Propagators for the wave operator on static Lorentzian manifolds with timelike boundary

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- There is a fairly accepted procedure for quantizing a given classical system by means of Canonical Commutation Relations (CCR):

$$[\phi(x), \phi(y)] = i\hbar G(x, y) \,.$$

- CCR require a good understanding of the classical theory.
  - $\Rightarrow$ done for the wave operator $\Box_g$ on (M, g) globally hyperbolic **[Riesz - '49; Bär, Ginoux, Pfäffle - '07;...]**;
  - $\Rightarrow$ missing (and required) for generic Lorentzian manifolds **[Dappiaggi, Nosari, Pinamonti - '14; Dappiaggi, Ferreira - '17; Zahn - '18;...]** ;
- In this talk[1]: propagators for the wave operator $\Box_g$ on Lorentzian manifolds (M, g) with timelike boundary.

[1] Based on a joint work with C. Dappiaggi and H. Ferreira – arXiv:1804.03434.

1. Propagators on globally hyperbolic (ultrastatic) Lorentzian manifolds

2. Boundary triples

3. Propagators on Lorentzian manifolds with timelike boundary

Propagators on globally hyperbolic Lorentzian manifolds

Simplest case: $(M,g)=(\mathbb{R}\times\Sigma,\mathrm{d}t^2-h)$, $\Box_g=\partial_t^2+\Delta_h$.
(Σ,h) complete Riemannian manifold, Δ_h Laplace-Beltrami operator.

Definition

The advanced/retarded propagators associated with $\Box_g$ are the unique maps

$$G^\pm\colon C^\infty_{\mathrm{c}}(\mathbb{R},H^\infty(\Sigma))\to C^\infty(\mathbb{R},H^\infty(\Sigma))\,,$$
$$\Box_g\circ G^\pm=G^\pm\circ\Box_g=\mathrm{id}_{C^\infty_{\mathrm{c}}(\mathbb{R},H^\infty(\Sigma))}\,,$$
$$\mathrm{supp}(G^\pm f)\subseteq J^\pm\big(\mathrm{supp}(f)\big)\,.$$

Explicitly

$$\big[G^\pm f\big](t):=\int\pm\theta(\pm(t-s))\sin\big[(t-s)\Delta_h^{\frac{1}{2}}\big]\Delta_h^{-\frac{1}{2}}f(s)\mathrm{d}s\,.$$

Simplest case: $(M,g) = (\mathbb{R}\times\Sigma, \mathrm{d}t^2 - h)\,,\ \Box_g = \partial_t^2 + \Delta_h\,.$
Causal propagator: $G := G^+ - G^+$. There is a short exact sequence

$$0 \to C^\infty_{\mathrm{c}}(\mathbb{R}, H^\infty(\Sigma)) \overset{\Box_g}{\to} C^\infty_{\mathrm{c}}(\mathbb{R}, H^\infty(\Sigma)) \overset{G}{\to} C^\infty(\mathbb{R}, H^\infty(\Sigma)) \overset{\Box_g}{\to} C^\infty(\mathbb{R}, H^\infty(\Sigma)) \to 0\,.$$

In particular

$$\ker\Box_g \simeq GC^\infty_{\mathrm{c}}(\mathbb{R}, H^\infty(\Sigma)) \simeq \frac{C^\infty_{\mathrm{c}}(\mathbb{R}, H^\infty(\Sigma))}{\Box_g C^\infty_{\mathrm{c}}(\mathbb{R}, H^\infty(\Sigma))}\,.$$

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Causal propagator: $G:=G^+-G^+\,.$ There is a short exact sequence

$$0\to C^\infty_{\mathrm{c}}(\mathbb{R},H^\infty(\Sigma))\overset{\Box_g}{\to}C^\infty_{\mathrm{c}}(\mathbb{R},H^\infty(\Sigma))$$
$$\overset{G}{\to}C^\infty(\mathbb{R},H^\infty(\Sigma))\overset{\Box_g}{\to}C^\infty(\mathbb{R},H^\infty(\Sigma))\to 0\,.$$

In particular

$$\ker\Box_g\simeq GC^\infty_{\mathrm{c}}(\mathbb{R},H^\infty(\Sigma))\simeq\frac{C^\infty_{\mathrm{c}}(\mathbb{R},H^\infty(\Sigma))}{\Box_g\,C^\infty_{\mathrm{c}}(\mathbb{R},H^\infty(\Sigma))}\,.$$

What if M has a boundary?

