Modular theory and entropy bounds

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Partly based on joint work with Feng Xu

Thermal equilibrium states

Thermodynamics concerns heat and temperature and their relation to energy and work. A primary role is played by the equilibrium distribution.

Gibbs states

Finite quantum system: $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad} e^{itH}$. Equilibrium state φ at inverse temperature β is given by the Gibbs property

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H} X)}{\mathrm{Tr}(e^{-\beta H})}$$

What are the equilibrium states at infinite volume where there is no trace, no inner Hamiltonian?

KMS states (HHW, Baton Rouge conference 1967)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ $F_{XY} \in A(S_\beta)$ s.t.

(a) $F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$

(b) $F_{XY}(t + i\beta) = \varphi\big(\tau_t(Y)X\big)$

where $A(S_\beta)$ is the algebra of functions analytic in the strip $S_\beta = \{0 < \Im z < \beta\}$, bounded and continuous on the closure $\bar{S}_\beta$.

KMS states have been so far the central objects in equilibrium Quantum Statistical Mechanics, for example in the analysis of phase transition.

Tomita-Takesaki modular theory

Let $\mathcal{M}$ be a von Neumann algebra and φ a normal faithful state on $\mathcal{M}$. The Tomita-Takesaki theorem gives a *canonical evolution*:

$$t \in \mathbb{R} \mapsto \sigma_t^\varphi \in \mathrm{Aut}(\mathcal{M})$$

By a remarkable historical accident, Tomita announced the theorem at the 1967 Baton Rouge conference. Soon later Takesaki completed the theory and charcterised the modular group by the KMS condition.

• σ^φ is a purely noncommutative object
• σ^φ does not depend on φ up to inner automorphisms by Connes' Radon-Nikodym theorem
• σ^φ is characterised by the KMS condition at inverse temperature $\beta = -1$ with respect to the state φ.
• σ^φ is intrinsic modulo scaling, the inverse temperature given by β the rescaled group $t \mapsto \sigma^\varphi_{-t/\beta}$ is physical

Bekenstein-Hawking entropy formula

If A is the surface area of a black hole (area of the event horizon), then the black hole entropy is given by

$$S_{BH} = A/4$$

(up to Boltzmann's constant).

For a spherically symmetric (Schwarzschild) black hole with mass M, the horizon's radius is $R = 2GM$, and its area is naturally given by $4\pi R^2$ (with $G = 1$)

Bekenstein's bound

For decades, modular theory has played a central role in the operator algebraic approach to QFT, very recently many physical papers in other QFT settings are dealing with the modular group, although often in a naive and heuristic (yet powerful) way!

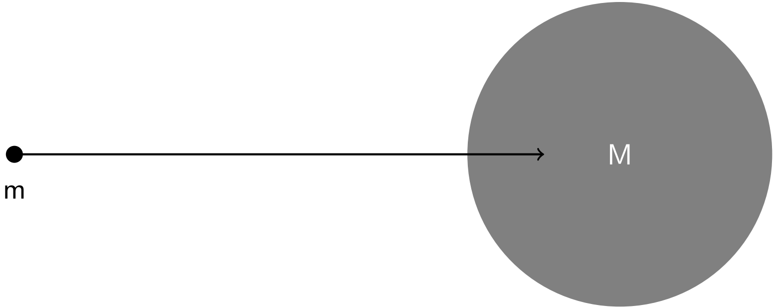
We will discuss the Bekenstein bound, a universal limit on the entropy that can be contained in a physical system with given size and given total energy

If R is the radius of a sphere that can enclose our system, while E is its total energy including any rest masses, then its entropy S is bounded by

$$S \leq \lambda R E$$

(with $\hbar = 1$, $c = 1$). The constant λ is often proposed $\lambda = 2\pi$.

Sketch of the original idea



Inferring the Bekenstein bound

Drop a small object of mass m with entropy S into a Schwarzschild black hole of mass M much larger than m.

