Haag Duality for the Sine Gordon Model ?

Klaus Fredenhagen [1]
II. Institut für Theoretische Physik, Hamburg

[1]based on joint work with Dorothea Bahns and Katarzyna Rejzner

Haag Duality

Basic insight (Wick-Wightman-Wigner):

Not all selfadjoint operators of a Hilbert space correspond to observables.

Observables restricted by

Principle of Locality (Haag)

Formalization:

Net of unital C*-algebras $\mathfrak{A}(\mathcal{O})$ for bounded spacetime regions $\mathcal{O}$.

$\mathfrak{A}(\mathbb{M})$ algebra of all local observables (inductive limit).

Important consequence: existence of inequivalent representations

Implications:

- Haag's Theorem (problems with the interaction picture)
- Superselection sectors
- Phase transitions

π irreducible representation of $\mathfrak{A}(\mathbb{M})$

Local commutativity implies

$$\pi(\mathfrak{A}(\mathcal{O})) \subset \pi(\mathfrak{A}(\mathcal{O}_1))'$$

for all spacelike separated double cones $\mathcal{O}_1$.

Haag duality:

$$\pi(\mathfrak{A}(\mathcal{O})) = \bigcap_{\mathcal{O}_1} \pi(\mathfrak{A}(\mathcal{O}_1))'$$

Haag duality of the vacuum representation crucial for the DHR theory of superselection sectors.

Haag duality holds for

- vacuum representation of the free field (Araki)
- vacuum representation of many models of 2d conformal field theory (Buchholz,Schulz-Mirbach, Longo et al.,...)
- models constructed from factorizing S-matrices (Lechner,...)

Weaker version: essential Haag duality (Roberts)

Dual net

$$\mathfrak{A}^d(\mathcal{O})) = \bigcap_{\mathcal{O}_1} \pi(\mathfrak{A}(\mathcal{O}_1))'$$

satisfies Haag duality

Implied by the Bisogano Wichmann property of the vacuum.

Infravacuum for the free massless scalar field in 2d

Free massless scalar field in 2d: Vacuum state does not exist.

Way out: Restrict to subalgebra generated by Weyl operators $W(f)$ with $\int f = 0$.

Vacuum state

$$\omega_0(W(f)) = e^{-\frac{1}{2}\int \frac{dp}{|p|}|\tilde{f}(|p|,p)|^2}$$

and induced representation π_0 on Hilbert space $\mathfrak{H}_0$.

Representation of the full algebra (Derezinski-Meissner 2006):

Choose ψ with $\int \psi = 1$. Then define a representation π on $\mathfrak{H} = \mathfrak{H}_0 \otimes \mathrm{L}^2(\mathbb{R})$ by

$$\pi(W(f)) = \pi_0(W(f - \psi \int f)) \otimes e^{iq\int f - p\int f\Delta\psi}$$

Properties of π:

- The unitary equivalence class of π does not depend on the choice of ψ.
- π is irreducible and is induced by a quasifree Hadamard state (Schubert 2013, Bahns-F-Rejzner 2017).
- π is conformally covariant with positive energy.
- As a representation of the CCR algebra of time zero fields it is locally quasiequivalent to the vacuum representation of the massive free scalar field (Bahns-F-Rejzner 2017).
- It satisfies Haag duality (Bahns-F-Rejzner, in preparation).

Thirring model

The observables of the (massless) Thirring model are generated by the fields

$$\overline{\psi}\gamma_\mu\psi = \frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial^\nu\Phi \ , \overline{\psi}\psi =: \cos\beta\Phi : \ , \overline{\psi}\gamma_5\psi = i :\sin\beta\Phi :$$

with the coupling constant $g = \frac{4\pi^2}{\beta^2} - \pi$.
They are relatively local to the observables of the free scalar field and therefore, as a proper subtheory, violate Haag duality. They satisfy instead essential Haag duality, namely the dual net (in the representation π) is local. This follows from the fact that it contains the theory of the derivatives of the free field which satisfies essential Haag duality.

