The relative Drinfeld commutant of a fusion category, orbifold subfactors and α-induction

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A relative version of the quantum double

The Drinfeld center is well understood in the context of Longo-Rehren subfactors. We now study its relative version, the relative Drinfeld commutant.

It is also known the Drinfeld center automatically involves orbifold subfactors for fusion subcategories of modular tensor categories. We study its relative version.

Outline of the talk:

1. The Drinfeld center and subfactors
2. The relative Drinfeld commutant and subfactors
3. α-induction and the Drinfeld center
4. α-induction and the relative Drinfeld commutant
5. The relative Drinfeld commutant and orbifold subfactors

The Drinfeld center and subfactors

Let $\mathcal{C}$ be a unitary fusion category. We may and do assume that it is realized as a full subcategory of the category of finite index endomorphisms of a type III factor M. Let $\{\lambda_i\}$ be a set of representatives of irreducible sectors in $\mathcal{C}$.

We then have the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$ so that we have $[\bar{\iota}\iota] = \bigoplus_i [\lambda_i \otimes \lambda_i^{\mathrm{opp}}]$, where ι is the inclusion map.

Then it is known that the fusion category of R-R sectors arising from this subfactor has a nondegenerate braiding. (So it is a modular tensor category.) The passage from $\mathcal{C}$ to this modular tensor category is known as the Drinfeld center construction.

A half-braiding and the Drinfeld center

Let $\mathcal{C}$ be a fusion category as before and consider a (possibly reducible) object σ of $\mathcal{C}$. If a set of intertwiners $\{\mathcal{E}_\sigma(\lambda_i)\}_i$ with $\mathcal{E}_\sigma(\lambda_i) \in \mathrm{Hom}(\sigma\lambda_i, \lambda_i\sigma)$ satisfies a certain type of a braiding-fusion equation with respect to λ_i, we say it is a half-braiding.

Pairs of σ and its half-braiding give a fusion category and it turns out that this coincides with the fusion category of the R-R sectors arising from the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$. This give a concrete description and a conceptual understanding of the Drinfeld center. This also leads to why we have a modular tensor category for the Drinfeld center.

The Drinfeld center and the tube algebra

Let $\mathcal{C}$ be a fusion category as before. We define Ocneanu's tube algebra $\mathbf{Tube}(\mathcal{C})$ by setting it to be $\bigoplus_{i,j,k} \mathbf{Hom}(\lambda_i\lambda_j, \lambda_j\lambda_k)$ as a linear space and putting a structure of a finite dimensional C^*-algebra on it.

Then it turns out that the minimal central projections of this algebra correspond to the irreducible R-R sectors of the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$.

It is generally hard to compute the R-R sectors, but this method based on the tube algebra is effective in actual computations. For example, Izumi computed the Drinfeld center of a fusion category arising from the Haagerup subfactor.

The Drinfeld center and topological quantum field theory

Let $\mathcal{C}$ be a fusion category as before. It produces Ocneanu's Turaev-Viro type topological quantum field theory (TQFT) of dimension 3 using triangulations. The minimal central projections of $\mathbf{Tube}(\mathcal{C})$ give a natural basis of the (finite dimensional) Hilbert space $H_{S^1\times S^1}$ of the torus in this TQFT.

The action of $SL(2,\mathbb{Z})$ on this Hilbert space $H_{S^1\times S^1}$ is the same as the one arising from the braiding on the R-R sectors.

Using the braiding on the R-R sectors, we can prove that this TQFT is equal to the Reshetikhin-Turaev type TQFT arising from the Drinfeld center of $\mathcal{C}$.
(K-Sato-Wakui)

A half-braiding and the relative Drinfeld commutant

Let $\mathcal{D}$ be a fusion category and $\mathcal{C}$ its full subcategory. Let $\{\lambda_i\}$ be a set of representatives of irreducible sectors in $\mathcal{C}$ as before. Consider a (possibly reducible) object σ of $\mathcal{D}$ with intertwiners $\{\mathcal{E}_\sigma(\lambda_i)\}_i$ satisfying the same conditions as before. (Now the braiding-fusion equation is with respect to λ_i.)

This defines a relative half-braiding of $\mathcal{D}$ with respect to $\mathcal{C}$. In this way, we obtain the relative Drinfeld commutant $\mathcal{C}' \cap \mathcal{D}$ of $\mathcal{C}$ in $\mathcal{D}$. It turns out this has a natural structure of a fusion category. This also has a description using the Longo-Rehren subfactor arising from $\mathcal{C}$, using also sectors from $\mathcal{D}$.

The relative Drinfeld commutant and the relative tube algebra

Let $\mathcal{C} \subset \mathcal{D}$ be fusion categories as before. We define a relative version of the tube algebra $\mathbf{Tube}(\mathcal{C}, \mathcal{D})$ by setting it to be $\bigoplus_{i,j,k} \mathbf{Hom}(\mu_i \lambda_j, \lambda_j \mu_k)$, where $\{\mu_i\}$ is a set of representatives of irreducible objects of $\mathcal{D}$, as a linear space and putting a structure of a finite dimensional C^*-algebra on it in the same way as before.