The black hole's mass will grow to $M + m$. Since initially the hole's entropy was $S_{BH} = 4\pi M^2$, it will have grown by $8\pi Mm$ plus a negligible term of order m^2. By the generalized second law the sum of ordinary entropy outside black hole and total black hole entropy does not decrease. Therefore

$$-S + 8\pi Mm \geq 0$$

The initial Schwarzschild radius is $R = 2M$, so the above inequality can be written as

$$S \leq 4\pi Rm$$

....

Information point of view

On a dual point of view, the Bekenstein bound gives maximum amount of information needed to describe a given physical system down to the quantum level

Cloning a human brain (calculation in Wikipedia)

An average human brain has a mass of 1.5 kg , volume 1260 cm^3 and is approximately a sphere with 6.7 cm radius

The information contained is $\approx 2.6 \times 10^{42}$ bits and represents the maximum information needed to recreate an average human brain down to the quantum level.

This means that the number of states of the human brain must be less than $\approx 10^{7.8\times10^{41}}$ (with mass energy equivalence)

Casini's argument

Subtract to the bare entropy of the local state the entropy corresponding to the vacuum fluctuations. V bounded region.

The restriction ρ_V of a global state ρ to von Neumann algebra $\mathcal{A}(V)$ has formally entropy given by

$$S(\rho_V) = -\operatorname{Tr}(\rho_V \log \rho_V)\ ,$$

known to be infinite. So subtract the vacuum state entropy

$$S_V = S(\rho_V) - S(\rho_V^0)$$

with ρ_V^0 the density matrix of the restriction of the vacuum state.

Similarly, K Hamiltonian for V, consider

$$K_V = \operatorname{Tr}(\rho_V K) - \operatorname{Tr}(\rho_V^0 K)$$

Bekenstein bound is now $S_V \leq K_V$ which is equivalent to the positivity of the relative entropy

$$S(\rho_V|\rho_V^0) \equiv \operatorname{Tr}\left(\rho_V(\log \rho_V - \log \rho_V^0)\right) \geq 0\,,$$

Araki's relative entropy and Connes' spatial derivative

An infinite quantum system, possibly with a classical part too, is described by a von Neumann algebra $\mathcal{M}$; the von Neumann entropy of a normal state φ on $\mathcal{M}$ makes no sense in this case, unless $\mathcal{M}$ is of type I; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are the vector representatives of φ, ψ in the natural cone $L^2_+(\mathcal{M})$ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

Relative entropy is one of the key concepts. We take the view that relative entropy is a primary concept and all entropy notions are derived concepts

Relative entropy is more intrinsic by Connes' spatial detivative

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta(\varphi|\psi')\eta)$$

ψ' state on $\mathcal{M}'$

Comment

Now, $\mathcal{A}(V)$ is a factor of type III so no trace Tr and no density matrix ρ is definable. Yet, modular theory and Araki's relative entropy $S(\varphi|\psi)$ are definable in general.

As said, relative entropy is a primary concept, indeed von Neumann entropy is

$$S(\varphi) = \sup_{(\varphi_i)} \sum_i S(\varphi|\varphi_i)$$

sup on all finite families of positive linear functionals φ_i of $\mathcal{M}$ with $\sum_i \varphi_i = \varphi$. Clearly $S(\varphi)$ cannot be finite unless $\mathcal{M}$ is of type I.

Here we are going to rely on the positivity of the incremental free energy, or conditional entropy, which can be obtained in two ways: by the monotonicity of the relative entropy in relations to Connes-Størmer's entropy, or by linking it to Jones' index.

Analog of the Kac-Wakimoto formula (L. '97)

The root of our work relies in this formula for the incremental free energy of a black hole (cf. the Kac-Wakimoto formula, Kawahigashi, Xu, L.)

