Invariant states on Weyl algebras for the action of the symplectic group

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AQFT: WHERE OPERATOR ALGEBRA MEETS MICROLOCAL ANALYSIS

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Joint project with Federico Bambozzi and Nicola Pinamonti

DYNAMICAL SYSTEM: ($\mathfrak{A}$ unital $*$-algebra, $\mathcal{G}$ group, Φ ergodic group of $*$-automorphisms)

QUESTION: How many **invariant states** we can find?

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ANY MODELS ARE LEFT OUT FROM THIS SCENARIO?

- Orientation preserving automorphisms of the noncommutative torus $(\mathfrak{A}^{nc}_{\mathbb{T}}, Sl(2,\mathbb{Z}), \Phi)$
- Symplectomorphim of the quantum Hall effect $(\mathfrak{A}^{QHE}, Sp(2g,\mathbb{Z}), \Phi)$

POSSIBLE OBSTRACTION!

♣ infinite Sl$(2,\mathbb{Z})$-invariant states ω on the (commutative) C^*-algebra $\mathfrak{A}_{\mathbb{T}}$

♢ following Waldmann's ideas we can "deform" ω so that ω_W is state on $\mathfrak{A}^{NC}_{\mathbb{T}^2}$

♠ the noncommutative torus $\mathfrak{A}^{NC}_{\mathbb{T}^2}$ is $*$-isomorphic to Weyl C^*-algebra

♡ GOAL: CLASSIFY $Sp(2g,\mathbb{Z})$-INVARIANT STATES ON WEYL ALGEBRAS

Outline of the Talk

- **Weyl algebras**
  - (I) **Construction for $\mathbb{Z}^{2g}$**
  - (II) **Automorphism induced by $Sp(2g,\mathbb{Z})$**
- $Sp(2g,\mathbb{Z})$**-invariant states**
  - (I) **Definition and main theorem**
  - (II) **Sketch of the proof**

- Based on:
  Invariant states on Weyl algebras for the action of the symplectic group
  Federico Bambozzi , S.M., Nicola Pinamonti - (arXiv:1802.02487 [math.OA])

Weyl algebras I: Construction for $\mathbb{Z}^{2g}$

- Fix $h \in \mathbb{R}$ s.t. $\hbar := h/2\pi \in \mathbb{R} \setminus \mathbb{Q}$
- Choose a skew-symmetric, bilinear map $\sigma : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}$

$$\sigma := \begin{pmatrix} 0 & -1_{g\times g} \\ 1_{g\times g} & 0 \end{pmatrix}$$

- $\mathbb{Z}^2 \ni m \mapsto W_m$ linear operator on $C^0(\mathbb{Z}^2, \mathbb{C})$ defined by

$$(W_m v)(n) := e^{\imath h \sigma(m,n)} v(n+m)$$

- **Weyl $*$-algebra $\mathcal{A}$** is obtained by endowing $\mathcal{V}=$ $\mathrm{span}_{\mathbb{C}}\{W_m \,|\, m \in \mathbb{Z}^2\}$ with

$$(\star)\quad W_m W_n = e^{\imath h \sigma(m,n)} W(n+m) \qquad\qquad (*)\quad (W_m)^* = W_{-m}$$

Remark:

- Weyl C^*-algebra $\mathfrak{A} = \overline{(\mathcal{A}, \|\ \|)}$ where the C^*-norm $\|\cdot\|$ is given by

$$\|\mathfrak{a}\| := \sup_{\omega \in S_{\mathcal{A}}} \sqrt{\omega(\mathfrak{a}^*\mathfrak{a})}$$

- By setting $g = 1$ and $W_{(1,0)} = U$, $W_{(0,1)} = V$, we obtain NC torus $UV = e^{2\imath h} VU$
- Since $\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^{2g}$ by $m = (m_1, m_2) \mapsto \widetilde{m} = (m_1, 0 \ldots, 0, m_2, 0, \ldots, 0)$, we set $g = 1$.

