Marco Falconi
Eberhard Karls Universität Tübingen.

Semiclassical Analysis in AQFT

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The basic ideas of semiclassical analysis

Q vs. C

∗ [Q-Observables]	$A(\hat{x}_\hbar, \hat{p}_\hbar)$	$A(x,p)$	[C-Observables] ∗
∗ [Q-States]	$\lvert\, \psi_\hbar \,\rangle\langle\, \psi_\hbar \,\rvert$	$\mu(\mathrm{d}x, \mathrm{d}p)$	[C-States] ∗
∗ [Q-Evolution]	$U_\hbar(t)\,(\,\cdot\,)\,U_\hbar^*(t)$	$\Phi(t)_{\,*}\,(\,\cdot\,)$	[C-Evolution] ∗

$\hbar \to 0$: Q $\to$ C

$$
\begin{array}{lr}
* \text{ [Q-Observables]} & A(\hat{x}_\hbar, \hat{p}_\hbar) \\
* \text{ [Q-States]} & |\,\psi_\hbar\,\rangle\langle\,\psi_\hbar\,| \\
* \text{ [Q-Evolution]} & U_\hbar(t)\,(\,\cdot\,)\,U_\hbar^*(t)
\end{array}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}
\xrightarrow[\hbar\to 0]{}
\left\{\begin{array}{l}
A(x,p) \\
\mu(\mathrm{d}x, \mathrm{d}p) \\
\Phi(t)_*(\,\cdot\,)
\end{array}\right.
\begin{array}{r}
\text{[C-Observables] } * \\
\text{[C-States] } * \\
\text{[C-Evolution] } *
\end{array}
$$

- Applications in non-relativistic quantum mechanics, and in non-relativistic and semi-relativistic *bosonic* QFT (Many body, Nelson, Pauli-Fierz):
    - Wigner measures;
    - Egorov-type theorems;
    - Weyl eigenvalue asymptotics;
    - Mean-field many-boson systems;
    - Quasi-classical particle-radiation systems;
    - …
- All the results are proved either in the Schrödinger ($d < \infty$) or Fock ($d = \infty$) representation of CCR, and depend heavily on the explicit action of canonical observables (either position and momentum or creation and annihilation operators).

Semiclassical analysis for bosonic states

Transitioning to the AQFT language

- We restrict our attention to algebras of bosonic observables $\mathfrak{B}_\hbar$ that embed a Weyl C*-algebra $\mathbb{W}_\hbar(X,\varsigma)$. Therefore, we can w.l.o.g. restrict to the study of states on $\mathbb{W}_\hbar(X,\varsigma)$.

- $\mathbb{W}_\hbar : \mathbf{Symp}_\mathbb{R} \to \mathbf{C^*alg}$ (*"Segal quantization" functor*)
  - $\mathbb{W}_\hbar({}^\forall X, {}^\forall\varsigma) = \mathrm{C}^*\{W_\hbar(x),\, x \in X\}$
    1. $W_\hbar({}^\forall x) \neq 0$;
    2. $(W_\hbar({}^\forall x))^* = W_\hbar(-x)$;
    3. $W_\hbar({}^\forall x) W_\hbar({}^\forall y) = e^{-i\hbar\varsigma(x,y)} W_\hbar(x+y)$.
  - $\mathbb{W}_\hbar({}^\forall s) = \mathfrak{s}_\hbar : \mathbb{W}_\hbar(X,\varsigma) \to \mathbb{W}_\hbar(Y,\tau)$ (induced by the action of the linear symplectomorphism $s : (X,\varsigma) \to (Y,\tau)$ on generators).

- Let $\mathbb{D}_+ : \mathbf{C^*alg} \to \mathbf{BanCone}$ be the (contravariant) functor mapping a C*-algebra in the Banach cone of positive states and *-homomorphisms to the transposed maps (that are linear, continuous and positivity-preserving).

- $\mathbb{S}_\hbar := \mathbb{D}_+ \circ \mathbb{W}_\hbar$ (*Bosonic quantum states functor*)
  - $\mathbb{S}_\hbar : \mathbf{Symp}_\mathbb{R} \to \mathbf{BanCone}$
  - $\mathbb{S}_\hbar$ is *contravariant*

Remarks

- It is tempting to interpret $(X, \varsigma) \in \mathbf{Symp}_\mathbb{R}$ as the phase space or one-particle space (due to its symplectic structure). However, *it is not*, it is the space of test functions, not of fields.
- We could have used the resolvent algebra instead of the CCR algebra [Buchholz-Grundling], however that would be equivalent since regular states on the CCR and resolvent algebras are isomorphic.

