

ALGEBRAIC HOLOGRAPHY IN ASYMPTOTICALLY ADS SPACE-TIMES: FUNCTIONAL FRAMEWORK, EXAMPLES AND STEPS TOWARDS RIGOROUS BULK RECONSTRUCTION

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1 CHANT TO THE MUSE

- AdS/CFT duality
- AAdS space-times
- Causal diamonds and causal wedges
- Rehren correspondence

2 QFT ON AADS

- Algebraic holography
- Klein-Gordon on AAdS
- The holographic Hadamard condition

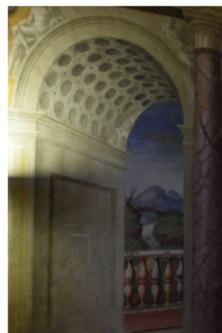
3 DECODING THE HOLOGRAM

4 CODA

CHANT TO THE MUSE: AdS/CFT DUALITY AND ITS AVATARS

The hologram simply does not have the intelligence of the trompe l'œil, which is one of seduction, of always proceeding, according to the rules of appearances, through allusion to and ellipsis of presence. It veers, on the contrary, into fascination, which is that of passing to the side of the double. If, according to Mach, the universe is that of which there is no double, no equivalent in the mirror, then with the hologram we are already virtually in another universe: which is nothing but the mirrored equivalent of this one. But which universe is this one?

(—Jean Baudrillard, *Simulacra and Simulation* (1981))



- **Maldacena's Conjecture (1997):** (Type IIB) string theory in the gravitational background $AdS_5 \times S^5$ is dual to ($\mathcal{N} = 4$, large- N SUSY $SU(N)$ Yang-Mills) conformal field theory in $\mathbb{R}^{1,3}$ (= (part of) conformal infinity/boundary of AdS_5).
- Result of Maldacena expressed in terms of k -point (Schwinger) functions (Gubsser-Klebanov-Polyakov (1998), Witten (1998)):

AdS/CFT DUALITY

Correlators of the dual theory at the boundary are given (in a certain effective limit) by functional derivatives of the **classical** (“*on-shell*”) (super)gravity action in the bulk under variation of boundary conditions at infinity.

⇒ expressed only in terms of **field theories!**

Axiomatic versions of AdS/CFT:

- In terms of Wightman k -point functions (Bertola-Bros-Moschella-Schaeffer (2000));
- In terms of (C*-)algebras of local observables (Rehren (2000)) \Rightarrow **algebraic holography** or **Rehren duality**.
- Both versions are extendable to **asymptotically AdS** space-times! **However...** where is (bulk) gravity encoded, **if at all?** (no backreaction here!)
- **Answer:** deviations from exact AdS geometry affect the **causal structure** in the bulk, therefore should be visible in the boundary dual QFT!
- This is a **truly holographic** phenomenon: to encode bulk geometry features relative to a certain background (AdS) in a boundary "screen".

KEY QUESTION

How to reconstruct the (deviations from AdS of the) causal structure of the bulk from the boundary dual QFT? How to decode this hologram?



b -GLOBALLY HYPERBOLIC SPACE-TIMES, AAdS SPACE-TIMES

Let's first provide our geometrical context (all our manifolds are smooth, Hausdorff, paracompact, second countable and oriented, and all our metrics are smooth unless otherwise indicated). Consider a ($3 \leq d$)-dimensional connected **space-time** (\mathcal{M}, g) (i.e. a (connected) time oriented Lorentzian manifold). We use the signature convention $(+ - \cdots -)$ for Lorentz metrics.

DEFINITION

A **conformal completion** or **conformal closure** of (\mathcal{M}, g) is a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{g})$ with (not necessarily connected) boundary $\partial \overline{\mathcal{M}} \doteq \mathcal{I}$ s. t. there is a diffeomorphism j of \mathcal{M} onto the interior of $\overline{\mathcal{M}}$ and a smooth $0 \leq z \in \mathcal{C}^\infty(\overline{\mathcal{M}})$ with

$$\begin{aligned} z^{-1}(0) &= \mathcal{I} , \\ i^* dz &\neq 0 \text{ everywhere,} \\ j^* \overline{g} &= z^2 g , \end{aligned}$$

where $i : \mathcal{I} \hookrightarrow \overline{\mathcal{M}}$ is the natural inclusion. We identify \mathcal{M} with $j(\mathcal{M})$. Such a z is called a **boundary defining function** for $(\overline{\mathcal{M}}, \overline{g})$. If $g_0 \doteq i^* \overline{g}$ is a pseudo-Riemannian metric on \mathcal{I} , we say that (\mathcal{I}, g_0) is a **conformal infinity** or **conformal boundary** for (\mathcal{M}, g) .