Boundary triples

$A\colon D(A) \subset \mathsf{H} \to \mathsf{H}$ closed and symmetric.

Definition (Gorbachuk, Derkach, Malamud)

$(\mathsf{h}\,,\gamma_0\,,\gamma_1)$ is a boundary triple for A^* if

1. $(\gamma_0\,,\gamma_1)\colon D(A^*) \to \mathsf{h}\times\mathsf{h}$ is linear (bounded) and surjective;
2. $(A^*\psi\,|\,\phi) - (\psi\,|\,A^*\phi) = (\gamma_1\psi\,|\,\gamma_0\phi) - (\gamma_0\psi\,|\,\gamma_1\phi) \qquad \forall\psi,\phi\in D(A^*)$.

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Example (Von Neumann deficiency indexes)

$D(A^*) = D(A)\oplus_A N_+ \oplus_A N_-\,, \quad \pi_\pm\colon D(A^*) \to N_\pm := \ker(A^* \mp i)\,.$

$$(A^*\psi\,|\,\phi) - (\psi\,|\,A^*\phi) = (2i)\big[(\pi_-\psi\,|\,\pi_-\phi) - (\pi_+\psi\,|\,\pi_+\phi)\big]\,.$$

If $V\colon N_- \to N_+$ is unitary then

$$\gamma_0 := \pi_+ - V\circ\pi_-\,, \quad \gamma_1 := i(\pi_+ + V\circ\pi_-)\,,$$

identify a boundary triple $(N_+\,,\gamma_0\,,\gamma_1)$ for A^*.

Example

- $A := -i\frac{\mathrm{d}}{\mathrm{d}x}\colon H^1_0(-\ell,\ell) \to L^2(-\ell,\ell)$, $D(A^*) = H^1(-\ell,\ell)$.

  $$(\gamma_0\,,\,\gamma_1)\colon H^1(-\ell,\ell) \ni \psi \mapsto \frac{1}{\sqrt{2}}\Big(i[\psi(\ell)-\psi(-\ell)]\,,\psi(\ell)+\psi(-\ell)\Big) \in \mathbb{C}^2\,.$$

  $(\mathbb{C}\,,\,\gamma_0\,,\,\gamma_1)$ is a boundary triple for A^*.

- $A := -\partial_i a^{ij}\partial_j + a_0\colon H^2_0(\Omega) \to L^2(\Omega)$, unif. elliptic $(D(A^*) = H^2(\Omega))$.

  $$(\gamma_0\,,\,\gamma_1)\colon H^2(\Omega) \ni \psi \mapsto (\psi|_{\partial\Omega}, -\partial_n\psi|_{\partial\Omega}) \in L^2(\partial\Omega)\,.$$

  $(L^2(\partial\Omega)\,,\,\gamma_0\,,\,\gamma_1)$ is a boundary triple for A^*.

With some care, one can consider A elliptic.
Problem: $H^2(\Omega) \subsetneq D(A^*)$ e.g. $A := -x^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2$, $x \in (0,1)$.

Theorem (Derkach and Malamud, '95)

Let $(\mathsf{h}, \gamma_0, \gamma_1)$ be a boundary triple for A^* and let $\Theta \colon D(\Theta) \subseteq \mathsf{h} \to \mathsf{h}$ be closed. Then

- $D(A) = \ker \gamma_0 \cap \ker \gamma_1$;
- The operator

$$A_\Theta := A^*|_{\ker(\gamma_1 - \Theta\gamma_0)} \colon \ker(\gamma_1 - \Theta\gamma_0) \to \mathsf{H} \,,$$

  is a closed extension of A ($A \subset A_\Theta \subset A^*$);
- $(A_\Theta)^* = A_{\Theta^*}$;
- There is a 1-1 correspondence

$$\{\textit{closed operators on } \mathsf{h}\} \ni \Theta \mapsto A_\Theta \in \{\textit{closed extensions of } A\} \,.$$

$A_\infty := A^*|_{\ker \gamma_0}$ and $A_0 := A^*|_{\ker \gamma_1}$ are selfadjoint extensions of A.

What about $\sigma(A_\Theta)$?