H_ρ be the Hamiltonian for a uniformly accelerated observer in the Minkowski spacetime with acceleration $a > 0$ in representation ρ (localised in the wedge for H_ρ)

$$(\Omega, e^{-tH_\rho}\Omega)\big|_{t=\beta} = d(\rho)$$

with Ω the vacuum vector and $\beta = \frac{2\pi}{a}$ the inverse Hawking-Unruh temperature. $d(\rho)^2$ is Jones' index.

The left hand side is a generalised partition formula, so $\log d(\rho)$ has an entropy meaning in accordance with Pimsner-Popa work.

The proof of formula is based on a tensor categorical and spacetime symmetries analysis. Here we generalise this formula without any reference to a given KMS physical flow

CP maps, quantum channels

$\mathcal{N}, \mathcal{M}$ vN algebras. A linerar map $\alpha : \mathcal{N} \to \mathcal{M}$ is completely positive is

$$\alpha \otimes \mathrm{id}_n : \mathcal{N} \otimes \mathrm{Mat}_n(\mathbb{C}) \to \mathcal{M} \otimes \mathrm{Mat}_n(\mathbb{C})$$

is positive $\forall n$. We always assume α to be unital and normal.
ω faithful normal state of $\mathcal{M}$ and $\alpha : \mathcal{N} \to \mathcal{M}$ CP map as above.
Set

$$\mathrm{H}_\omega(\alpha) \equiv \sup_{(\omega_i)} \sum_i S(\omega|\omega_i) - S(\omega \cdot \alpha|\omega_i \cdot \alpha)$$

supremum over all ω_i with $\sum_i \omega_i = \omega$.
The conditional entropy $H(\alpha)$ of α is defined by

$$\mathrm{H}(\alpha) = \inf_\omega \mathrm{H}_\omega(\alpha)$$

infimum over all "full" states ω for α. Clearly $\mathrm{H}(\alpha) \geq 0$ because $\mathrm{H}_\omega(\alpha) \geq 0$ by the monotonicity of the relative entropy.
α is a quantum channel if its conditional entropy $\mathrm{H}(\alpha)$ is finite.

Generalisation of Stinespring dilation

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, completely positive unital map between the vN algebras $\mathcal{N}$, $\mathcal{M}$. A pair (ρ, v) $\rho : \mathcal{N} \to \mathcal{M}$ a homomorphism, $v \in \mathcal{M}$ an isometry s.t.

$$\alpha(n) = v^* \rho(n) v \ , \quad n \in \mathcal{N} \ .$$

(ρ, v) is *minimal* if the left support of $\rho(\mathcal{N})v\mathcal{H}$ is qual to 1.

Thm Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, CP unital map with $\mathcal{N}$, $\mathcal{M}$ properly infinite. There exists a minimal dilation pair (ρ, v) for α. If (ρ_1, v_1) is another minimal pair, $\exists!$ unitary $u \in \mathcal{M}$ such that

$$u\rho(n) = \rho_1(n)u \, , \quad v_1 = uv, \qquad n \in \mathcal{N}$$

We have

$$H(\alpha) = \log Ind(\alpha) \qquad \text{(minimal index)}$$

Bimodules and CP maps

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a completely positive, normal, unital map and ω a faithful normal state of $\mathcal{M}$. $\exists!$ $\mathcal{N} - \mathcal{M}$ bimodule $\mathcal{H}_\alpha$, with a cyclic vector $\xi_\alpha \in \mathcal{H}$ and left and right actions ℓ_α and r_α, such that

$$(\xi_\alpha, \ell_\alpha(n)\xi_\alpha) = \omega_{\rm out}(n)\,, \quad (\xi_\alpha, r_\alpha(m)\xi_\alpha) = \omega_{\rm in}(m)\,,$$

with $\omega_{\rm in} \equiv \omega$, $\omega_{\rm out} \equiv \omega_{\rm in} \cdot \alpha$.

