Type III Representations and Modular Spectral Triples for the Noncommutative Torus

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abstract

We outline the explicit construction of non type II_1 representations and relative modular spectral triples of the noncommutative 2-torus $\mathbb{A}_\alpha$, provided α is special kind of Liouville number. For such representations, we define also appropriate Fourier transforms and show how those "diagonalise" appropriately some examples of the Dirac operators assocaited to such spectral triples.

introduction

For any irrational rotation number α, it is well known that the noncommutative torus $\mathbb{A}_\alpha$ must have representations π such that the generated von Neumann algebra $\pi(\mathbb{A}_\alpha)''$ is of type III. At the knowledge of the author, it seems that none of such representations of these kind is explicitly exhibited and investigated in detail, up to now. We also note that no nontrivial example of Spectral Triple associated to type III representations whose twist of the involved Dirac Operator and the corresponding twisted derivation arises from the Tomita modular operator (*i.e.* when the latter is neither bounded nor even) is explicitly known, at least for the noncommutative torus.

For the noncommutative 2-torus $\mathbb{A}_{2\alpha}$, we briefly outline how to fill such a gap at least when α is Liouville number (for non type II_1 representation), or when α is Liouville, with a faster approximation property by rationals (in order to construct also modular spectral triples as above).*

The idea is to look at C^∞ diffeomorphisms f of the unit circle $\mathbb{T}$ with rotation number, $\rho(f) =$

*The factor 2 is introduced only for the sake of convenience.

2α. By Denjoy theorem, f is conjugate to the rotation R_α of the angle $4\pi\alpha$:

$$f = h_f \circ f \circ h_f^{-1},$$

where h_f is a (essentially unique) homeomorphism of $\mathbb{T}$. If α is diophantine, then h_f is necessarily smooth and therefore our construction provides still the type II_1 hyperfinite factor. If α is Liouville, things go differently. In this situation, we are looking at diffeomorphisms as above for which the unique probability measure, invariant under the rotations, satisfies

$$m \circ h_f^{-1} \perp m,$$

$m = \frac{\mathrm{d}\theta}{2\pi}$ being the Haar measure on $\mathbb{T}$. Such diffeomorfisms are explicitly constructed in "S. Matsumoto: Nonlinearity 26 (2013), 1401-1414" for any prescribed Krieger-Araki-Woods ratio-set.†

†For the dynamical system (X, T, ν) for which the probability measure ν (supposed to be atom-less) is quasi-invariant and ergodic for the action of the automorphism T, the ratio-set $r(T)$ determines the type of the factor $L^\infty(X, \nu) \ltimes_T \mathbb{Z}$, provided it is not of type II_1.

For our construction, the probability measure $\mu := m \circ h_f$ plays a crucial role. It is then possible to exhibit a state ω_μ canonically associated to the measure μ, and our non type II_1 representations π_{ω_μ} produce hyperfinite von Neumann factors isomorphic to the crossed product

$$\pi_{\omega_\mu}(\mathbb{A}_{2\alpha})'' \sim L^\infty(\mathbb{T},\mu) \ltimes_{R_{2\alpha}} \mathbb{Z} \sim L^\infty(\mathbb{T},m) \ltimes_\rho \mathbb{Z}.$$

Here, β is the dual action of f on functions: $\beta(g) := g \circ f$.

For such representations, we also construct a one-parameter family of modular spectral triple, one for each $\eta \in [0,1]$, whose twists are constructed by taking into account of the modular data, provided α is a Liouville number satisfying a faster approximation property by rationals.‡

‡In our preliminary construction, for such a purpose we consider only the Tomita modular operator Δ, the conjugation J will play a crucial role in a more refined analysis.