DEFINITION

We say that (\mathcal{M}, g) is **b -globally hyperbolic** if it admits a conformal completion $(\overline{\mathcal{M}}, \overline{g})$ such that:

- (\mathcal{I}, g_0) is a space-time (of dimension $d - 1$), i.e., \mathcal{I} is **timelike** w.r.t. \overline{g} ;
- $(\overline{\mathcal{M}}, \overline{g})$ admits a **Cauchy time function** $\tau \in \mathcal{C}^\infty(\overline{\mathcal{M}})$ surjective such that $d\tau$ is everywhere timelike and any inextendible causal curve in $(\overline{\mathcal{M}}, \overline{g})$ crosses $\tau^{-1}(t)$ exactly once for all $t \in \mathbb{R}$.

(\mathcal{M}, g) is said to be **proper** if τ as above can be chosen proper. This means that the Cauchy hypersurfaces of $(\overline{\mathcal{M}}, \overline{g})$ are **compact**.

It follows immediately that $\tau|_{\mathcal{I}}$ is a Cauchy time function on \mathcal{I} . We can (and will without further notice) exploit the freedom of choice of boundary defining functions z and Cauchy time functions τ on $(\overline{\mathcal{M}}, \overline{g})$ so as to have:

- $\overline{g}^{-1}(dz, dz) \equiv -1$ on an open collar $\text{ngb } \mathcal{I} \times [0, \epsilon) \cong U \supset \mathcal{I} \cong \mathcal{I} \times \{0\}$, $\epsilon > 0$;
- $\overline{g}^{-1}(dz, d\tau) \equiv 0$ on U .

Notice that only the **conformal class** of the conformal completion and the conformal infinity of a b -globally hyperbolic space-time are uniquely defined, i.e. they are uniquely defined **only up to a conformal diffeomorphism**.



Remarks:

- b -globally hyperbolic space-times are **causally simple** (i.e. (\mathcal{M}, g) is causal and $J^\pm(p)$ are closed for all $p \in \mathcal{M}$) but **never** globally hyperbolic, even in the proper case;
- As an example of a b -globally hyperbolic space-time, one can consider **timelike tubes** of a globally hyperbolic space-time $(\mathcal{N}, \tilde{g}) =$ open submanifolds \mathcal{M} of \mathcal{N} such that $\partial\mathcal{M} = \mathcal{I}$ is a **timelike** hypersurface. In that case, $(\mathcal{M}, g = \tilde{g}|_{\mathcal{M}})$ is proper iff $\overline{\mathcal{M}}$ is spatially compact.

DEFINITION

An **AdS-type** space-time is a proper b -globally hyperbolic space-time (\mathcal{M}, g) such that \mathcal{I} is diffeomorphic to $\mathbb{R} \times \mathbb{S}^{d-2}$ with $\mathbb{R} \times \{\theta\}$ timelike for all $\theta \in \mathbb{S}^{d-2}$. If, in addition, g_0 is in the conformal class of the metric $d\tau^2 - h_0$, where h_0 is the standard metric on \mathbb{S}^{d-2} , we say that (\mathcal{M}, g) is an **asymptotically AdS** space-time (or **AAdS** space-time for short). If, more generally, (\mathcal{M}, g) is a b -globally hyperbolic space-time such that any $p \in \mathcal{I}$ has an open ngb in $\overline{\mathcal{M}}$ which embeds isometrically onto an open ngb of a point of the conformal infinity of an AAdS space-time, we say that (\mathcal{M}, g) is a **locally AAdS** space-time.

- In other words, an AAdS spacetime is a proper b -globally hyperbolic space-time whose conformal infinity is conformal to the $(d - 1)$ -dimensional Einstein static universe ESU_{d-1} .
- AdS space-times (by which we always mean the universal cover), being conformal to an open half of ESU_d , are of course AAdS. Locally AAdS space-times comprise all known black hole space-times which are asymptotic to AdS.



CAUSAL DIAMONDS AND CAUSAL WEDGES

DEFINITION

Let $\emptyset \neq \mathcal{O} \subset \mathcal{M}$. We say that \mathcal{O} is a **future/past set** if $I^{+/-}(\mathcal{O}) \subset \mathcal{O}$.