The functor for classical states

- $\mathbb{S}_0 : \mathbf{Symp}_{\mathbb{R}} \to \mathbf{CylM}$ is a *Wigner measures functor* iff

  1. $\mathbb{S}_0({}^{\forall}X, {}^{\forall}\varsigma) = \mathcal{M}_{\mathrm{cyl}}\big({}^{\exists}A, X\big) \quad (X \subseteq \mathbb{R}^A)$
  2. ${}^{\forall}s(x) = x \circ \mathbb{S}_0(s) \quad (\forall x \in X)$

Proposition

A Wigner measures functor *exists and it is unique up to natural isomorphisms.*

Concrete Example

$A := X^*$, $\mathbb{S}_0(s) = {}^{\mathrm{t}}s$.

X^* (or A more in general) is the space of *classical fields*, acting on test functions by duality ("rough" distributions).

Semiclassical states and topologies of convergence

Definition (Semiclassical state)

A state $\omega_\hbar \in \mathbb{W}_\hbar(X,\varsigma)'_+$ is *semiclassical* iff

1. $\Big(\lambda \mapsto \omega_\hbar\big(W_\hbar(\lambda\,{}^\forall x)\big)\Big) \in \mathcal{C}^0(\mathbb{R},\mathbb{C})$ (Regular state);
2. $\sup_{\hbar\in(0,1)} \omega_\hbar\big(W_\hbar(0)\big) < \infty$ (Uniformly bounded norm w.r.t. small $\hbar$).

Definition ($\mathfrak{P}_X$ topology)

$\omega_{\hbar_\beta} \xrightarrow[\hbar_\beta \to 0]{\mathfrak{P}_X} M$ iff

$$\lim_{\hbar_\beta \to 0} \omega_{\hbar_\beta}\bigl(\mathrm{Op}_{1/2}^{\hbar_\beta}(\forall f)\bigr) = \int_{X^*} f(\xi) M(\mathrm{d}\xi) \; ,$$

where the functions f are *cylindrical*, smooth, and compactly supported (w.r.t. the finite dimensional base of the cylinder). M is a *cylindrical Wigner measure* of $\omega_\hbar$.

Definition ($\mathfrak{T}_X$ topology)

$\omega_{\hbar_\beta} \xrightarrow[\hbar_\beta \to 0]{\mathfrak{T}_X} M$ iff

$$\lim_{\hbar_\beta} \omega_{\hbar_\beta}\bigl(W_{\hbar_\beta}(\forall x)\bigr) = \int_{X^*} e^{2i\xi(x)} \mathrm{d}M(\xi) = \hat{M}(x) \; .$$

Remarks

- If $\omega_{\hbar_\beta} \xrightarrow[\hbar_\beta \to 0]{\mathfrak{P}_X} M$ and $\omega_{\hbar_\beta}\big(W_{\hbar_\beta}(0)\big) \xrightarrow[\hbar_\beta \to 0]{} M(X^*)$, then $\omega_{\hbar_\beta} \xrightarrow[\hbar_\beta \to 0]{\mathfrak{T}_X} M$.
- The convergence $\mathfrak{T}_X$ is useful in concrete examples to identify explicitly the limit measure M.

"$\mathbb{S}_\hbar \xrightarrow[\hbar\to 0]{} \mathbb{S}_0$"

- *Semiclassical convergence in objects*

  For any small $\mathbf{S} \subset \mathbf{Symp}_{\mathbb{R}}$, the family of maps $\{(X,\varsigma) \mapsto \omega_{\hbar,X}, \hbar \in (0,1)\}$, where all $\omega_{\hbar,X}$ are semiclassical, is compact in the $\mathfrak{P} = \prod_{(X,\varsigma)\in\mathbf{S}} \mathfrak{P}_X$ topology.

  In addition, all its limit points for $\hbar \to 0$ are maps $(X,\varsigma) \mapsto M_X$, where M_X is a Wigner measure of $\omega_{\hbar,X}$.

- *Semiclassical pointwise convergence of morphisms*

  For any small $\mathbf{S} \subset \mathbf{Symp}_{\mathbb{R}}$, if $\Big((X,\varsigma) \mapsto \omega_{\hbar_\beta,X}\Big) \xrightarrow[\hbar_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} \Big((X,\varsigma) \mapsto M_X\Big)$, then for any symplectic morphism $s : (X,\varsigma) \to (Y,\tau)$,

$$\Big((Y,\tau) \mapsto \mathbb{S}_{\hbar_\beta}(s)\omega_{\hbar_\beta,Y}\Big) \xrightarrow[\hbar_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} \Big((Y,\tau) \mapsto \mathbb{S}_0(s)M_Y\Big)\,.$$

- *"Surjectivity" of the semiclassical limit*

  For any small $\mathbf{S} \subset \mathbf{Symp}_{\mathbb{R}}$, every element $M_{\exists X} \in \mathrm{Ran}\big(\mathbb{S}_0\big|_{\mathbf{S}}\big)$ is the limit point in the $\mathfrak{P}_X \vee \mathfrak{T}_X$ topology of at least one semiclassical state $\omega_{\hbar, M_X} \in \mathrm{Ran}\big(\mathbb{S}_\hbar\big|_{\mathbf{S}}\big)$.