Converse is true, any $\mathcal{N} - \mathcal{M}$ bimodule $\mathcal{H}$ with a cyclic vector $\xi \in \mathcal{H}$, with $\omega = (\xi, r(\cdot)\xi)$ faithful state of $\mathcal{M}$ comes from a unique completely positive, unital, normal map $\alpha : \mathcal{N} \to \mathcal{M}$

Connes relative tensor product

Properly infinite case. An $\mathcal{H}$ an $\mathcal{N} - \mathcal{M}$ bimodule is of the form

$$\mathcal{H} \simeq {}_{\rho}L^2(\mathcal{M}) \simeq L^2(\mathcal{N})_{\rho'}$$

$\rho : \mathcal{N} \to \mathcal{M}$, $\rho' : \mathcal{M} \to \mathcal{N}$ homomorphisms, L^2 the identity bimodule.

$\mathcal{K} \simeq L^2(\mathcal{M})_{\theta}$ an $\mathcal{M} - \mathcal{L}$ bimodule. Choose φ faithful normal state of $\mathcal{M}$. The composition of $\mathcal{H}$ and $\mathcal{K}$ is given by

$$\mathcal{H} \otimes_{\varphi} \mathcal{K} = {}_{\rho}L^2(\mathcal{M}) \otimes L^2(\mathcal{M})_{\theta} \equiv {}_{\rho}L^2(\mathcal{M})_{\theta}$$

Note: $L^2(\mathcal{M})$ is unique up to unitary equivalence, φ gives us a specific element in the unitary equivalence class (GNS).

Modular Hamiltonian

$\mathcal{H}$ Connes' $\mathcal{N}-\mathcal{M}$-bimodule with finite Jones' index $\text{Ind}(\mathcal{H})$

Given faithful, normal, positive linear functional φ, ψ on $\mathcal{N}$ and $\mathcal{M}$, we define the modular operator $\Delta_{\mathcal{H}}(\varphi|\psi)$ of $\mathcal{H}$ with respect to φ, ψ as

$$\Delta_{\mathcal{H}}(\varphi|\psi) \equiv d(\varphi \cdot \ell^{-1})/d(\psi \cdot r^{-1} \cdot \varepsilon) \ ,$$

Connes' spatial derivative for the pair $r(\mathcal{M})', r(\mathcal{M})$ w.r.t. the states $\varphi \cdot \ell^{-1}$ and $\psi \cdot r^{-1} \cdot \varepsilon$ and $\varepsilon : \ell(\mathcal{N})' \to r(\mathcal{M})$ is the minimal conditional expectation

$\log \Delta_{\mathcal{H}}(\varphi|\psi)$ is called the modular Hamiltonian of the bimodule $\mathcal{H}$, or of the quantum channel α if $\mathcal{H}$ is associated with α.

Properties of the modular Hamiltonian

If $\mathcal{N}$, $\mathcal{M}$ factor

$$\Delta^{it}_{\mathcal{H}}(\varphi|\psi)\ell(n)\Delta^{-it}_{\mathcal{H}}(\varphi|\psi) = \ell\big(\sigma^{\varphi}_t(n)\big)$$

$$\Delta^{it}_{\mathcal{H}}(\varphi|\psi)r(m)\Delta^{-it}_{\mathcal{H}}(\varphi|\psi) = r\big(\sigma^{\psi}_t(m)\big)$$

(implements the dynamics)

$$\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2) \otimes \Delta^{it}_{\mathcal{K}}(\varphi_2|\varphi_3) = \Delta^{it}_{\mathcal{H}\otimes\mathcal{K}}(\varphi_1|\varphi_3)$$

(additivity of the energy)

$$\Delta^{it}_{\bar{\mathcal{H}}}(\varphi_2|\varphi_1) = \mathrm{Ind}(\mathcal{H})^{-it}\,\overline{\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2)}$$

If $T : \mathcal{H} \to \mathcal{H}'$ is a bimodule intertwiner, then

$$T\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2) = (d_{\mathcal{H}'}/d_{\mathcal{H}})^{it}\Delta^{it}_{\mathcal{H}'}(\varphi_1|\varphi_2)T$$

Connes's bimodule tensor product w.r.t. φ_2.