Basic properties of future and past sets:

- If \mathcal{O} is a future/past set, then \mathcal{O}^c is a past/future set;
- If \mathcal{O} is **open**, then \mathcal{O} is a future/past set iff $I^{+/-}(\mathcal{O}) = \mathcal{O}$;
- If \mathcal{O} is a future/past set, then

$$\overline{\mathcal{O}} = \{p \in \mathcal{M} \mid I^{+/-}(p) \subset \mathcal{O}\} \Rightarrow \text{int}\mathcal{O} = \bigcup_{p \in \overline{\mathcal{O}}} I^{+/-}(p) = \bigcup_{p \in \mathcal{O}} I^{+/-}(p).$$

In particular, both $\overline{\mathcal{O}}$ and $\text{int}\mathcal{O}$ are future/past sets;

- If \mathcal{O} is a future set or a past set, then $\partial\mathcal{O}$ is a closed, achronal and embedded locally Lipschitz hypersurface.

Future and past sets can be decomposed as a union of “simple” pieces.

DEFINITION

Let \mathcal{O} be a future (resp. past) set. We say that \mathcal{O} is an **indecomposable** future (resp. past) set, or an **IF** (resp. **IP**), if there are no future (resp. past) sets $\emptyset \neq \mathcal{O}_1, \mathcal{O}_2 \neq \mathcal{O}$ such that $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$.

It can be shown that \mathcal{O} is an IF/IP iff

$$\mathcal{O} = I^{+/-}(\gamma), \quad \gamma \text{ a timelike curve.}$$

(actually, one may choose γ to be only causal) Now, if:

- γ has a past/future endpoint $p \Rightarrow$ we say that $\mathcal{O} = I^{+/-}(p)$ is a **proper indecomposable future/past set**, or **PIF/PIP**;
- γ is past/future inextendible \Rightarrow we say that \mathcal{O} is a **terminal indecomposable future/past set**, or **TIF/TIP**.

We denote the **causal complement** of $\emptyset \neq \mathcal{O} \subset \mathcal{M}$ by

$$\mathcal{O}^\perp = \text{int}((J^+(\mathcal{O}) \cup J^-(\mathcal{O}))^c) = (\overline{I^+(\mathcal{O})} \cup \overline{I^-(\mathcal{O})})^c = \overline{I^+(\mathcal{O})}^c \cap \overline{I^-(\mathcal{O})}^c.$$

Recall now that a subset $U \subset \mathcal{M}$ of a space-time (\mathcal{M}, g) is said to be **causally convex** if given any $p, q \in U$ we have that any causal curve segment from p to q is contained in U . This concept extends to Lorentzian manifolds with boundary such as $(\overline{\mathcal{M}}, \overline{g})$ without change. **Strongly causal** space-times are those which admit a topological basis made of causally convex subsets – from now on, all our space-times (with or without boundary) (\mathcal{M}, g) are strongly causal.



DEFINITION

If $\emptyset \neq \theta \subset \mathcal{M}$ is open and causally convex, we say that θ is a **generalized causal diamond**. Equivalently, θ is a generalized causal diamond if

$$\theta = P \cap Q, \quad P \text{ an open future set, } Q \text{ an open past set.}$$

θ is said to be:

- **proper** if $\theta^\perp \neq \emptyset$;
- **simple** if P, Q can be chosen to be indecomposable;
- a **causal diamond** if P (resp. Q) can be chosen to be a *PIF* (resp. a *PIP*);
- a **causal wedge** if it is proper and P (resp. Q) can be chosen to be a union of TIF's (resp. TIP's).

One immediately sees that θ^\perp is a generalized causal diamond if nonvoid, for **all** $\emptyset \neq \theta \subset \mathcal{M}$ (not even necessarily open, let alone generalized causal diamond). An useful, alternative characterization of generalized causal diamonds is that it's enough to check the definition only for the "smallest" possible choice of P, Q .

LEMMA

$\emptyset \neq \theta \subset \mathcal{M}$ is a generalized causal diamond iff $\theta = I^+(\theta) \cap I^-(\theta)$. Moreover,

- θ is simple $\Leftrightarrow I^+(\theta)$ is an IF and $I^-(\theta)$ is an IP;
- θ is a causal diamond $\Leftrightarrow I^+(\theta)$ is a PIF and $I^-(\theta)$ is a PIP;
- θ is a **simple** causal wedge $\Leftrightarrow I^+(\theta)$ is a TIF and $I^-(\theta)$ is a TIP.