Weyl algebra II: automorphism induced by $Sp(2,\mathbb{Z})$

- Symplectic group $Sp(2,\mathbb{Z})\big(\equiv Sl(2,\mathbb{Z})\big)$ acts on $\mathbb{Z}^2 \ni m \mapsto \Theta m$, with $\det \Theta{=}1$
- $Sp(2,\mathbb{Z}) \ni \Theta \longmapsto \Phi_\theta \in \mathrm{Aut}(\mathfrak{A})$ by linearity: $\Phi_\Theta W_m = W_{\Theta m}$
- Set of fixed points $= \{(0,0) \in \mathbb{Z}^2\} \Longrightarrow$ the action of Φ_Θ is **ergodic** on $\mathcal{A}$

$$\Phi_\Theta\big(\lambda W_{(0,0)}\big) = \lambda W_{(0,0)} \qquad \text{for any } \lambda \in \mathbb{C}\,, \Theta \in Sp(2,\mathbb{Z})$$

Proposition: characterization of $Sp(2,\mathbb{Z})$-orbits

(1) {set of orbits of the symplectic group $Sp(2,\mathbb{Z})$ } $\leftrightarrow \mathcal{E} := \{(0,j)\,|\,j \in \mathbb{N}\}$

(2) Every $Sp(2,\mathbb{Z})$-orbit of $\mathbb{Z}^2$ contains an element of the form (j,j) with $j \in \mathbb{N}$

Sketch of the proof - part (1):

- $\Theta \in Sp(2,\mathbb{Z}) \equiv Sl(2,\mathbb{Z})$ takes the form $\Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$
- $\forall q = (q_1, q_2) \in \mathbb{Z}^2$ we have $\Theta q = (aq_1 + bq_2, cq_1 + dq_2) \xrightarrow[\exists\, a,b,c,d]{\forall q_1, q_2} \Theta q = \begin{pmatrix} 0 \\ q_3 > 0 \end{pmatrix}$
- Let be $n_i = (0, m_i)$ s.t. $m_i \geq 0$ and $m_1 \neq m_2$ and <u>assume</u> $\exists \Theta \in Sp(2,\mathbb{Z})$ s.t. $\Theta n_1 = n_2$
- $\Theta n_1 = n_2 \Longrightarrow b = 0 \xrightarrow[m_i > 0]{\det \Theta = 1} a = d = 1 \xrightarrow{\Theta n_1 = n_2} \Theta = Id \Longrightarrow m_1 = m_2$ ⚡

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Sketch of the proof - part (2):

- Let $\mathcal{O}$ be a $Sp(2,\mathbb{Z})$-orbit
- (1) $\Longrightarrow \mathbf{j} = (0,j) \in \mathcal{O}$
- Choosing $\Theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have $\Theta\,\mathbf{j} = (j,j) \in \mathcal{O}$

$Sp(2,\mathbb{Z})$-invariant states

Definitions: A $Sp(2,\mathbb{Z})$-**invariant state** ω if for any $\Phi_\Theta \in \mathrm{Aut}\,(\mathfrak{A})$, $\Theta \in Sp(2,\mathbb{Z})$ it holds

- $\omega(\mathfrak{a}^*\mathfrak{a}) \geq 0$
- $\omega(1_{\mathfrak{A}}) = 1$
- $\omega \circ \Phi_\Theta = \omega$

N.B.: To construct ω, it is enough to prescribe its values on the generators of $\mathfrak{A}$

$$\omega(W_m) = \begin{cases} 1 & \text{if } m = (0,0) \\ p^{(m)} \in \mathbb{C} & \text{else} \end{cases}$$

for a sequence of values $p^{(m)}$ and then extend it by linearity to any $\mathfrak{a} \in \mathfrak{A}$

Theorem

The only $Sp(2,\mathbb{Z})$-invariant state on $\mathfrak{A}$ is the **trace state** τ defined by

$$\tau(W_m) = \begin{cases} 1 & \text{if } m = (0,0) \\ 0 & \text{else} \end{cases}$$