We can identify the minimal central projections in the relative tube algebra $\mathbf{Tube}(\mathcal{C}, \mathcal{D})$ with representatives of the irreducible objects of the relative Drinfeld commutant $\mathcal{C}' \cap \mathcal{D}$. They again naturally correspond to generalized R-R sectors arising from the Longo-Rehren subfactor.

α-induction

Let $N \subset M$ be a finite index subfactor and suppose its dual canonical endomorphism is an object of a modular tensor category $\mathcal{C}$ of endomorphisms of N. For an object λ in $\mathcal{C}$, we can define an endomorphism $\alpha_\lambda^\pm$ of M using the braiding of $\mathcal{C}$. (Longo-Rehren, Xu, Ocneanu, Böckenhauer-Evans-K)

A typical situation this arises is an inclusion of completely rational local conformal nets $\{A(I) \subset B(I)\}_{I \subset S^1}$. Then the modular tensor category $\mathcal{C}$ is the one of the DHR sectors of $\{A(I)\}$. The $\alpha^\pm$-induction produces fusion categories $\mathcal{D}^\pm$ and their intersection $\mathcal{D}^0$ gives the category of the DHR sectors of $\{B(I)\}$. The fusion categories $\mathcal{D}^\pm$ generate a larger fusion category $\mathcal{D}$.

The Drinfeld center and α-induction

Let $\mathcal{C}$ be the modular tensor category as above and $\mathcal{D}^0, \mathcal{D}^\pm, \mathcal{D}$ be as above arising from the α-induction. We have computed the Drinfeld centers of $\mathcal{D}^0, \mathcal{D}^\pm, \mathcal{D}$ as follows. (Böckenhauer-Evans-K)

The Drinfeld center of $\mathcal{D}^0$ is trivially $\mathcal{D}^0 \boxtimes \mathcal{D}^{0,\mathrm{opp}}$, because $\mathcal{D}^0$ is a modular tensor category. The Drinfeld center of $\mathcal{D}^\pm$ is $\mathcal{C}^0 \boxtimes \mathcal{D}^{0,\mathrm{opp}}$, which is the main non-trivial result. The Drinfeld center of $\mathcal{D}$ is easily seen to be $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{opp}}$, because $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent and $\mathcal{C}$ is a modular tensor category. The second result is called the boundary-bulk duality in another context of α-induction for anyon condensation.

The relative Drinfeld commutant and α-induction

Let $\mathcal{C}$ be the modular tensor category as above and $\mathcal{D}^0, \mathcal{D}^\pm, \mathcal{D}$ be as above arising from the α-induction applied to a subfactor. Then we compute the relative Drinfeld commutants for $\mathcal{D}^0 \subset \mathcal{D}^\pm \subset \mathcal{D}$ as follows.

The relative Drinfeld commutant $(\mathcal{D}^+)' \cap \mathcal{D}$ is given by $\mathcal{C} \boxtimes \mathcal{D}^-$. The relative Drinfeld commutant $(\mathcal{D}^0)' \cap \mathcal{D}^+$ is given by $\mathcal{D}^+ \boxtimes \mathcal{D}^0$. The relative Drinfeld commutant $(\mathcal{D}^0)' \cap \mathcal{D}$ is given by $\mathcal{D}^+ \boxtimes \mathcal{D}^-$. These are identifications as fusion categories. The last one is the most subtle.

Note that $\mathcal{C}' \cap \mathcal{C}$ is a full fusion subcategory of $\mathcal{C}' \cap \mathcal{D}$, which is of course compatible with the above.

The relative Drinfeld commutant and α-induction

The methods to get these results are as follows.

We have a relative braiding between $\mathcal{D}^+$ and $\mathcal{D}^-$. (Ocneanu, Böckenhauer-Evans) On the one hand, we obtain appropriate half-braidings explicitly using this. On the other hand, we have an estimate $\dim(\mathcal{C}' \cap \mathcal{D}) = \dim(\mathcal{C}) \dim(\mathcal{D})$, where $\dim(\mathcal{C}) = \sum_i \dim(\lambda_i)^2$ is the global index of a fusion category. These together show that our relative braidings exhaust all possible ones.

For the last identification, the above argument gives identification on the object level. In order to have identification on the fusion category level, we need to compare the two sets of the $6j$-symbols.

The Drinfeld centers and orbifold subfactors

Consider the modular tensor category corresponding to the WZW-model $SU(2)_{2n}$. The irreducible sectors are numbered as $0, 1, 2, \ldots, 2n$. The irreducible sectors of the Drinfeld center are naturally labeled with (i, j) with $i, j = 0, 1, 2, \ldots, 2n$.