Moreover, one can show that the **boundary** $\partial\theta$ of a generalized causal diamond $\theta = P \cap Q$ can be written as the disjoint union

$$\partial\theta = \partial_+\theta \dot{\cup} \partial_-\theta \dot{\cup} \partial_0\theta ,$$

where

$$\partial_+\theta = I^+(\theta) \cap \partial I^-(\theta) = P \cap \partial Q \Rightarrow \text{future horizon of } \theta ;$$

$$\partial_-\theta = I^-(\theta) \cap \partial I^+(\theta) = \partial P \cap Q \Rightarrow \text{past horizon of } \theta ;$$

$$\partial_0\theta = \partial I^+(\theta) \cap \partial I^-(\theta) \Rightarrow \text{edge of } \theta .$$

It follows that

$$\overline{\partial_\pm\theta} = \partial_\pm\theta \cup \partial_0\theta \text{ is achronal;}$$

$$\partial_\pm\theta^\perp = \partial I^\pm(\theta) \setminus \overline{I^\mp(\theta)} ;$$

$$\partial_0\theta^\perp = \partial_0\theta ;$$

$$(\partial_0\theta)^\perp = \theta \cup \theta^\perp .$$



Since a future/past set is a union of TIF's/TIP's iff its boundary is ruled by future inextendible null geodesics (Geroch-Kronheimer-Penrose (1972)), one infers the following crucial

THEOREM

Let \mathcal{O} be a proper generalized causal diamond. Then \mathcal{O} is a causal wedge iff $\partial_{+/-}\mathcal{O} \cup \partial_0\mathcal{O}$ is ruled by future/past inextendible null geodesics.

Our definition of causal wedges subsumes essentially all known examples in the literature. The above results allow one to define candidates for simple causal wedges in any b -globally hyperbolic space-time (\mathcal{M}, g) rather easily, provided (\mathcal{M}, g) satisfies the following

HYPOTHESIS (ACS) = "ABSENCE OF CAUSAL SHORTCUTS"

Let $p, q \in \mathcal{I}$. Then

$$q \in J^+(p, \overline{\mathcal{M}}) \Rightarrow q \in J^+(p, \mathcal{I}) ;$$

$$q \in I^+(p, \overline{\mathcal{M}}) \Rightarrow q \in I^+(p, \mathcal{I}) .$$

Moreover, any causal curve segment from p to q in \mathcal{I} maximizing the Lorentzian arc length in (\mathcal{I}, g_0) also does so in $(\overline{\mathcal{M}}, \overline{g})$.



REHREN CORRESPONDENCE

If (\mathcal{M}, g) is a b -globally hyperbolic satisfying (ACS) and $p, q \in \mathcal{I}$ satisfy $q \in I^+(p, \mathcal{I})$, set

$$\begin{aligned}\mathcal{D}_{p,q} &= I^+(p, \mathcal{I}) \cap I^-(q, \mathcal{I}) ; \\ \mathcal{W}_{p,q} &= I^+(p, \overline{\mathcal{M}}) \cap I^-(q, \overline{\mathcal{M}}) \cap \mathcal{M} .\end{aligned}$$

Obviously, $\mathcal{D}_{p,q}$ is a causal diamond in (\mathcal{I}, g_0) . It follows from (ACS) that

$\mathcal{D}_{p,q} = I^+(p, \overline{\mathcal{M}}) \cap I^-(q, \overline{\mathcal{M}}) \cap \mathcal{I}$. Moreover, if $\mathcal{D}_{p,q}$ is **proper**, so is $\mathcal{W}_{p,q}$ – hence, in this case $\mathcal{W}_{p,q}$ is a **simple wedge** since for any causal curve $\gamma : [0, 1] \rightarrow \overline{\mathcal{M}}$ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma((0, 1)) \subset \mathcal{M}$ we have that $\gamma|_{(0,1)}$ is an **inextendible** causal curve in (\mathcal{M}, g) and $\mathcal{W}_{p,q} = I^+(\gamma((0, 1))) \cap I^-(\gamma((0, 1)))$.

DEFINITION (REHREN (2000), PLR (2007))

Let

$$\mathcal{D} = \{(p, q) \in \mathcal{I} \times \mathcal{I} \mid q \in I^+(p, \mathcal{I}) , (J^+(p, \mathcal{I}) \cup J^-(q, \mathcal{I}))^c \neq \emptyset\} .$$

The **Rehren correspondence** is the bijection

$$\mathcal{W}_{p,q} \leftrightarrow \mathcal{D}_{p,q} , \quad (p, q) \in \mathcal{D} .$$

Remarks:

- Thanks to **gravitational time delay** theorems (Gao-Wald (2000), Page-Surya-Woolgar (2002), PLR (2007)), any AAdS space-time (\mathcal{M}, g) satisfying the **null energy condition**

$$g(k, k) = 0 \Rightarrow \text{Ric}(g)(k, k) \geq 0$$

satisfies (ACS). Moreover, any causal curve $\gamma : [0, 1] \rightarrow \overline{\mathcal{M}}$ in $(\overline{\mathcal{M}}, \bar{g})$ such that $\gamma^{-1}(\mathcal{I}) = \{0, 1\}$ is endpoint-homotopic to a causal curve in $(\mathcal{I}, g_0) \Rightarrow$ **topological censorship** (Galloway-Schleich-Witt-Woolgar (1999,2001));