By considering the von Neumann algebra $M := \pi_{\omega_\mu}(\mathbb{A}_{2\alpha})''$ equipped with the "measure" $\omega := \langle \cdot\, \xi_{\omega_\mu}, \xi_{\omega_\mu}\rangle$, we consider the various noncommutative L^p spaces $L^p(M)$, and define the Fourier transform(s). At level of Hilbert spaces (*i.e.* $L^2(M)$), we show that they "diagonalise" the involved deformed Dirac operators $D^{(\eta)}$, for $\eta = 0, 1/2, 1$, corresponding to the left, symmetric and right embeddings

$$
\begin{aligned}
&L^\infty\big(\pi(\mathbb{A}_\alpha)''\big) \equiv L^\infty(M) \equiv M \\
\hookrightarrow &M_* \equiv L^1(M) \equiv L^1\big(\pi(\mathbb{A}_\alpha)''\big)\,,
\end{aligned}
$$

of M in M_* associated to the underlying state ω.

The present talk is based on the following papers:

(1) F. Fidaleo and L. Suriano: *Type III representations and modular spectral triples for the noncommutative torus*, to appear.

(2) F. Fidaleo: *Fourier analysis for the noncommutative 2-torus*, in preparation.

the noncommutative torus and type III representations

For a fixed $\alpha \in \mathbb{R}$, the *noncommutative torus* $\mathbb{A}_{2\alpha}$ (the factor 2 is pure matter of convenience), *i.e.* that associated with the rotation by the angle $4\pi\alpha$, is the universal C^*-algebra with identity I generated by the commutation relations involving two noncommutative unitary indeterminates U, V:

$$\begin{aligned} &UU^* = U^*U = I = VV^* = V^*V, \\ &UV = e^{4\pi\imath\alpha}VU. \end{aligned} \tag{1}$$

We express $\mathbb{A}_{2\alpha}$ in the so called *Weyl form*. Let $\mathbf{a} := (m, n) \in \mathbb{Z}^2$ be a double sequence of integers, and define

$$W(\mathbf{a}) := e^{-2\pi\imath\alpha mn}U^m V^n, \quad \mathbf{a} \in \mathbb{Z}^2.$$

Obviously, $W(\mathbf{0}) = I$, and the commutation relations (1) become

$$W(\mathbf{a})W(\mathbf{A}) = W(\mathbf{a}+\mathbf{A})e^{2\pi\imath\alpha\sigma(\mathbf{a},\mathbf{A})},$$
$$W(\mathbf{a})^* = W(-\mathbf{a}), \quad \mathbf{a},\mathbf{A} \in \mathbb{Z}^2, \tag{2}$$

where the symplectic form σ is defined as

$$\sigma(\mathbf{a},\mathbf{A}) := (mN - Mn),$$
$$\mathbf{a} = (m,n),\ \mathbf{A} = (M,N) \in \mathbb{Z}^2.$$

We now fix a function $f \in \mathcal{B}(\mathbb{Z}^2)$, which we may assume to have finite support. The element $W(f) \in \mathbb{A}_{2\alpha}$ is then defined as

$$W(f) := \sum_{\mathbf{a}\in\mathbb{Z}^2} f(\mathbf{a})W(\mathbf{a}).$$

The set $\{W(f) \mid f \in \mathcal{B}(\mathbb{Z}^2)\,\text{with finite support}\}$ provides a dense $*$-algebra of $\mathbb{A}_{2\alpha}$. We also recall that $\mathbb{A}_{2\alpha}$ is simple and has a necessarily unique and faithful trace

$$\tau(W(f)) := f(\mathbf{0}), \quad W(f) \in \mathbb{A}_{2\alpha}.$$

Any element $A \in \mathbb{A}_{2\alpha}$ is uniquely determined by the corresponding Fourier coefficients

$$f(\mathbf{a}) := \tau(W(-\mathbf{a})A), \quad \mathbf{a} \in \mathbb{Z}^2.$$

The relations (2) transfer on the generators $W(f)$ as follows

$$W(f)^* = W(f^\star), \quad W(f)W(g) = W(f *_{2\alpha} g),$$

where

$$f^\star(\mathbf{a}) := \overline{f(-\mathbf{a})},$$
$$(f *_{2\alpha} g)(\mathbf{a}) = \sum_{\mathbf{A}\in\mathbb{Z}^2} f(\mathbf{A})g(\mathbf{a}-\mathbf{A})e^{-2\pi\imath\alpha\sigma(\mathbf{a},\mathbf{A})}.$$

To construct type III representations, we restrict α to be a so called **L**-numbers (L stands for Liouville), which are numbers, automatically irrational, which admit a "fast approximation by rationals". Here, we report the definition of Liouville and that we call "ultra-liouville" numbers.