τ is obviously invariant: $\tau(\Phi_\Theta W_m) = \tau(W_{\theta m}) = \begin{cases} 1 & \text{if } m = (0,0) \\ 0 & \text{else} \end{cases}$

Sketch of the proof I

Assume by contraddiction there exists $Sp(2,\mathbb{Z})$-invariant state ω (different from τ!)

$$\omega(W_\chi) = \begin{cases} 1 & \text{if } \chi = (0,0) \\ p^{(\chi)} \in \mathbb{C} & \text{else} \end{cases}$$

Goal: $Sp(2,\mathbb{Z})$-invariance $\Longleftrightarrow$ $p^{(\chi)} = 0$ for any $\chi \neq 0 \Longrightarrow \omega \equiv \tau$

(1) By positivity of $\omega \Longrightarrow \overline{\omega(W_\chi)} = \omega(W_\chi^*)$

(2) By $Sp(2,\mathbb{Z})$-invariance $\Longrightarrow \overline{\omega(W_\chi)} = \omega(W_\chi^*) = \omega(W_{-\chi}) = \omega(W_{-Id\chi}) = \omega(W_\chi)$

(3) Choosing $\mathfrak{a} = W_0 + W_\chi \xrightarrow{\omega(\mathfrak{a}^*\mathfrak{a})\geq 0} 1 - p^2 \leq 0$

(4) **Hence, any $Sp(2,\mathbb{Z})$-invariant states reads as**

$$\omega(W_\chi) = \begin{cases} 1 & \text{if } \chi = (0,0) \\ p^{(\chi)} \in [-1,1] & \text{else} \end{cases}$$

Next, we choose a more suitable χ without loosing of generality

(5) Fix $\chi \neq 0$.

(6) By previous Prop.: For any χ there exists $\Theta \in Sp(2,\mathbb{Z})$ s.t. $\Theta\chi = \xi = (\xi_1, \xi_1)$

(7) In particular $Sp(2,\mathbb{Z})$-invariance $\Longrightarrow \omega(W_\chi) = \omega(W_\xi) = p$

Sketch of the proof II

(8) Let $m, n \in \mathbb{N}$ s.t. $\frac{m}{n} \in \mathbb{N}$ and $n > 1$ and consider $\mathcal{V}_{\xi;\, m,n} \subset \mathcal{A}$ with elements of the form

$$\mathfrak{a} = \alpha_0 W_{(0,0)} + \sum_{j\geq 1} \alpha_j W_{\Theta_j \xi} \qquad \text{with } \Theta_j := \begin{pmatrix} 1 + \frac{m}{n}(n-1)j & \frac{m}{n}j \\ n-1 & 1 \end{pmatrix} \in Sp(2, \mathbb{Z})$$

(9) For any $\mathcal{V}_{\xi;\, m,n}$, the map $\mathfrak{a} \mapsto \omega(\mathfrak{a}^* \mathfrak{a})$ is a quadratic form $\omega(\mathfrak{a}^* \mathfrak{a}) = \overline{\alpha}^t\, \mathbf{H}\, \alpha$

$$\mathbf{H} = \begin{pmatrix} 1 & p & p & p & p & p & \cdots \\ p & 1 & q_m e^{\imath\varphi_{m,n}} & q_{2m} e^{2\imath\varphi_{m,n}} & q_{3m} e^{3\imath\varphi_{m,n}} & q_{4m} e^{4\imath\varphi_{m,n}} & \cdots \\ p & q_m e^{-\imath\varphi_{m,n}} & 1 & q_m e^{\imath\varphi_{m,n}} & q_{2m} e^{2\imath\varphi_{m,n}} & q_{3m} e^{3\imath\varphi_{m,n}} & \cdots \\ p & q_{2m} e^{-2\imath\varphi_{m,n}} & q_m e^{-\imath\varphi_{m,n}} & 1 & q_m e^{\imath\varphi_{m,n}} & q_{2m} e^{2\imath\varphi_{m,n}} & \cdots \\ p & q_{3m} e^{-3\imath\varphi_{m,n}} & q_{2m} e^{-2\imath\varphi_{m,n}} & q_m e^{-\imath\varphi_{m,n}} & 1 & q_m e^{\imath\varphi_{mn}} & \cdots \\ p & q_{4m} e^{-4\imath\varphi_{m,n}} & q_{3m} e^{-3\imath\varphi_{m,n}} & q_{2m} e^{-2\imath\varphi_{m,n}} & q_m e^{-\imath\varphi_{m,n}} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\quad q_{(i-j)m} := \omega(W_{\Theta_i \xi - \Theta_j \xi}) \quad$ and $\quad \varphi_{m,n} := hmn\xi_2^2$