- ESU_{d-1} has the property that all future directed null generators of $\partial I^+(p)$ meet at a **single** future endpoint \bar{p} , called the **future antipode** of p , and a similar result holds for the past. It turns out that $p \ll \bar{q} \ll \bar{\bar{p}}$ in ESU_{d-1} implies $(p, \bar{q}) \in \mathcal{D}$ and therefore

$$\mathcal{D}_{\bar{p}, \bar{q}} = \mathcal{D}_{p, \bar{q}}^\perp \neq \emptyset$$

(notice that $\mathcal{D}_{p, \bar{\bar{p}}}$ is isometric to the image of the conformal embedding of $(d-1)$ -dimensional Minkowski space-time into ESU_{d-1}). Moreover, since the same result holds for ESU_d , we conclude as well in this case that **in AdS**

$$\mathcal{W}_{\bar{p}, \bar{q}} = \mathcal{W}_{p, \bar{q}}^\perp.$$

For **more general AAdS space-times** satisfying (ACS), we have instead

$$\mathcal{W}_{\bar{p}, \bar{q}} \subsetneq \mathcal{W}_{p, \bar{q}}^\perp$$

due to **gravitational time delay!**



QFT ON AAdS – ALGEBRAIC HOLOGRAPHY

Consider a presheaf \mathfrak{A} of $*$ -algebras

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$$

indexed by all generalized causal diamonds $\mathcal{O} \subset \mathcal{M}$ on a b -globally hyperbolic space-time (\mathcal{M}, g) satisfying (ACS). **Algebraic holography** uses the Rehren correspondence to define a presheaf \mathfrak{B} of $*$ -algebras

$$\mathcal{D}_{p,q} \mapsto \mathfrak{B}(\mathcal{D}_{p,q}), \quad (p, q) \in \mathcal{D}$$

on (\mathcal{I}, g_0) indexed by the latter's causal diamonds, called the **boundary dual** presheaf. More precisely,

$$\mathfrak{B}(\mathcal{D}_{p,q}) = \mathfrak{A}(\mathcal{W}_{p,q}), \quad (p, q) \in \mathcal{D}.$$

It follows from (ACS) that \mathfrak{B} is **causal** if \mathfrak{A} is. Moreover, in the case of AdS this correspondence intertwines any covariant action of the bulk isometry group on \mathfrak{A} with a covariant action of the boundary conformal group on \mathfrak{B} (Rehren (2000)) if the former exists.

More generally, any isometry ψ of a b -globally hyperbolic space-time (\mathcal{M}, g) satisfying (ACS) which extends smoothly to $\overline{\mathcal{M}}$ is a conformal isometry of $(\overline{\mathcal{M}}, \overline{g})$ and therefore uniquely defines a conformal isometry of (\mathcal{I}, g_0) . In this case, ψ preserves both bulk wedges and boundary diamonds, and algebraic holography intertwines any covariant action of ψ on \mathfrak{A} with a covariant action of ψ on \mathfrak{B} .



THE KLEIN-GORDON FIELD ON AAdS SPACE-TIMES

We shall construct as an example of a bulk precosheaf of $(C)^*$ -algebras the Weyl system associated to the **Klein-Gordon equation**

$$(\square_g + \lambda)\phi = 0, \quad \square_g = g^{-1}\nabla^2, \quad \lambda \in \mathbb{R},$$

on a proper b -globally hyperbolic space-time (\mathcal{M}, g) .

- Off-shell field configuration space: $\mathcal{Q} = \mathcal{C}^\infty(\overline{\mathcal{M}}, \mathbb{R})$,
- Weyl elements = functionals on \mathcal{Q}

$$W_f(\phi) = \exp\left(i \int_{\mathcal{M}} f \phi \, d\mu_{\bar{g}}\right), \quad f \in \dot{\mathcal{D}}(\mathcal{M})$$

($d\mu_{\bar{g}} = z^d d\mu_g =$ volume measure induced by \bar{g}), where $\dot{\mathcal{D}}(\mathcal{M}) = \overline{\mathcal{D}(\mathcal{M})\mathcal{D}(\overline{\mathcal{M}})}$ = elements of $\mathcal{D}(\overline{\mathcal{M}})$ which **vanish to infinite order at \mathcal{I}** \Rightarrow we can still have $\text{supp} f \cap \mathcal{I} \neq \emptyset$!