L A *Liouville number* $\alpha \in (0,1)$ is a real number such that for each $N \in \mathbb{N}$ the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^N}$$

has an infinite number of solutions for $p, q \in \mathbb{N}$ with $(p,q) = 1$.

UL A *Ultra-Liouville number* $\alpha \in (0,1)$ is a real number such that, for each $\lambda > 1$ and $N \in \mathbb{N}$, the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\lambda^{q^N}}$$

again admits infinite number of solutions for p, q as above.

Any **L**-number is irrational, and in addition each **UL**-number is a **L**-number.

Our construction is based upon the explicit knowledge of special C^{∞}-diffeomorphisms (simply denoted by "diffeomorphisms) of the circle such that $\rho(f) = 2\alpha$. Such diffeomorfisms are constructed in the above mentioned paper of Matsumoto for a prescribed ratio-set F, but without any control on the so-called growth sequence, playing also a crucial role in constructing modular spectral triples.

The *growth sequence* is defined as

$$\Gamma_n(f) := \|Df^n\|_\infty \vee \|Df^{-n}\|_\infty, \quad n \in \mathbb{N}.$$

It was proven (*cf.* "N. Watanabe: Geom. Funct. Anal. 17 (2007), 320-331) that, in our situation, the growth sequence is not too wild: $\Gamma_n(f) = o(n^2)$. We prove that:

(i) for all diffeomorphisms f in Proposition 2.1 of **M** (constructed by fixing any preassigned ratio-set), we have $\Gamma_n(f) = o(n)$;

(ii) if in addition α is a **UL**-number, with the same method in **M** it is possible to construct diffeomorpisms with any preassigned ratio-set, and in addition $\Gamma_n(f) = o(\ln n)$.

Our construction of non type II_1 representations of the noncommutative torus starts by noticing that, if $\nu \in \mathcal{S}(C(\mathbb{T}))$ (*i.e.* ν is a probability measure on the circle) then

$$\omega_\mu(W(f)) := \sum_{m\in\mathbb{Z}} \check{\mu}(m) f(m,0) \tag{3}$$

defines a state on $\mathbb{A}_{2\alpha}$.[§]

For any diffeomorphism f as before, and $\mu = m \circ h_f$, the crucial fact is that

$$\mu \circ R_\alpha^{2n} \sim \mu, \quad n \in \mathbb{Z}, \tag{4}$$

[§] The normalised trace corresponds to $\nu = \frac{\mathrm{d}\theta}{2\pi}$.

and therefore the support of ω_μ in the bidual is central: $s(\omega_\mu) \in Z(\mathbb{A}_{2\alpha}^{**})$.¶

Put for simplicity $\omega = \omega_\mu$ We notice that the modular structure (i.e. Δ_ω, J_ω) is explicitly expressed by the Radon-Nikodym derivatives $\mathrm{d}\mu \circ R^{2n}/\mathrm{d}\mu$, $n \in \mathbb{Z}$. The representations we are searching for (including the type III ones) are those π_ω associated to the states ω_μ previously described. In this case

$$\pi_\omega(\mathbb{A}_{2\alpha})'' \sim L^\infty(\mathbb{T}, \mathrm{d}\mu) \ltimes_{R_{2\alpha}} \mathbb{Z} \sim L^\infty(\mathbb{T}, \mathrm{d}\theta/2\pi) \ltimes_\beta \mathbb{Z}.$$

is an hyperfinite factor acting in standard form with a copy of L^∞ as MASA, whose type is determined by the Krieger-Araki-Woods *ratio-set*.