(10) *Notation:* $\mathbf{H} = [\, p \,,\, 1 \,,\, q_m e^{\imath\varphi_{m,n}} \,,\, q_{2m} e^{\imath 2\varphi_{m,n}} \,,\, q_{3m} e^{\imath 3\varphi_{m,n}} \,,\, \ldots]$

Sketch of the proof III

(11) **On $(d+1)$-D subspace $\mathcal{V}_{d;n} \subset \mathcal{V}_{\xi;\,m,n}$ the restriction of H to $\mathcal{V}_{d;n}$ reads**

$$\mathbf{H}'_n = [\, p \,,\, 1 \,,\, q_m e^{\imath \frac{2\pi}{d} n} \,,\, q_{2m} e^{2\imath \frac{2\pi}{d} n} \,,\, q_{3m} e^{3\imath \frac{2\pi}{d} n} \,,\, \dots \,,\, q_{(d-1)m} e^{(d-1)\imath \frac{2\pi}{d} n}] + \text{“}\varepsilon\text{”}$$

$\left(\text{idea: } \hbar := \frac{h}{2\pi} \text{ is irrational} \Rightarrow \exists m \in \mathbb{N} \text{ big enough s.t. } \frac{m}{d!} \in \mathbb{N} \text{ and } \left| (h\, m\, \xi_2^2) \mod (2\pi) - \frac{2\pi}{d} \right| < \frac{\varepsilon}{4d^2} \right)$

We can now argue that p has to be 0:

(12) We can notice that the set of positive Hermitian matrices form a convex cone

(13) The matrix $\mathbf{P}'_d := [p; 1; 0; 0; \dots; 0]$ can be obtained as the convex combination

$$\mathbf{P}'_d = \sum_{n=1}^{d} \frac{1}{d} \mathbf{H}'_n \,.$$

(14) For d “big enough” $\det(\mathbf{P}'_d) = 1 - dp^2 < 0 \Rightarrow \mathbf{P}'_n$ not positive $\Rightarrow \mathbf{H}'_n$ not positive

(15) Hence $p = 0$ is a necessary condition for ω being an $Sp(2,\mathbb{Z})$-invariant state

(16) This holds for every $m \in \mathbb{Z}^2 \Longrightarrow$ therefore the only $Sp(2,\mathbb{Z})$-invariant state is τ

□

Resume & Outlook

- *Weyl $*$-algebra* useful used in QM and noncommutative geometry

$$\mathcal{A} = \text{span}_{\mathbb{C}}\{W_m \,|\, m, n \in \mathbb{Z}^{2g}\,,\ W_m W_n = e^{\imath h \sigma(m,n)} W(n+m)\,,\ (W_m)^* = W_{-m}\}$$

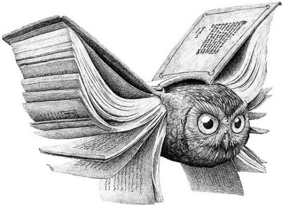
- *Unique $Sp(2,\mathbb{Z})$-invariant states* on Weyl algebras is

$$\tau(W_m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{else} \end{cases}$$

`What comes next?`

- Weyl $*$-algebra for generic for symplectic abelian group
- Other noncommutative spaces: Moyal space, Connes-Landi Sphere, ...

...and most importantly: `KLAUS' BOOK!`



`THANKS for your attention!`