- If $\mathcal{O} \subset \mathcal{M}$ is a generalized causal diamond, set $\tilde{\mathfrak{A}}(\mathcal{O}) =$ complex vector space generated by $W_f, f \in \dot{\mathcal{D}}(\mathcal{O})$, where

$$\dot{\mathcal{D}}(\mathcal{O}) = \overline{\mathcal{D}(\mathcal{O})\mathcal{D}(\overline{\mathcal{M}})} = \{f \in \dot{\mathcal{D}}(\mathcal{M}) \mid \text{supp} f \subset \mathcal{O} \cup (\overline{\mathcal{O}} \cap \mathcal{I})\} \subset \dot{\mathcal{D}}(\mathcal{M})$$

(= $\mathcal{D}(\mathcal{O})$ if \mathcal{O} globally hyperbolic, using $f \in \dot{\mathcal{D}}(\mathcal{O})$ is necessary to cope with **non-globally hyperbolic \mathcal{O} 's!**).



Here we have endowed $\tilde{\mathfrak{A}}(\mathcal{O})$ for each \mathcal{O} with the standard “point” (=field)-wise vector space operations (w.r.t. which the W_f 's are linearly independent) and with the antilinear involution $W_f^* = W_{-f}$, setting $\mathbb{1} = W_0$. We still need an antisymmetric distribution kernel Δ on $\dot{\mathcal{D}}(\mathcal{M})$ which allows us to define a (Weyl) product

$$W_f W_{f'} = e^{\frac{i\hbar}{2} \Delta(f, f')} W_{f+f'}$$

so that $\mathbb{1}$ is a unit and $W_f W_f^* = W_f^* W_f = \mathbb{1}$. This, on its turn, will allow us to define a maximal C^* -norm $\|\cdot\|$ on $\tilde{\mathfrak{A}}(\mathcal{O})$ with respect to which $\|W_f\| = 1$ for all f (Manuceau-Sirigie-Testard-Verbeure (1973)).

In analogy with the globally hyperbolic case, we want to define Δ as

$$\Delta = \Delta_R - \Delta_A ,$$

where (identifying Δ_R and Δ_A with their respective operators):

- Δ_R and Δ_A (denoted by the same symbols) are **fundamental solutions** of $\square_g + \lambda$ in $\dot{\mathcal{D}}(\mathcal{M})$:

$$(\square_g + \lambda)\Delta_{R/A} = \Delta_{R/A}(\square_g + \lambda) = \mathbb{1}_{\dot{\mathcal{D}}(\mathcal{M})} ;$$

- Δ_R (resp. Δ_A) maps $\dot{\mathcal{D}}(\mathcal{M})$ into elements of $\mathcal{C}^\infty(\overline{\mathcal{M}})$ with **past compact** (resp. **future compact**) support.

As usual, since $\square_g + \lambda$ is formally self-adjoint w.r.t. the L^2 scalar product induced by g , one sees that Δ_R and Δ_A as above are mutually $L^2(d\mu_g)$ -adjoint in $\dot{\mathcal{D}}(\mathcal{M})$, hence assuring the antisymmetry of Δ .



The center of the local Weyl C*-algebra $\mathfrak{A}(\mathcal{O}) = \overline{\tilde{\mathfrak{A}}(\mathcal{O})}^{\|\cdot\|}$ is given by

$$\mathfrak{Z}(\mathcal{O}) = \overline{\tilde{\mathfrak{Z}}(\mathcal{O})}^{\|\cdot\|}, \quad \tilde{\mathfrak{Z}}(\mathcal{O}) = \left\{ \sum_k \alpha_k W_{f_k} \mid \alpha_k \in \mathbb{C}, f_k \in \ker \Delta \right\},$$

$$\ker \Delta = \{f \mid \Delta(f, f') = 0 \text{ for all } f'\}.$$

Maximal closed *-ideals of $\mathfrak{A}(\mathcal{O})$ are uniquely defined by their intersection with $\mathfrak{Z}(\mathcal{O})$. More precisely, consider the linear map

$$m \left(\sum_{k=1}^n \alpha_k W_{f_k} \right) = \sum_{k: f_k \in \ker \Delta} \alpha_k W_{f_k}$$