¶This simply means that the GNS vector ξ_{ω_μ} is also separating.

the Fourier transform

The key-point for the construction of noncommutative L^p spaces is the modular theory. [‖] In fact, for $x \in M$ the map

$$t \in \mathbb{R} \mapsto \sigma_t^\omega(x)\omega \equiv \omega\big(\cdot\, \sigma_t^\omega(x)\big) \in M_*$$

extends to a bounded and continuous map on the strip $\{z \in \mathbb{C} \mid -1 \leqslant \text{Imz} \leqslant 1\}$, analytic in the interior. After putting $L^1_\theta(M) := M_*$, $\theta \in [0, 1]$, and $L^\infty_\theta(M) := \iota^\theta_{\infty,1}(M) \sim M$ and checking that the complex interpolation functor C' based on such analytic functions coincides (equal norms) with the standard one C, it is possible to define $L^p_\theta(M) := C_{1/p}(\iota^\theta_{\infty,1}(M), M_*)$, $1 < p < +\infty$. In such a way, for $1 \leqslant p \leqslant q \leqslant +\infty$ there are contractive embeddings

$$\iota^\theta_{q,p} : L^q_\theta(M) \hookrightarrow L^p_\theta(M).$$

[‖] See *e.g.* "H Kosaki: J. Funct. Anal. 56 (1984), 29-78", "M. Terp: J. Operator Theory 9 (1982), 327-360". Another equivalent approach is in "U. Hageerup: Colloques Internationaux C.N.R.S. 274 (1979), 175-184".

We also remark (*cf.* "F. Fidaleo: J. Funct. Anal. 169 (1999), 226-250") that one can equip the L^p-spaces with a canonical operator space structure arising from the canonical embedding $\iota^{\theta}_{\infty,1} : M \to M_*$, which incidentally is also completely bounded. It is possible to show that such an operator space structure coincides with the Pisier OH-structure at level of Hilbert spaces.

Concerning a suitable definition of the Fourier transform for such cases (*i.e.* when the reference measure is not the trace), we have to understand what is the best replacement of the "characters" $e^{\imath(n_1\theta_1+n_2\theta_2)}$, which for the tracial case correspond to $U_1^{n_1}U_2^{n_2}$ (where $U_1 \equiv U$, $U_2 = V$). In the cases we are managing, the choices are reported as follows.

We put $f_k(m,n) := \delta_{m,0}\delta_{n,k}$, $g_l(m,n) := \widehat{(h_f)^l}(m)\delta_{n,0}$, and define $\{u_{kl} \mid k,l \in \mathbb{Z}\} \subset \mathbb{A}_{2\alpha}$ where

$$u_{kl} := \pi_\omega\big(W(f_k)W(g_l)\big)\,, \quad k,l \in \mathbb{Z}.^{**}$$

We can embed (left one) such elements in $\mathcal{H}_\omega = \bigoplus_{\mathbb{Z}} L^2(\mathbb{T}, \mathrm{d}m)$ by defining $e^{kl} := u_{kl}\xi_\omega$. We get

$$e^{kl}_n(z) = z^l \delta_{n,k}\,, \quad k,l \in \mathbb{Z}\,.$$

It is easily seen that

$$\langle e^{rs}, e^{RS}\rangle = \delta_{r,R}\oint z^{s-S}\frac{\mathrm{d}z}{2\pi\imath z} = \delta_{r,R}\delta_{s,S}\,, \qquad (5)$$

that is the e^{kl} form and orthonormal system.