We next consider the Drinfeld center of the fusion category generated by $0, 2, 4, \ldots, 2n$. Then they are labeled with (i, j) with $i, j = 0, 1, 2, \ldots, 2n$ and $i + j \in 2\mathbb{Z}$ together with the identification $(i, j) = (2n - i, 2n - j)$ and splitting of (n, n) into two irreducible sectors $(n, n)_+$ and $(n, n)_-$ of the same dimension. (Ocneanu, Evans-K)

This phenomenon is known as an orbifold subfactor.

The relative Drinfeld commutants for A_{2n+1}

Let $\mathcal{D}$ be the fusion category corresponding to the WZW-model $SU(2)_{2n}$. We label the irreducible sectors as $0, 1, 2, \ldots, 2n$. Let $\mathcal{C}$ be its fusion subcategory generated by $0, 2, 4, \ldots, 2n$. We consider the relative Drinfeld commutant $\mathcal{C}' \cap \mathcal{D}$.

We show that the irreducible sectors of $\mathcal{C}' \cap \mathcal{D}$ are labeled with (i, j) with $i, j = 0, 1, 2, \ldots, 2n$ together with the identification $(i, j) = (2n - i, 2n - j)$ and splitting of (n, n) into two irreducible sectors $(n, n)_+$ and $(n, n)_-$ of the same dimension. We already have an orbifold subfactor here.

When we further look at the Drinfeld center, we see only half of the sectors.

The relative Drinfeld commutants for D_{2n}

We now consider the fusion category $\mathcal{D}$ whose irreducible sectors correspond to the vertices of the Dynkin diagram D_{2n}. This is the category $\mathcal{D}^{\pm}$ arising from the α-induction for the simple current extension of order 2 of the WZW-model $SU(2)_{4n-4}$. Let $\mathcal{C}$ be the fusion subcategory of $\mathcal{D}$ generated by the even vertices of the bipartite graph D_{2n}. We compute the relative Drinfeld commutant $\mathcal{C}' \cap \mathcal{D}$.

It is equal to $(\mathcal{D}^0)' \cap \mathcal{D}^+$ in the notation for the relative Drinfeld commutant arising from the α-induction, so it is identified with $\mathcal{D}^+ \boxtimes \mathcal{D}^0$. This is a direct product $\mathcal{C} \boxtimes \mathcal{D}$. We can regard also this as an orbifold.

The relative Drinfeld commutant for E_6

We now consider the fusion category $\mathcal{D}$ whose irreducible sectors correspond to the vertices of the Dynkin diagram E_6. This is the category $\mathcal{D}^{\pm}$ arising from the α-induction for the conformal embedding $SU(2)_{10} \subset SO(5)_1$. Let $\mathcal{C}$ be the fusion subcategory of $\mathcal{D}$ generated by the even vertices of the bipartite graph E_6. We compute the relative Drinfeld commutant $\mathcal{C}' \cap \mathcal{D}$.

Now the irreducible sectors of $\mathcal{C}' \cap \mathcal{D}$ are labeled with (i, j) with $i = 0, 1, 2$ and $j = 0, 1, 2, \ldots, 10$ together with identification $(i, j) = (2 - i, 10 - j)$ and splitting of $(1, 5)$ into two irreducible sectors $(1, 5)_+$ and $(1, 5)_-$ of the same dimension.

The relative Drinfeld commutant for E_8

We now consider the fusion category $\mathcal{D}$ whose irreducible sectors correspond to the vertices of E_8. This is the category $\mathcal{D}^{\pm}$ arising from the α-induction for the conformal embedding $SU(2)_{28} \subset (G_2)_1$. Let $\mathcal{C}$ be the fusion subcategory of $\mathcal{D}$ generated by the even vertices of the bipartite graph E_8. We compute the relative Drinfeld commutant $\mathcal{C}' \cap \mathcal{D}$.

Now the irreducible sectors of $\mathcal{C}' \cap \mathcal{D}$ are labeled with (i, j) with $i = 0, 2$ and $j = 0, 1, 2, \ldots, 28$ together with identification $(i, j) = (i, 28 - j)$ and splittings of $(0, 14)$ into two irreducible sectors $(0, 14)_+$ and $(0, 14)_-$ of the same dimension and $(2, 14)$ into two irreducible sectors $(2, 14)_+$ and $(2, 14)_-$.

Future problem and topological quantum field theory

Compare the descriptions of the Drinfeld center and the relative Drinfeld commutant in the context of (relative) half-braiding and the (relative) tube algebra. What is missing is a "relative" counterpart of the description $H_{S^1 \times S^1}$ using topological quantum field theory.

We expect some kind of a "relative" version of a topological quantum field theory based on triangulation in dimension 3 for an inclusion of fusion categories $\mathcal{C} \subset \mathcal{D}$. Such a relative version would give some meaning to the center of the relative tube algebra as some Hilbert space attached to the relative TQFT.

Such a description would provide a deeper understanding of the relative Drinfeld commutant in terms of TQFT.