The physical unitary evolution

$\mathcal{M}_k$ vN algebras with finite-dim centers and φ_k faithful normal states of $\mathcal{M}_k$. $\mathcal{H}, \mathcal{H}'$ finite index $\mathcal{M}_1 - \mathcal{M}_2$ bimodules and $\mathcal{K}$ a finite index $\mathcal{M}_2 - \mathcal{M}_3$ bimodule, $U^{\mathcal{H}}(\varphi_1|\varphi_2)$ is one par. group on $\mathcal{H}$, natural on $\mathcal{H}, \varphi_1, \varphi_2$:
$U^{\mathcal{H}}(\varphi_1|\varphi_2)$ implements the modular dynamics:

$$U_t^{\mathcal{H}}(\varphi_1|\varphi_2)\ell_{\mathcal{H}}(m_1)U_{-t}^{\mathcal{H}}(\varphi_1|\varphi_2) = \ell_{\mathcal{H}}\big(\sigma_t^{\varphi_1}(m_1)\big), \quad m_1 \in \mathcal{M}_1\,,$$
$$U_t^{\mathcal{H}}(\varphi_1|\varphi_2)r_{\mathcal{H}}(m_2)U_{-t}^{\mathcal{H}}(\varphi_1|\varphi_2) = r_{\mathcal{H}}\big(\sigma_t^{\varphi_2}(m_2)\big), \quad m_2 \in \mathcal{M}_2\,,$$

and the following hold (with $\mathcal{H} \otimes \mathcal{K} \equiv \mathcal{H} \otimes_{\varphi_2} \mathcal{K}$):

$U_t^{\mathcal{H}\otimes\mathcal{K}}(\varphi_1|\varphi_3) = U_t^{\mathcal{H}}(\varphi_1|\varphi_2) \otimes U_t^{\mathcal{K}}(\varphi_2|\varphi_3)$ (additiv. of energy);

$U_t^{\bar{\mathcal{H}}}(\varphi_2|\varphi_1) = \overline{U_t^{\mathcal{H}}(\varphi_1|\varphi_2)}$ (conjugation symmetry);

$TU_t^{\mathcal{H}}(\varphi_1|\varphi_2) = U_t^{\mathcal{H}'}(\varphi_1|\varphi_2)T$ (functoriality).

($T : \mathcal{H} \to \mathcal{H}'$ bimodule intertwiner)

Matrix dimension (with L. Giorgetti)

$\mathcal{N}, \mathcal{M}$ vN algebras, $\mathcal{H}$ finite index bimodule. Assume $\dim Z(\mathcal{N}' \cap \mathcal{M}) < \infty$ (equiv. $\dim Z(\mathcal{N}) < \infty$ equiv. $\dim Z(\mathcal{M}) < \infty$) on $\mathcal{H}$.

The matrix dimimension $D_{\mathcal{H}}$ is the matrix

$$D_{ij} \equiv d_{\mathcal{H}_{ij}} \qquad \mathcal{H}_{ij} = p_i \mathcal{H} q_j$$

p_i, q_j atoms of $Z(\mathcal{N})$, $Z(\mathcal{M})$, d^2 = minimal index. Then

$D_{\mathcal{H}\otimes\mathcal{K}} = D_{\mathcal{H}} \otimes D_{\mathcal{K}}$ (multiplicativity);

$D_{\mathcal{H}\oplus\mathcal{K}} = D_{\mathcal{H}} \oplus D_{\mathcal{K}}$ (additivity);

$D_{\bar{\mathcal{H}}} = D_{\mathcal{H}}$ (conjugation symmetry).

moreover $d_{\mathcal{H}} = ||D_{\mathcal{H}}||$ but $d_{\mathcal{H}}$ not multiplicative!