Proposition (Derkach and Malamud, '91)

$A_\infty := A^*|_{\ker\gamma_0}$, $\lambda \in \rho(A_\infty)$, $\Theta\colon D(\Theta) \subseteq \mathsf{h} \to \mathsf{h}$ closed. Then:

- There is an algebraic decomposition

$$D(A^*) = D(A_\infty) \dot{+} N_\lambda\,, \qquad N_\lambda := \ker(A^* - \lambda)\,.$$

  The function

$$M\colon \rho(A_\infty) \ni \lambda \mapsto \gamma_1 \circ (\gamma_0|_{N_\lambda})^{-1}$$

  is called Weyl function;

- For all $\lambda \in \rho(A_\infty)$

$$\lambda \in \rho(A_\Theta) \Longleftrightarrow 0 \in \rho(\Theta - M(\lambda))\,,$$
$$\lambda \in \sigma_\sharp(A_\Theta) \Longleftrightarrow 0 \in \sigma_\sharp(\Theta - M(\lambda))\,, \quad \sharp \in \{\mathrm{p}, \mathrm{c}, \mathrm{r}\}\,.$$

Some examples

Example

- For $(N_+\,,\,\pi_+ - V\circ\pi_-\,,\,i(\pi_+ + V\circ\pi_-))$ we have

$$\gamma_1\psi = \Theta\gamma_0\psi \quad\Longleftrightarrow\quad \pi_+\psi = C(-\Theta)V\pi_-\psi\,.$$

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- $A := -i\frac{\mathrm{d}}{\mathrm{d}x}\colon H^1_0(-\ell,\ell)\to L^2(\ell,\ell)$, $(\mathbb{C}\,,\,\gamma_0\,,\,\gamma_1)$ as above.

$$D(A_\Theta) = \{\psi\in H^1(-\ell,\ell)\,|\,\psi(\ell)+\psi(-\ell) = \Theta[\psi(\ell)-\psi(-\ell)]\}\,,\ \Theta\in\mathbb{R}\,.$$

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- $A := -\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\colon H^2_0(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$, $(\mathbb{C}\,,\,\psi\mapsto\psi(0)\,,\,\psi\mapsto\psi'(0))\,.$

$$D(A_\Theta) = \{\psi\in H^2(\mathbb{R}_+)|\quad \psi'(0) = \Theta\psi(0)\}\,,\quad \Theta\in\mathbb{R}\,.$$

$\sigma(A_\infty) = (0,+\infty)$ and $\Theta - M(\lambda) = \Theta + |\lambda|^{1/2}$. Thus

$$\Theta\geq 0\Rightarrow\sigma(A_\Theta) = (0,+\infty)\,;\,\Theta<0\Rightarrow\sigma(A_\Theta) = \{-\Theta^2\}\cup(0,+\infty)\,.$$

Lorentzian manifolds with timelike boundary

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Definition (manifold of bounded geometry)

$(\widetilde{\Sigma}, \widetilde{h})$, $\partial\widetilde{\Sigma} = \emptyset$, is of bounded geometry if:

- $r_{\mathrm{inj}}(\widetilde{\Sigma}) > 0$;
- $\|\nabla^k R\|_{L^\infty(\widetilde{\Sigma})} < +\infty\ \forall k \in \mathbb{N}$.

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$(Y, \iota^*\widetilde{h}) \xhookrightarrow{\iota} (\widetilde{\Sigma}, \widetilde{h})$ is a submanifold of bounded geometry if

- all covariant derivatives of K_Y are bounded;
- $\exists \epsilon > 0$ such that $(-\epsilon, \epsilon) \times Y \ni (s, y) \mapsto \exp_y\left[s\nu_y\right] \in \widetilde{\Sigma}$ is injective.

Lorentzian manifolds with timelike boundary

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(Σ, h), $\partial\Sigma \neq \emptyset$, is of bounded geometry if

- $\exists(\widetilde{\Sigma}, \widetilde{h})$, $\partial\widetilde{\Sigma} = \emptyset$, of bounded geometry such that $\Sigma \subseteq \widetilde{\Sigma}$, $\widetilde{h}|_\Sigma = h$;
- $(\partial\Sigma, \iota^*\widetilde{h}) \overset{\iota}{\hookrightarrow} (\widetilde{\Sigma}, \widetilde{h})$ is a submanifold of bounded geometry.