Sine Gordon model

The formal power series for the time ordered exponential of $\int : \cos \beta\phi(x) : g(x) d^2x$ in perturbative AQFT converges for $\beta^2 < 4\pi$ (Bahns-Rejzner 2016) and defines a unitary operator $S(g)$ on $\mathfrak{H}$ which satisfies the Bogoliubov relation

$$S(f + g + h) = S(f + g)S(g)^{-1}S(g + h)$$

if supp$f$ does not intersect the past of supp$g$. (Bahns-F-Rejzner 2017)

Construction of the interacting Haag-Kastler net:

Functionals on the configuration space $\mathcal{C}^\infty(\mathbb{R}^2)$ of the scalar field

$$F(c,s,f)[\phi] = \int \cos\beta\phi(x)c(x) + \sin\beta\phi(x)s(x) + \phi(x)f(x)$$

c, s, f compactly supported, real and smooth densities.

$$S(c,s,f) := T\exp i :F(c,s,f):$$

unitary operators on $\mathfrak{H}$ satisfying Bogoliubov's factorization relation and the unitary Schwinger Dyson equation

$$S(c,s,f) = T\exp i(:F_\varphi(c,s,f) + \int \mathcal{L}_0(\phi+\varphi)(x) - \mathcal{L}_0(\phi)(x):)$$

with the Lagrangian $\mathcal{L}_0$ of the free theory, test functions φ and

$$F_\varphi(c,s,f)[\phi] = F(c,s,f)[\phi+\varphi]$$
$$= F((c+s)\cos\varphi,(s-c)\sin\varphi,f)[\phi] + \int \varphi f$$

hence

$$S(c,s,f) = S((c+s)\cos\varphi,(s-c)\sin\varphi,f+\Box\varphi)e^{i\int \varphi f + \frac{1}{2}\varphi\Box\varphi}$$

Interaction Lagrangian of the Sine-Gordon model:

$$\mathcal{L}_1(\phi)(x) = \cos\beta\phi(x)$$

Local observables $A \in \mathfrak{A}^{\mathrm{SG}}(\mathcal{O})$ are generated by the unitary valued functions

$$S^{\mathcal{L}_1}(c,s,f)[g] = S(g,0,0)^{-1}S(g+c,s,f)$$

with $g \equiv dtdx$ on a neighborhood of $\overline{\mathcal{O}}$ and $\mathrm{supp} c,s,f \subset \mathcal{O}$.

$S^{\mathcal{L}_1}$ satisfies the factorization equation and the unitary Schwinger-Dyson equation for the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

i.e.

$$S^{\mathcal{L}_1}(c,s,f) = S^{\mathcal{L}_1}(\gamma+c+s)\cos\varphi-\gamma,(s-c-\gamma)\sin\varphi, f+\Box\varphi)e^{i\int\varphi f+\frac{1}{2}\varphi\Box\varphi}$$

with $\gamma = dtdx$.

Representations of the interacting net can be obtained by using the validity of the time slice axiom (Chilian-F 2009).

Observables of the SG model localized at positive times are constructed from unitaries

$$S_\chi(c,s,f) = S(\chi g)^{-1} S(\chi g + c, s, f)$$

with suppc, s, f compact and contained in $t > 0$,
χ independent of x, $\chi(t) = 1$ for $t > -\epsilon$, $\chi(t) = 0$ for $t < -2\epsilon$
and supp$g$ compact and $g \equiv dtdx$ on

$$J_-(\text{supp} f \cap \text{supp} c \cup \text{supp} s) \cap \text{supp}\chi \ .$$

(Due to the Bogoliubov relation, the r.h.s. does not depend on the choice of g.)

Time-slice axiom: The observables $S_\chi(f)$ with $\mathrm{supp} f \subset (0 < t < \epsilon)$ generate the algebra $\mathfrak{A}^{\mathrm{SG}}(\mathbb{M})$ of all local observables of the Sine Gordon Model.

Representation on $\mathfrak{H}$:

$$\pi_\chi(S^{\mathcal{L}_1}(f)) := S_\chi(f) \ , \ \mathrm{supp} f \subset (0 < t < \epsilon)$$

(Regularized interaction picture)

π_χ is covariant under spatial translations but not under time translations (this avoids contradiction with Haag's Theorem).