It can be shown (Manuceau et al., *ibid.*, Th. 4.2) that m is **continuous** with respect to $\|\cdot\|$ and its continuous extension to $\mathfrak{A}(\mathcal{O})$ is a **conditional expectation** of $\mathfrak{A}(\mathcal{O})$ onto $\mathfrak{Z}(\mathcal{O})$. It follows that

$$\mathfrak{J} \mapsto \mathfrak{N}_{\mathfrak{J}} = \mathfrak{J} \cap \mathfrak{Z}(\mathcal{O}),$$

$$\mathfrak{N} \mapsto \mathfrak{J}_{\mathfrak{N}} = \{a \in \mathfrak{A}(\mathcal{O}) \mid m(aW_f) \in \mathfrak{N} \text{ for all } f \in \dot{\mathcal{D}}(\mathcal{O})\}$$

establishes a 1-1 correspondence between maximal closed *-ideals \mathfrak{J} of $\mathfrak{A}(\mathcal{O})$ and maximal closed *-ideals \mathfrak{N} of $\mathfrak{Z}(\mathcal{O})$ (Manuceau et al., *ibid.*, Th. 4.15 and Cor. 4.21). This allows us to locate “on-shell” ideals once Δ is known, just like in the globally hyperbolic case.



The construction of Δ_R, Δ_A in $\dot{\mathcal{D}}(\mathcal{M})$ for proper b -globally hyperbolic space-times (\mathcal{M}, g) was performed by Vasy (2012). Similarly to what happens in AdS (see e.g. Ishibashi-Wald (2004), Dappiaggi-Ferreira (2016,2018)), one has that

$$\square_g \phi = z^2 \square_{\bar{g}} \phi + (2-d)z\bar{g}^{-1}(dz, d\phi),$$

hence one is tempted to use a “Frobenius-like” *Ansatz* for solutions ϕ of the inhomogeneous Klein-Gordon equation

$$(\square_g + \lambda)\phi = f \in \dot{\mathcal{D}}(\mathcal{M})$$

similar to that employed for solving Fuchsian ODE's (see e.g. de Haro-Skenderis-Solodukhin (2001), Hollands-Ishibashi-Marolf (2005)). More precisely, in an open collar $U \supset \mathcal{I}$ such that $\bar{g}^{-1}(dz, dz) = -1$, one writes through Taylor's formula with remainder

$$\phi = z^s \sum_{k=0}^n z^k \phi_k + O(z^{s+k+1}), \quad \phi_k \in \mathcal{C}^\infty(\mathcal{I})$$

and exploits the fact that $f = o(z^\infty)$ together with the formula below (holding in U for all $\varphi \in \mathcal{C}^\infty(\overline{\mathcal{M}})$)

$$(\square_g + \lambda)(z^s \varphi) = z^{s+2} \square_{\bar{g}} \varphi + (2-d+2s)z^{s+1} \bar{g}^{-1}(dz, d\varphi) + sz^{s+1} (\square_{\bar{g}} z) \varphi + (\lambda + (d-1)s - s^2) z^s \varphi,$$

entailing that we must have $s^2 + (1-d)s - \lambda = 0$, therefore

$$s = s_{\pm} = \frac{d-1}{2} \pm \sqrt{\frac{(d-1)^2}{4} + \lambda}.$$



A few extra requirements on s are in order:

- We want our solutions ϕ to be real, hence s must be real $\Rightarrow \lambda \geq -\frac{(n-1)^2}{4}$ (Breitenlohner-Freedman bound);
- We want $\phi \in \mathcal{C}^\infty(\overline{\mathcal{M}})$, hence $s \geq 0$. Since

$$\begin{cases} \lambda > 0 & \Rightarrow s_- < 0 < s_+ \\ \lambda = 0 & \Rightarrow s_- = 0 \\ -\frac{(d-1)^2}{4} < \lambda < 0 & \Rightarrow 0 < s_- < s_+ \\ \lambda = -\frac{(d-1)^2}{4} & \Rightarrow s_- = s_+ = \frac{d-1}{2} \end{cases}$$

one has to exclude s_- if $\lambda > 0$. Both s_- and s_+ are admissible if $\lambda \leq 0$. Roughly, solutions with $s = s_+$ corresponds to ϕ_0 providing “Dirichlet” boundary conditions at \mathcal{I} (= Friedrichs extension of the spatial part of $\square_g + \lambda$ if g static (Ishibashi-Wald *ibid.*, see also N. Drago’s talk)), whereas $s = s_-$ corresponds to ϕ_0 providing “Neumann” boundary conditions at \mathcal{I} .

- The ϕ_k ’s for $k > 0$ are completely determined from ϕ_0 through recursion relations if $\lambda > 0$ or $\lambda \leq 0$ and $s_+ - s_-$ is not an integer. Otherwise, one has to add terms proportional to $z^k \log(z)$ for k starting at the order where the recursion relations break down if one wants to consider “oblique” = “Robin” boundary conditions.

