_1+a_2g_2)$, whenever $\mathrm{supp}(g_1) \times \mathrm{supp}(g_2)$

Outline

Linear symplectic forms and absence of topological charges

- Given a regular pure states ω on $\mathfrak{V}$ with GNS $(\pi, \mathcal{H}, \Omega)$, not required to be the vacuum, exists $A_\pi(g)$ self-adjoint as above
- from (3), the commutator $[A_\pi(g_1), A_\pi(g_2)]$ is affiliated with the centre of the weak closure $\pi(\mathfrak{V})^-$ as multiples of the identity and we may define, for (wick) disjoint-support 1-forms

$$\sigma_\pi(g_1, g_2) \doteq i\,\langle \Omega, [A_\pi(g_1), A_\pi(g_2)]\,\Omega\rangle,\ g_1, g_2 \in \mathscr{C}_1,\ \mathrm{supp}(g_1) \perp \mathrm{supp}(g_2)$$

- depending on the state ω, σ_π may be a bilinear and skew symmetric, i.e. a symplectic form on $\mathscr{C}_1$; in general σ_π is only spacelike linear, see (2)
- if σ_π vanishes for any pair of test functions $g_1, g_2 \in \mathscr{C}_1$ having supports in spacelike separated, linked loop-shaped regions, then the corresponding topological charges vanish

  In fact it is possible to prove, using co-cohomology:

1. For any loop-shaped region $\mathcal{L}$ and any function $g \in \mathscr{C}_1$, $\mathrm{supp}(g) \subset \mathcal{L}$, exists a loop function $l_{s,\gamma}$, as above, in the same co-cohomology class
2. Let $g_1, g_2 \in \mathscr{C}_1$ supported in $\mathcal{L}_1$ and $\mathcal{L}_2$, spacelike separated loop-shaped linked regions with retracts γ_1 linked to γ_2; then there are two loop functions l_{s_1,γ_1}, l_{s_2,γ_2} co-cohomologous to g_1, g_2, s.t. $\sigma_\pi(g_1, g_2) = \sigma_\pi(l_{s_1,\gamma_1}, l_{s_2,\gamma_2})$, for any bilinear σ_π.

Linear symplectic forms and absence of Topological Charges

3. For given loop γ, the co-cohomology classes are fixed by the class values which are given by the integral $\kappa \doteq \int dx\, s(x) \in \mathbb{R}$ of the scalar functions $s \in \mathscr{D}_0$ in the definition of $l_{s,\gamma}$. Thus for given loops γ_1, γ_2, the expression $\sigma_\pi(l_{s_1,\gamma_1},\ l_{s_2,\gamma_2})$ is proportional to the product $\kappa_1\kappa_2$.
4. It is possible to deform any two disjoint simple loops γ_1, γ_2 in $\mathbb{R}^4$ to corresponding disjoint simple loops β_1 and β_2 on the time-zero plain $\mathbb{R}^3$ s.t. $\sigma_\pi(l_{s_1,\gamma_1},\ l_{s_2,\gamma_2}) = \sigma_\pi(l_{s'_1,\beta_1},\ l_{s'_2,\beta_2})$.
5. For fixed product $\kappa_1\kappa_2$, the values of σ_π depend only on the homology class of β_1 in $\mathbb{R}^3 \backslash \mathrm{supp}(\beta_2)$.

Hence, for a symplectic form σ_π linear in both entries, i.e. whose representation gives a linear vector potential A_π on $\mathscr{C}_1$, the topological charges vanish:

Proposition

Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two spacelike separated loop-shaped regions which can continuously be retracted to spacelike linked loops γ_1 and γ_2, respectively. Moreover, let σ_π be linear in both entries. Then, for any $g_1, g_2 \in \mathscr{C}_1$ having support in $\mathcal{L}_1$, respectively $\mathcal{L}_2$, one has $\sigma_\pi(g_1, g_2) = 0$.