Now we pass to the definition of the Fourier transform, which will be defined by looking at the left embedding of M into M_* as follows. For $a = \pi_\omega(A) \in M$.

$$x := \omega(\,\cdot\, a) =: L_a \in M_* = L^1(M,\omega) \equiv L^1(M),$$

** Here, $h_f(z)$ replaces one of the coordinates corresponding to $g(z) = z$. Notice that the new coordinate $h_f(z)$ is in general not smooth.

we put

$$\widehat{x}(k,l) := \omega\big(u_{kl}^* a\big)\,.$$

For general $x \in M_* = L^1(M) \equiv L^1_0(M)$, the formula assumes the form

$$\widehat{x}(k,l) = x(u_{kl}^*). \tag{6}$$

Al level of $L^2_0(M) \sim \mathcal{H}_\omega$, for $x = a\xi_\omega$ we obtain $\widehat{x}(k,l) = \langle x, e^{kl}\rangle$. as expected. Therefore, for generic elements $\xi \in \mathcal{H}_\omega$ we still get

$$\widehat{x}(k,l) = \langle x, e^{kl}\rangle, \quad k,l \in \mathbb{Z}.$$

The other possibility (essentially related to the right embedding) assumes the form

$$\widehat{x}(k,l) := \omega(a u_{kl}), \quad k,l \in \mathbb{Z}. \tag{7}$$

At level of Hilbert spaces, for $\xi \in \mathcal{H}_\omega$ we get

$$\widehat{\xi}(k,l) := \langle x, E^{kl}\rangle, \quad k,l \in \mathbb{Z},$$

where the involved orthonormal bases is determined by $E^{kl} := J_\omega e^{kl}$, $k,l \in \mathbb{Z}$.

For both definitions, we first have the non-commutative generalisation of classical/tracial results (where # stands for each of the previously defined Fourier Transform, and for the corresponding left or right L^p-spaces):

(i) Riemann-Lebesgue Lemma: $(L^1(M,\omega))^{\#} \subset c_o(\mathbb{Z}^2)$ with $x^{\#} = 0 \Rightarrow x = 0$, $\|x^{\#}\|_{c_o} \leqslant \|x\|_{L^1}$.

(ii) The Fourier transform is a unitary map between $L^2_{\#}(M)$ and $\ell^2(\mathbb{Z}^2)$:

$$\|x^{\#}\|_{\ell^2} = \|x\|_{L^2}, \quad x \in \mathcal{H}_\omega.$$

Consequently, The formula is well defined for all $L^p_{\#}(M)$, $1 \leqslant p \leqslant 2$.

(iii) Hausdorfff-Young Theorem: for $p \in [1,2]$ and $q \in [2,+\infty]$ its conjugate exponent, the

Fourier transform extends to a complete contraction

$$\mathcal{F}^{\#}_{p,q} : L^p_{\#}(M) \to \ell^q(\mathbb{Z}^2)\,.$$

In addition, for $p = 2$ $\mathcal{F}^{\#}_{2,2}$ provides a complete isometry when the involving Hilbert spaces are equipped with Pisier OH-structure.

(iv) Fejer and Abel Theorems: for $1 \leqslant p \leqslant 2$ and $x \in L^p_{\#}(M)$, we have for the limit in the L^p-norm,

$$\sum_{|k|,|l|\leqslant N}\left(1-\frac{|k|}{N+1}\right)\left(1-\frac{|l|}{N+1}\right)x^{\#}(k,l)\iota^{\#}_{\infty,p}(u_{k,l})\to x.$$

$$\sum_{k,l\in\mathbb{Z}} r^{|k|+|l|}\; x^{\#}(k,l)\iota^{\#}_{\infty,p}(u_{k,l}) \to x,$$

when $N \to +\infty$, $r \uparrow 1$.

We can also define the corresponding Fourier anti-transforms

$$f\in\ell^1(\mathbb{Z}^2)\mapsto\left\{\begin{array}{ll}\check{f}= & \sum_{k,l\in\mathbb{Z}}f(k,l)u_{kl}\\ \bar{f}= & \sum_{k,l\in\mathbb{Z}}f(k,l)u^*_{kl}\end{array}\right.\in M,$$

and prove the results corresponding to the (complete) boundedness for the restrictions to such maps to ℓ^p, $p\in[1,2]$, and some other natural ones.

modular spectral triples

When α is a **UL**-number defined before, we can construct a "genuine" modular spectral triple for $\mathbb{A}_{2\alpha}$.