Physical Hamiltonian

$$K(\varphi_1|\varphi_2) = -\log \Delta_{\mathcal{H}}(\varphi_1|\varphi_2) - \log D$$

is the physical Hamiltonian (at inverse temperature 1). Here D is the operator $D_{ij} = d_{ij}$ on the factorial component $\mathcal{H}_{ij}$ of $\mathcal{H}$.

The physical Hamiltonian at inverse temperature $\beta > 0$ is given by

$$-\beta^{-1} \log \Delta - \beta^{-1} \log D$$

From the modular Hamiltonian to the physical Hamiltonian:

$$-\log \Delta \xrightarrow{\text{shifting}} -\log \Delta - \log d(\alpha) \xrightarrow{\text{scaling}} \beta^{-1}\big(-\log \Delta - \log D\big)$$

The shifting is **intrinsic**, the scaling is to be determined by the context!

Modular and Physical Hamiltonians for a quantum channel

We now are going to compare two states of a physical system, $\omega_{\rm in}$ is a suitable reference state, e.g. the vacuum in QFT, and $\omega_{\rm out}$ is a state that can be reached from $\omega_{\rm in}$ by some physically realisable process (quantum channel).

$\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel (normal, unital CP map with finite entropy) and $\omega_{\rm in}$ a faithful normal state of $\mathcal{M}$. $\omega_{\rm out} = \omega_{\rm in} \cdot \alpha$

$$\log \Delta_\alpha \equiv \log \Delta_{\mathcal{H}_\alpha}$$

$$K_\alpha = \beta^{-1} K_{\mathcal{H}_\alpha} = \beta^{-1}\big(-\log \Delta_{\mathcal{H}_\alpha} - \log D_{\mathcal{H}_\alpha}\big)$$

(physical Hamiltonian at inverse temperature β)

K_α may be considered as a local Hamiltonian associated with α and the state transfer with input state $\omega_{\rm in}$.

Thermodynamical quantities

The entropy $S \equiv S_{\alpha,\omega_{\rm in}}$ of α is

$$S = -(\hat{\xi}, \log \Delta' \hat{\xi})$$

where $\hat{\xi}$ is a vector representative of the state $\omega_{\rm in} \cdot r^{-1} \cdot \varepsilon'$ in $\mathcal{H}_\alpha$.

S is thus Araki's relative entropy $S \equiv S(\omega_\xi|_{\ell(\mathcal{N})}|\omega_\xi \cdot \varepsilon')$ w.r.t. the states $\omega_\xi|_{\ell(\mathcal{N})}$ of $\ell(\mathcal{N})$ and $\omega_\xi \cdot \varepsilon'$ of $\ell(\mathcal{N})'$, with $\xi \equiv \xi_\alpha$. Thus $S \geq 0$.

The quantity

$$E = (\hat{\xi}, K\hat{\xi})$$

is the relative energy w.r.t. the states $\omega_{\rm in}$ and $\omega_{\rm out}$.

The free energy F is now defined by the relative partition function

$$F = -\beta^{-1} \log(\hat{\xi}, e^{-\beta K} \hat{\xi})$$

F satisfies the thermodynamical relation

$$F = E - TS$$

A form of Bekenstein bound

As $F = \frac{1}{2}\beta^{-1}\mathrm{H}(\alpha)$, we have

$$F \geq 0 \quad \text{(positivity of the free energy)}$$

because

$$\mathrm{H}(\alpha) \geq 0 \quad \text{(monotonicity of the entropy)}$$

So the above thermodynamical relation

$$F = E - \beta^{-1}S$$

entails the following general version of the Bekenstein bound

$$S \leq \beta E\,.$$

Fixing the temperature in QFT

O a spacetime region s.t. the modular group σ_t^ω of the local von Neumann algebra $\mathcal{A}(O)$ associated with vacuum ω has a geometric meaning. So there is a geometric flow $\theta_s : O \to O$ and a re-parametrisation of σ_t^ω that acts covariantly w.r.t θ.