Outline

Non-trivial Topological Charges and spacelike linearity in vacuum

Merging the electric and magnetic parts of a free e.m. field in a non-linear manner gives a new field with spacelike linearity and topological charge

The Haag-Kastler net coincides with the net of the original e.m. field then this is a reinterpretation of the theory giving non trivial topological charges

- Regular, quasi-free, vacuum state ω_0 on $\mathfrak{V}$ with GNS denoted by $(\pi_0, \mathcal{H}_0, \Omega_0)$ gives both a vanishing electric current $j_0(h)$, for every $h \in \mathscr{D}_1$ and a linear free vector potential A_0 in the representation π_0

  Remind that $j(g) = A(\delta dg)$ for $g \in \mathscr{D}_1$, this result follow by using Reeh-Schlieder theorem and Källén-Lehmann representation of Wightman two-point functions

- For $g \in \mathscr{C}_1$ and $G \in \mathscr{D}_2$ a co-primitive of g we get the constant tensor

$$\overline{G}^{\mu\nu} \doteq \int dx \; G^{\mu\nu}(x)$$

  depends only on g, invariant under translations and covariant under Lorentz transformations of g

- $A_0(\delta \star G)$ also depend only on $g \in \mathscr{C}_1$ but not on the co-primitive G

  In fact, if $\delta k = G - G'$ for $k \in \mathscr{D}_3$ and $\delta G = \delta G' = g$, it holds $A_0(\delta \star \delta k) = -A_0(\delta d \star k) = -j_0(\star k) = 0$ in vacuum

- Let $\theta_\pm$ be the characteristic functions of the positive respectively negative reals and let $\overline{G}^2 \doteq \overline{G}_{\mu\nu}\,\overline{G}^{\mu\nu}$.
- Suppose for simplicity that $g \in \mathscr{C}_1$ is supported in a connected region then we obtain a topological potential A_T defining

$$A_T(g) \doteq \theta_+(\overline{G}^2)\, A_0(\delta G) + \theta_-(\overline{G}^2)\, A_0(\delta \star G)\,.$$

- For 1-forms $g_1, g_2 \in \mathscr{C}_1$ with (connected) spacelike separated, linked supports, i.e. $\mathrm{supp}(g_1) \perp \mathrm{supp}(g_2)$, we have the commutator

$$\begin{aligned}[A_T(g_1), A_T(g_2)] = {} & \big(\theta_+(\overline{G_1}^2)\theta_+(\overline{G_2}^2) + \theta_-(\overline{G_1}^2)\theta_-(\overline{G_2}^2)\big)\, \Delta(G_1, G_2)\, 1 \\ & + \big(\theta_+(\overline{G_1}^2)\theta_-(\overline{G_2}^2) - \theta_+(\overline{G_2}^2)\theta_-(\overline{G_1}^2)\big)\, \Delta(G_1, \star G_2)\, 1\end{aligned}$$

  where Δ is the commutator function of the free Maxwell field
- The term $\Delta(G_1, \star G_2)$, studied in the cited result [Roberts77], makes the commutator $[A_T(g_1), A_T(g_2)]$ non-trivial

and we have non-trivial topological charges, by spacelike linearity, in vacuum with or without currents

Outline

Non-trivial Topological Charges with electric current

- It is in general not possible to couple any given current to a Wightman field *via* a field equation [Araki, Haag, Schroer 1961]
- The vector potential, being spacelike linear is not a Wightman field so the inhomogeneous Maxwell equation does have solutions for almost any given current
- We may obtain a spacelike linear vector potential carrying a topological charge by combining a current j with the previous spacelike linear A_T
- For any given j, look for a spacelike linear potential A_J in a regular vacuum representation of the universal algebra $\mathfrak{V}$, satisfying the inhomogeneous Maxwell equation $A_J(\delta dh) = j(h)$, $h \in \mathscr{D}_1$
- Problem: A_J is already defined on the subspace $\delta d\mathscr{D}_1$ of $\mathscr{C}_1$;
how to extend A_J to all $\mathscr{C}_1$ and obtain its localization from the one of j

Lemma Let $g = \delta dh \in \delta d\mathscr{D}_1$ and call $h \in \mathscr{D}_1$ the pre-image of $g \in \delta d\mathscr{D}_1$.

(i) h is uniquely determined by g up to elements of $d\mathscr{D}_0$.

(ii) Let $\mathcal{O} \supset \mathrm{supp}(g)$ be s.t. the complement of $\widetilde{\mathcal{O}} \doteq (\mathcal{O} + V_+) \bigcap (\mathcal{O} + V_-)$ has trivial first homology, $H_1(\mathbb{R}^4 \backslash \widetilde{\mathcal{O}}) = \{0\}$. There exist pre-images h of g having support in any given neighbourhood of $\widetilde{\mathcal{O}}$.

(iii) The map from $\delta d\mathscr{D}_1$ into the classes $\mathscr{D}_1 / d\mathscr{D}_0$ of pre-images is continuous.