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- $(\partial\Sigma,\iota^*\widetilde{h})\overset{\iota}{\hookrightarrow}(\widetilde{\Sigma},\widetilde{h})$ is a submanifold of bounded geometry.

$(L^2(\partial\Sigma),u\mapsto u|_{\partial\Sigma},u\mapsto-\nabla_\nu u|_{\partial\Sigma})$ is a boundary triple for Δ_h.

Theorem (C. Dappiaggi, N. D., H. Ferreira,'18)

$(M,g)=(\mathbb{R}\times\Sigma,\mathrm{d}t^2-h)$, (Σ,h) of bounded geometry.
$\Theta\colon D(\Theta)\subseteq L^2(\partial\Sigma)\to L^2(\partial\Sigma)$ self-adjoint.
$\Delta_{h,\Theta}:=\Delta^*|_{\ker(\gamma_1-\Theta\gamma_0)}$ **bounded from below**. Then for all $f\in C^\infty_{\mathrm{c}}(M)$,

$$[G^\pm_\Theta f](t):=\int \pm\theta(\pm(t-s))\sin\left[(t-s)\Delta_{h,\Theta}^{\frac{1}{2}}\right]\Delta_{h,\Theta}^{-\frac{1}{2}}f(s)\mathrm{d}s\,,$$

satisfy

- $\Box_g\circ G^\pm_\Theta=G^\pm_\Theta\circ\Box_g=\mathrm{id}\,;$
- $\mathrm{supp}(G^\pm_\Theta f)\subseteq J^\pm(\mathrm{supp}(f))\,.$
- $G^\pm_\Theta\colon C^\infty_{\mathrm{c}}(\mathbb{R},H^\infty_\Theta(\Sigma))\to C^\infty(\mathbb{R},H^\infty_\Theta(\Sigma))\,,$
  where $H^\infty_\Theta(\Sigma):=\cap_k D((1+\Delta_{h,\Theta})^k)\,.$
- $G^\pm_\Theta$ are the unique adv./ret. propagators for $\Box_{g,\Theta}:=\partial_t^2+\Delta_{h,\Theta}\,;$

Theorem (C. Dappiaggi, N. D., H. Ferreira,'18)

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$\Theta\colon D(\Theta) \subseteq L^2(\partial\Sigma) \to L^2(\partial\Sigma)$ self-adjoint.
$\Delta_{h,\Theta} := \Delta^*|_{\ker(\gamma_1 - \Theta\gamma_0)}$ **bounded from below**.

- there is a short exact sequence

$$0 \to C^\infty_{\mathrm{c}}(\mathbb{R}, H^\infty_\Theta(\Sigma)) \overset{\Box_{g,\Theta}}{\to} C^\infty_{\mathrm{c}}(\mathbb{R}, H^\infty_\Theta(\Sigma)) \overset{G_\Theta}{\to} C^\infty(\mathbb{R}, H^\infty_\Theta(\Sigma)) \overset{\Box_{g,\Theta}}{\to} C^\infty(\mathbb{R}, H^\infty_\Theta(\Sigma)) \to 0\,,$$

  where $G_\Theta := G^+_\Theta - G^-_\Theta$.

Conclusions

$\surd$ Existence of advanced/retarded propagators for $\Box_g$ on Lorentzian manifold with timelike boundary;

$\rightarrow$ Boundary triples provides useful informations on the extension $\Delta_{h,\Theta}$;

$\surd$ Wentzell boundary conditions can be treated within this framework:

$$\begin{cases} \partial_t^2\phi + \Delta_h\phi = 0 \\ \gamma_1\phi = \Theta\gamma_0\phi + \partial_t^2\gamma_0\phi \end{cases} \longleftrightarrow \partial_t^2\widehat{\phi} + \begin{pmatrix} \Delta_h & 0 \\ -\gamma_1 & \Theta \end{pmatrix}\widehat{\phi} = 0\,;$$

- Different geometrical data and models
  - $\Rightarrow$ Dirac, Proca, . . . ;
  - $\Rightarrow$ Evolution equation approach **[Dereziński, Siemssen - 2017]**;
- Microlocal properties of $G_\Theta^\pm$?
  - $\Rightarrow$ Reflection at the boundary?
    [Dappiaggi, Nosari, Pinamonti - '14; Dappiaggi, Ferreira - '17]