A priori information on classical states given by quantum properties

Local structure (+ existence of local fields)

Definition (Local functor)

Let $\mathbf{L}$ be a small category (of local spacetimes), and $\mathbb{L} : \mathbf{L} \to \mathbf{Symp}_{\mathbb{R}}$. Then

$$\mathbb{LS}_\hbar := \mathbb{S}_\hbar \circ \mathbb{L} \,, \; \mathbb{LS}_0 := \mathbb{S}_0 \circ \mathbb{L}$$

are the local functors for quantum states and Wigner measures respectively.

Proposition

$$\mathbb{LS}_\hbar \xrightarrow[\hbar \to 0]{} \mathbb{LS}_0$$

Semiclassical implications of LCQFT axioms

- **Covariance**

  Both $\mathbb{LS}_\hbar$ and $\mathbb{LS}_0$ are contravariant. In other words, classical local fields are contravariant w.r.t. local test functions.

- **Isotony**

  Existence of global classical states: $\mathcal{M}_{\mathrm{cyl}}(\underline{X}^*, \underline{X}) \cong \varprojlim \mathcal{M}_{\mathrm{cyl}}\big(\mathbb{L}(L)^*, \mathbb{L}(L)\big)$, where the space of global test functions is defined as $\underline{X} = \varinjlim \mathbb{L}(L)$. In addition, the semiclassical structure is inherited: states on the global algebra $\varinjlim \mathbb{LW}(L)$ that are projective families of convergent semiclassical states converge semiclassically to the correspondent measure in $\mathcal{M}_{\mathrm{cyl}}(\underline{X}^*, \underline{X})$.

- **Einstein Causality**

  $\mathbb{LS}_\hbar^{\otimes\alpha} \xrightarrow[\hbar\to 0]{} \mathbb{LS}_0^{\otimes}$ (it is possible to assign tensor structures to quantum and classical target categories, and the functors respect such structures).

  In addition, *spacelike entanglement can only be destroyed in the limit* $\hbar \to 0$ (it is true for entanglement in general).

- **Time-Slice**

  One would like to use the time-slice axiom and time-zero fields to obtain Egorov-type results. This can only be done in concrete models (and has not been done yet for fully relativistic theories).

Invariant and KMS states

- **Invariant states**

Let G be a group, represented by $\big(s(g)\big)_{g\in G}$ on $(X,\varsigma)\in \mathbf{Symp}_{\mathbb{R}}$. Suppose that $\omega_{\hbar_\beta}\in \mathbb{S}_\hbar(X,\varsigma)$ is G-invariant:

$$\mathbb{S}_\hbar\big(s(g)\big)\omega_{\hbar_\beta} = \omega_{\hbar_\beta}$$

and

$$\omega_{\hbar_\beta} \xrightarrow[\hbar\to 0]{\mathfrak{P}_X\vee\mathfrak{T}_X} M\ .$$

Then M is G-invariant as well

$$\mathbb{S}_0\big(s(g)\big)M = M\ .$$

KMS states

Let $\omega_\hbar$ be a state on a C* or W* dynamical system $(\mathfrak{B}_\hbar, \alpha_\hbar)$ that embeds $\mathbb{W}_\hbar(X,\varsigma)$, such that it is both a β-KMS and a semiclassical state. We would like to prove that if

$$\omega_{\hbar_\beta} \xrightarrow[\hbar\to 0]{\mathfrak{P}_X\vee\mathfrak{T}_X} M\ ,$$

then M satisfies the classical KMS condition

$$\int \{a(\xi), b(\xi)\}\mathrm{d}M(\xi) = \beta \int b(\xi)\{a(\xi), \mathfrak{h}(\xi)\}\mathrm{d}M(\xi)\ ,$$

for any a, b in some class of classical observables, where $\mathfrak{h}$ is the classical Hamiltonian (related to the semiclassical limit of $\alpha_\hbar$).
It is possible to prove such result in concrete examples, *e.g.* if X is finite dimensional or for non-relativistic systems.

Possible future directions

- Pseudodifferential calculus for non-cylindrical classical symbols
- Fermionic semiclassical analysis
- Applications to concrete relativistic systems:
  - semiclassical scattering;
  - semiclassical Haag's theorem;
  - Wigner measure of specific states of QFT in curved spacetime (*e.g.* Hadamard states);
  - …
- …

Thank you for the attention