- Denote by $g_\smile$ elements in the subspace $\delta d\mathscr{D}_1 \subset \mathscr{C}_1$, i.e. $g_\smile \in \mathscr{C}_1$ and exists $h_\smile \in \mathscr{D}_1$ s.t. $g_\smile = \delta d h_\smile$
- We extend A_J to $\mathscr{C}_1$ by

$$A_J(g) \doteq \begin{cases} j(h_\smile) & \text{if } g = g_\smile \\ 0 & \text{otherwise} \end{cases}$$

  $A_J(g)$ is well defined by Lemma (i), since $j(ds) = 0$, $s \in \mathscr{D}_0$
- Then, using Lemma (iii), it is possible to extend consistently A_J to elements not in $\delta d\mathscr{D}$ and prove the existence of a regular vacuum state on $\mathfrak{V}$ with a spacelike linear vector potential A_J in its GNS representation with $A_J(g) = A_J(g_\smile) = j(h_\smile),\ g \in \mathscr{C}_1$ and $A_J(g) = 0$ otherwise

But the topological charge of the potential A_J associated with linked spacelike separated loop-shaped regions, are trivial. In fact

- the loop-shaped region $\mathcal{L}$ and $\widetilde{\mathcal{L}} \doteq \{\mathcal{L} + V_+\} \bigcap \{\mathcal{L} + V_-\} \supset \mathcal{L}$ both have as continuous retract the simple spacelike loop γ; so $\mathbb{R}^4 \backslash \widetilde{\mathcal{L}}$ is homotopic to $\mathbb{R}^4 \backslash \gamma$ and for their homology groups holds, by using Alexander duality

$$H_1(\mathbb{R}^4 \backslash \widetilde{\mathcal{L}}) \approx H_1(\mathbb{R}^4 \backslash \gamma) \approx H^2(\gamma) \approx H^2(S^1) = \{0\}$$

  so by (ii) of Lemma, the component $g_\smile$ of g supported in $\widetilde{\mathcal{L}}$ has a pre-image $h_\smile$ supported in any given neighbourhood of $\widetilde{\mathcal{L}}$

- thus if $g_1, g_2 \in \mathscr{C}_1$ have supports in spacelike separated loop-shaped regions $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, one obtains

$$[A_J(g_1), A_J(g_2)] = [A_J(g_1\smile), A_J(g_2\smile)] = [j(h_1\smile), j(h_2\smile)] = 0$$

However, combining the previous results for A_T with the one of A_J we obtain non-trivial topological charges with electric currents

(similar to definition of s-products in the Wightman framework of quantum field theories [Borchers 1984])

- $(\pi_T, \mathcal{H}_T, \Omega_T)$ be a regular vacuum representation with non-trivial topological charge but trivial electric current and $(\pi_J, \mathcal{H}_J, \Omega_J)$ be a regular vacuum representation with non-trivial electric current but trivial topological charge

- Construct the representation π_{TJ} on $\mathcal{H}_T \otimes \mathcal{H}_J$ putting for $a \in \mathbb{R}$, $g \in \mathscr{C}_1$

$$\pi_{TJ}(V(a,g)) \doteq \pi_T(V(a,g)) \otimes \pi_J(V(a,g))$$

Restricting the algebra $\pi_{TJ}(\mathfrak{V})$ to $\mathcal{H}_{TJ} \doteq \pi_{TJ}(\mathfrak{V})\,\Omega_{TJ} \subset \mathcal{H}_T \otimes \mathcal{H}_J$, where $\Omega_{TJ} \doteq \Omega_T \otimes \Omega_J$ one obtains a regular vacuum representation $(\pi_{TJ}, \mathcal{H}_{TJ}, \Omega_{TJ})$ of $\mathfrak{V}$ with generating function

$$\omega_{TJ}(V(a,g)) \doteq \langle \Omega_T, \pi_T(V(a,g))\Omega_T\rangle\, \langle \Omega_J, \pi_J(V(a,g))\Omega_J\rangle,\ a \in \mathbb{R},\ g \in \mathscr{C}_1$$

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The C*-algebras of the e.m. field

Linear symplectic forms and absence of topological charges

Non-trivial topological charges and spacelike linearity in vacuum

Non-trivial topological charges with spacelike linearity and electric current

Topological charges of multiplets of electromagnetic fields

Conclusions and outlook

Topological charges of multiplets of electromagnetic fields

Discuss the appearance of topological charges for multiplets of e.m. fields, that is non-abelian gauge theories, in short distance (scaling) limit:

in the asymptotically free case the fields become non-interacting and transform as tensors under the adjoint action of some global gauge group