For the corresponding twisted Dirac operators, we put

$$\mathsf{D}_n=\begin{pmatrix}0 & \mathsf{L}_n\\ \mathsf{L}_n^* & 0\end{pmatrix}:=\begin{pmatrix}0 & \left(\imath z\frac{\mathsf{d}}{\mathsf{d}z}-a_nI\right)\\ \left(-\imath z\frac{\mathsf{d}}{\mathsf{d}z}-a_nI\right) & 0\end{pmatrix}$$

with

$$a_n := \mathsf{sign}(n) \sum_{l=1}^{|n|} \frac{1}{\Gamma_{l-\frac{1-\mathsf{sign}(n)}{2}}(f)}, \qquad n \in \mathbb{Z}.$$

For $\eta \in [0,1]$, we then define deformed Dirac Operators as

$$\begin{aligned} \mathsf{D}^{(\eta)} &= \bigoplus_{n\in\mathbb{Z}} \begin{pmatrix} 0 & M_{\delta_n^{\eta-1}} \mathsf{L}_n M_{\delta_n^{-\eta}} \\ M_{\delta_n^{-\eta}} \mathsf{L}_n^* M_{\delta_n^{\eta-1}} & 0 \end{pmatrix} \\ &= \bigoplus_{n\in\mathbb{Z}} \mathsf{D}_n^{(\eta)} = \begin{pmatrix} 0 & \Delta_\omega^{\eta-1} \mathsf{L} \Delta_\omega^{-\eta} \\ \Delta_\omega^{-\eta} \mathsf{L}^* \Delta_\omega^{\eta-1} & 0 \end{pmatrix}, \end{aligned}$$

with the associated twisted derivation

$$\mathcal{D}^{(\eta)} = \imath \begin{pmatrix} 0 & \Delta_\omega^{\eta-1} [\mathsf{L}, \cdot\,] \Delta_\omega^{-\eta} \\ \Delta_\omega^{-\eta} [\mathsf{L}^*, \cdot\,] \Delta_\omega^{\eta-1} & 0 \end{pmatrix}.$$

Here, Γ is the growth sequence previously defined, and

$$\delta_n(z) := \frac{\mathsf{d}m \circ f^n}{\mathsf{d}m}(z) = \frac{z(Df^n)(z)}{f^n(z)}, \qquad z \in \mathbb{T},\, n \in \mathbb{Z}$$

are nothing else the previous Radon-Nikodym derivatives describing the modular operator.

We can prove the following results, crucial to construct modular spectral triples:

(i) the $\mathsf{D}^{(\eta)}$ have compact resolvent, provided that $\Gamma_n(f) = o(\ln n)$;

(ii) the twisted commutators $\mathcal{D}^{(\eta)}$ admit in their domain a dense $*$-algebra $\mathbb{A}^o_{2\alpha} \subset \mathbb{A}_{2\alpha}$, stable under the entire functional calculus.

As an application of the above Fourier transforms, we can show that those "diagonalise" the deformed twisted operators $D^{(0)}$, $D^{(1)}$, and finally $D^{(1/2)}$, obtaining for $k, l, r, s \in \mathbb{Z}$,

$$\begin{aligned}
\big\langle \Delta_\omega^{-1} \mathsf{L} e^{kl}, e^{rs} \big\rangle =& (\imath l - a_k) \widehat{1/\delta_k}(s-l)\delta_{k,r},\\
\big\langle \mathsf{L} \Delta_\omega^{-1} e^{kl}, e^{rs} \big\rangle =& (\imath s - a_k) \widehat{1/\delta_k}(s-l)\delta_{k,r},\\
\big\langle \Delta_\omega^{-1/2} \mathsf{L} \Delta_\omega^{-1/2} E^{kl}, E^{rs} \big\rangle =& -\big(\imath l \delta_{l,s} + a_{-k}\widehat{\delta_k}(l-s)\big)\delta_{k,r}.
\end{aligned}$$