Well known illustration concerns a Rindler wedge region O of the Minkowski spacetime. The vacuum modular group Δ^{-it} of $\mathcal{A}(O)$ w.r.t. the vacuum state is here equal to $U(\beta t)$, with U the boost unitary one-parameter group acceleration a and β the Unruh inverse temperature. Re-parametrisation of the geometric flow is the rescaling by inverse temperature $\beta = 2\pi/a$.

In general, the re-parametrisation is not just a scaling. Connes and Rovelli suggest to define locally the inverse temperature by

$$\beta_s = \left\| \frac{d\theta_s}{ds} \right\|$$

the Minkowskian length of the tangent vector to the modular orbit. Namely $d\tau = \beta_s ds$ with τ proper time

Schwarzschild black hole

Schwarzschild-Kruskal spacetime of mass $M > 0$, namely the region inside the event horizon, and $\mathcal{N} \equiv \mathcal{A}(O)$ the local von Neumann algebra associated with O on the underlying Hilbert space $\mathcal{H}$, O Schwarzschild black hole region.

We consider the Hartle-Hawking vacuum state ω

$\mathcal{H}$ is a $\mathcal{N} - \mathcal{N}$ bimodule, indeed the identity $\mathcal{N} - \mathcal{N}$ bimodule $L^2(\mathcal{N})$ associated with ω.
The modular group of $\mathcal{A}(O)$ associated with ω is geometric and corresponds to the geodesic flow. KMS Hawking temperature is

$$T = 1/8\pi M = 1/4\pi R$$

with $R = 2M$ the Schwarzschild radius, then

$$S \leq 4\pi RE$$

with S the entropy associated with the Hartle-Hawking state and the output state transferred by a quantum channel, and E the corresponding relative energy.

Conformal QFT

Conformal Quantum Field Theory on the Minkowski spacetime, any spacetime dimension. O_R double cone with basis a radius $R > 0$ sphere centered at the origin and $\mathcal{A}(O_R)$ associated local vN algebra. The modular group of $\mathcal{A}(O_R)$ w.r.t. the vacuum state ω has a geometrical meaning (Hislop, L.):

$$\Delta_{O_R}^{-is} = U\big(\Lambda_{O_R}(2\pi s)\big)$$

with U is the representation of the conformal group and Λ_{O_R} is a one-parameter group of conformal transformation leaving O_R globally invariant and conjugate to the boost one-parameter group of pure Lorentz transformations.

$$\Lambda_{O_R}(s) = \delta_R \cdot \Lambda_{O_1}(s) \cdot \delta_{1/R}$$

with δ_R the dilation by R.

Compare the proper time at a point $\mathbf{x}$ with parameter of the flows

$$d\tau = \Big|\Big|\frac{d}{ds}\Lambda_{O_R}(s)\mathbf{x}\Big|\Big|ds = \Big|\Big|\frac{d}{ds}\delta_R\cdot\Lambda_{O_1}(s)\cdot\delta_{1/R}\mathbf{x}\Big|\Big|ds = R\Big|\Big|\frac{d}{ds}\Lambda_{O_1}(s)\frac{\mathbf{x}}{R}\Big|\Big|ds$$

(Minkowskian norm); in particular, in the center $\mathbf{0}$ of the sphere, the proper time τ_R of Λ_{O_R} is R times the one of Λ_{O_1}.

The inverse temperature $\beta_R = \big|\big|\frac{d}{ds}\Lambda_{O_R}(s)\mathbf{x}\big|\big|_{s=0}$ in O_R is maximal on the time-zero basis of O_R, in fact at the origin $\mathbf{x} = \mathbf{0}$. Thus the maximal inverse temperatures in O_R and β_1 in O_1 are related by

$$\beta_R = R\beta_1$$

We fix the KMS inverse temperature for Λ_{O_R} as $\beta_R = R\beta_1$.
$\beta_1 = \pi$, half of the Unruh value, so $\beta_R = \pi R$.