- Consider the case of two e.m. fields with corresponding intrinsic vector potentials defined on $\mathscr{C}_1 \oplus \mathscr{C}_1$. The resulting *-algebra generated by the unitaries $V_2(a, g)$ with $a \in \mathbb{R}$ and $g \in \mathscr{C}_1 \oplus \mathscr{C}_1$ has a C*-norm induced by all of its GNS representations; its completion w.r.t. this norm denoted by $\mathfrak{V}_2$ is the universal C*-algebra in the scaling limit
- $\mathfrak{V}_2$ is not isomorphic to the completion of $\mathfrak{V} \otimes \mathfrak{V}$ since the two types of fields need not commute with each other
- We show that there exist regular vacuum representations of $\mathfrak{V}_2$ with non-trivial topological charge i.e. for $g_1, g_2 \in \mathscr{C}_1 \oplus \mathscr{C}_1$ with supp $\mathrm{supp}(g_1) \perp \mathrm{supp}(g_2)$ and $a_1, a_2 \in \mathbb{R}$ it holds $[A(g_1), A(g_2)] \neq 0$
- Observe that now the underlying electromagnetic fields are (linear) Wightman fields so we shall proceed using a generalized free field

- Denote up and down subspaces of $\mathscr{C}_1 \oplus \mathscr{C}_1$ by $\mathscr{C}_u = \mathscr{C}_1 \oplus \{0\}$ and $\mathscr{C}_d = \{0\} \oplus \mathscr{C}_1$ with elements g_u, g_d and primitives G_u, G_d respectively
- For $g \in \mathscr{C}_1$ and classes of co-primitives $G \in \mathscr{D}_2$ s.t. $g = \delta G$ we use the scalar product on the one-particle space of the free Maxwell field

$$\langle G_1, G_2 \rangle_0 \doteq (2\pi)^{-3} \int dp\, \theta(p_0)\, \delta(p^2)\, \overline{(p\, \widehat{G_1}(p))}(p\, \widehat{G_2}(p)), \quad G_1, G_2 \in \mathscr{D}_2$$

- for fixed $-1 \leq \zeta \leq 1$ we give a sesquilinear form on $\mathbb{C}\,\mathscr{C}_1 \oplus \mathscr{C}_1$

$$\langle g_1, g_2 \rangle_\zeta \doteq \langle G_{1u}, G_{2u} \rangle_0 + \langle G_{1d}, G_{2d} \rangle_0 + \zeta \langle G_{1u}, \star G_{2d} \rangle_0 - \zeta \langle G_{1d}, \star G_{2u} \rangle_0$$

- hence $\langle \cdot, \cdot \rangle_\zeta$ defines a positive (semidefinite) scalar product on $\mathbb{C}\,\mathscr{C}_1 \oplus \mathscr{C}_1$ with a corresponding regular quasi-free vacuum states ω_ζ on $\mathfrak{V}_2$ with generating function

$$\omega_\zeta(V_2(a, g)) \doteq e^{-a^2 \langle g, g \rangle_\zeta / 2}, \quad a \in \mathbb{R},\ g \in \mathscr{C}_1 \oplus \mathscr{C}_1$$

- In the GNS representations $(\pi_\zeta, \mathcal{H}_\zeta, \Omega_\zeta)$, we have that $A_\zeta(g)$ is a (linear) Wightman field on $\mathscr{C}_1 \oplus \mathscr{C}_1$ with a global internal symmetry group $SO(2)$
- Nevertheless, the potential A_ζ carries a non-trivial topological charge obtained by the commutator

$$[A_\zeta(g_1), A_\zeta(g_2)] = (\langle g_1, g_2 \rangle_\zeta - \langle g_2, g_1 \rangle_\zeta)\, 1$$
$$= (\Delta(G_{1u}, G_{2u}) + \Delta(G_{1d}, G_{2d}) + \zeta \Delta(G_{1u}, \star G_{2d}) - \zeta \Delta(G_{1d}, \star G_{2u}))\, 1$$

similarly to the case of A_T.

Outline

Conclusions and outlook

- Representations with topological charges at finite scale in asymptotically free, non-abelian gauge theories
- Topological charges for massive theories: look not to depend on masses, so are intrinsically related to the non-massive e.m. field
- Relation between Topological charges and electric charge and currents: e.g. can we measure the electric charge by the topological one? . . .
- Universal algebra for models with e.m. field in low dimensional spacetime
- Superselection of topological charges
- Connection representations and potential systems [CRV2012, 2015] for the intrinsic potential with topological charges
- Universal algebra and topological aspects of non-abelian, local-gauge theories
- How topological charges would manifest themselves experimentally e.g. by interference patterns, . . .

Thank You for Your Attention