$\Longrightarrow$ The free ($\mathfrak{A}_0$) and the interacting ($\mathfrak{A}_\chi = \pi_\chi \circ \mathfrak{A}^{\mathrm{SG}}$) net restricted to $t = 0$ satisfy

$$\mathfrak{A}_0(a + 4\epsilon, b - 4\epsilon) \subset \mathfrak{A}_\chi(a, b) \subset \mathfrak{A}_0(a - 4\epsilon, b + 4\epsilon)$$

Conclusion: Haag duality for the free field in the DM representation π implies Haag duality for the interacting field in the representation π_χ up to 4ϵ:

$$\mathfrak{A}_\chi((-\infty, a) \cup (b, \infty))' \subset \mathfrak{A}_\chi(a - 4\epsilon, b + 4\epsilon)$$

Problem: The representations π_χ for different χ are locally equivalent, but not necessarily globally equivalent.

Therefore a local version of Haag duality is needed.

Relative Haag Duality (Camassa 2006):
$M \subset \mathbb{M}$ globally hyperbolic, relatively compact sub-spacetime

$$\text{Version 1:} \quad \mathfrak{A}(\mathcal{O}) = \bigcap_{\mathcal{O}_1 \subset M \cap \mathcal{O}'} \mathfrak{A}(\mathcal{O}_1)' \cap \mathfrak{A}(M)$$

$$\text{Version 2:} \quad \mathfrak{A}(\mathcal{O})' \cap \mathfrak{A}(M) = \bigvee_{\mathcal{O}_1 \subset M \cap \mathcal{O}'} \mathfrak{A}(\mathcal{O}_1)$$

Proposition

The net $\mathfrak{A}_0$ of the free massless scalar field in the representation π satisfies both versions of relative Haag duality.

This follows from the fact that it holds true for the massive theory (Camassa 2006) and from local quasiequivalence of the DM representation.

Consequence: Also the net $\mathfrak{A}^{\mathrm{SG}}$ of the Sine Gordon model satisfies both versions if it is outer regular.

$$\bigcap_{\mathcal{O}_1 \supset\supset \mathcal{O}} \mathfrak{A}^{\mathrm{SG}}(\mathcal{O}_1) = \mathfrak{A}^{\mathrm{SG}}(\mathcal{O})$$

($\mathcal{O}_1 \supset\supset \mathcal{O}$ means that $\mathcal{O}_1$ is a neighborhood of the closure of $\mathcal{O}$)

Massive Thirring Model

The observables of the massless Thirring model

$$j^\mu = \frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi \ , \ N(\overline{\psi}\psi) =: \cos\beta\phi : \text{ and } N(\overline{\psi}\gamma_5\psi) =: \sin\beta\phi : \ .$$

are invariant under the automorphism $\phi \to \phi + \frac{2\pi}{\beta}$.

Therefore the massless Thirring model satisfies version 1 of relative Haag duality, but violates version 2.

The observables of the massive Thirring model form a subnet

$$\mathfrak{A}^{\mathrm{MT}} \subset \mathfrak{A}^{\mathrm{SG}}$$

which is mapped by π_χ into the net of the massless Thirring model such that the time zero nets coincide up to ϵ.

Therefore also $\mathfrak{A}^{MT}$ satisfies version 1 and violates version 2 of relative Haag duality.

Conclusions and Outlook

The Sine Gordon model is an example of an interacting theory satisfying the axioms of AQFT. Its relation to the massive Thirring model can be analyzed in terms of concepts related to Haag duality.

It is generated by the time ordered exponentials of the fields 1, Φ, $:\cos\beta\Phi:$ and $:\sin\beta\Phi:$ satisfying

- Bogoliubov's factorization condition and
- the unitary Schwinger Dyson equation for the Sine-Gordon action

The generated group is represented by unitary operators in the DM Hilbert space.

Open questions:

- Is the net outer regular?
- How to construct the vacuum?
- How to see the integrable structure?
- Computation of the S-Matrix and comparison with "exact" S-matrices?