We have $S \leq \beta_R E = \beta_1 RE \leq \pi RE$, so

$$S \leq \pi RE$$

with S and E the entropy and energy associated with any quantum channel by the vacuum state.

Boundary CFT

The analysis in this section is rather interlocutory, less complete than the previous ones. Yet it shows up new aspects as the temperature depends on the distance from the boundary.

1+1 dimensional Boundary CFT on the right Minkowski half-plane $x > 0$. The net $\mathcal{A}_+$ of von Neumann algebras on the half-plane is associated with a local conformal net $\mathcal{A}$ of von Neumann algebras on the real line (time axes) by

$$A_+(O) = \mathcal{A}(I) \vee \mathcal{A}(J)\,;$$

Here I, J are intervals of the real line at positive distance with $I > J$ (and $O = I \times J$).

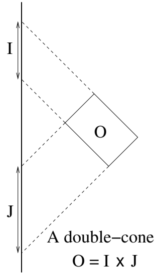


Figure: BCFT

(More generally a finite-index extension of $\mathcal{A}$ is needed). The following discussion is the same.

There is a natural state with geometric modular action (Martinetti, Rehren, L.), that corresponds to the chiral “2-interval state” and geometric action of the double covering of the Möbius group.

With $I = (a_1, b_1)$, $J = (a_2, b_2)$, in chiral coordinates $u = x + t$, $v = x - t$, the flow $\theta_s^O(u, v) = (u_s, v_s)$ has velocity field $(\partial u_s, \partial v_s)$

$$\partial_s u_s = 2\pi \frac{(u_s - a_1)(u_s - b_1)(u_s - a_2)(u_s - b_2)}{Lu_s^2 - 2Mu_s + N} \equiv -2\pi V^O(u_s)\,,$$

with $L = b_1 - a_1$, $M = b_1 b_2 - a_1 a_2$,
$N = b_2 a_2(b_1 - a_1) + b_1 a_1(b_2 - a_2)$, and similarly for v_s.

Let us fix a double cone O with basis of unit length (say O is Lorentz conjugate to a double cone with basis on the space axis with length one).

With $R > 0$, let O_R be the double cone associated with the intervals RI, RJ, namely $O_R = \delta_R O$, with δ_R the dilation by R on the half-plane. Then $\theta^{O_R} = \delta_R \cdot \theta^O \cdot \delta_{R^{-1}}$.

As above, the maximal inverse temperatures are related by

$$\beta^{O_R} = R\,\beta^O\,.$$

By choosing the KMS inverse temperatures equal to the maximal temperature, with S and E the entropy and energy in O_R with respect to the geometric state and a quantum channel, we have

$$S \leq \lambda_O R E$$

where the constant λ_O is equal to β_O.

Related work: Landauer's bound for infinite systems

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel between quantum systems $\mathcal{N}$, $\mathcal{M}$. If α is irreversible, then

$$F_\alpha \geq \frac{1}{2} kT \log 2$$

The original lower bound for the incremental free energy is $F_\alpha \geq kT \log 2$, it remains true for finite-dimensional systems $\mathcal{N}$, $\mathcal{M}$.

Related work: Relative entropy in CFT (with Feng Xu)

Computation of the relative entropy for free fermions on the circle and related models

Let A, B intervals with disjoint closures of S^1 and $\mathcal{A}_r$ the conformal net generated by r-fermions. Then the mutual information

$$F_{\mathcal{A}_r}(A, B) = S(\omega|\omega \otimes \omega)$$

is given by

$$F_{\mathcal{A}_r}(A, B) = -\frac{r}{6} \log \eta$$

with η the cross ratio of A, B

First rigorous computation of entropy in QFT!