Therefore, if $-\frac{(d-1)^2}{4} \leq \lambda \leq 0$, Δ_R and Δ_A are not unique! Different choices of boundary conditions do lead to the same restrictions of Δ_R, Δ_A to globally hyperbolic \mathcal{O} , though. Through appropriate energy estimates, Vasy managed to show that the above Ansatz does lead to solutions of $(\square_g + \lambda)\phi = f \in \dot{\mathcal{D}}(\mathcal{M})$ with zero Cauchy data in the far past (resp. far future, by time reflection).



THE HOLOGRAPHIC HADAMARD CONDITION

Vasy also analyzed the **propagation of singularities** of solutions of $(\square_g + \lambda)\phi = f \in \dot{\mathcal{D}}(\mathcal{M})$ (see as well Wrochna (2017)). Since the principal symbol of $\square_g + \lambda$ **vanishes identically at \mathcal{I}** , standard propagation of singularities doesn't apply. The idea is to replace $\text{WF}(\phi)$ with a subset $\text{WF}_b(\phi)$ (= the **b -wave front set** of ϕ) of the so-called **compressed cotangent bundle** $\dot{T}^*\overline{\mathcal{M}}$, which essentially consists in identifying points $(p, \xi), (p, \xi') \in T^*_{\mathcal{I}}\overline{\mathcal{M}}$ whenever

$$\xi - \xi' = \alpha dz(p)$$

for some $\alpha \in \mathbb{R}$ and keeping $T^*_{\mathcal{M}}\overline{\mathcal{M}} \cong T^*\mathcal{M}$ unchanged. This accounts for the **reflection** of bicharacteristics. Vasy essentially states that

$$(\square_g + \lambda)\phi \in \dot{\mathcal{D}}(\mathcal{M}) \Rightarrow \text{WF}_b(\phi) \subset \dot{\Sigma} = \{[(x, \xi)] \in \dot{T}^*\overline{\mathcal{M}} \setminus 0 \mid \bar{g}^{-1}(\xi, \xi) = 0\}.$$

Moreover, in this case $\text{WF}_b(\phi)$ is a union of (lifts to $\dot{T}^*\overline{\mathcal{M}}$ of) inextendible **broken null geodesics** γ in $\overline{\mathcal{M}}$ whose only breaking points are in \mathcal{I} and the corresponding discontinuity jumps of $\bar{g}^\sharp \dot{\gamma}$ are proportional to $dz \circ \gamma$. b -global hyperbolicity ensures that the set of breaking points is discrete.

Vasy's results were later employed by Wrochna (2017) to construct Hadamard two-point functions ω_2 **on (\mathcal{M}, \bar{g})** and **restrict them to (\mathcal{I}, g_0)** . It can be shown that $\text{WF}(\omega|_{\mathcal{I} \times \mathcal{I}})$ consists of points $(p, q; \xi, \eta) \in T^*(\mathcal{I} \times \mathcal{I}) \setminus 0$ such that there is a null geodesic segment $\gamma : [0, 1] \rightarrow \overline{\mathcal{M}}$ (possibly broken if $\gamma([0, 1]) \cap \mathcal{M} \neq \emptyset$) such that

$$\begin{aligned} \gamma(0) = p, \gamma(1) = q \text{ and} \\ \dot{\gamma}(0) = \xi + \alpha dz, \dot{\gamma}(1) = -\eta + \beta dz \text{ for some } \alpha, \beta \in \mathbb{R}. \end{aligned}$$



DECODING THE HOLOGRAM

The boundary two-point function $\omega|_{\mathcal{I} \times \mathcal{I}}$ has some new features when compared to Hadamard two-point functions:

- Singular directions may be **timelike** – expected from non-canonical, conformally invariant generalized free fields which arise as boundary duals of Klein-Gordon fields on AdS;
- Due to **gravitational time delay** in AAdS space-times, there may be pairs of (timelike related) singular points in ESU_{d-1} which **cannot** be connected by a boundary null geodesic because bulk null geodesics are “delayed” with respect to the boundary. **This cannot happen in AdS!** \Rightarrow **holographic data** about bulk geometry is encoded **here!**

There are a few possible ways to recover the bulk geometry of an AAdS space-time satisfying the Einstein equations from $WF(\omega|_{\mathcal{I} \times \mathcal{I}})$:

- **Light observation sets** – bulk points $p \in \mathcal{M}$ are completely determined by $\partial I^+(p, \overline{\mathcal{M}}) \cap \mathcal{I}$ (= **future light observation set** of p) or $\partial I^-(p, \overline{\mathcal{M}}) \cap \mathcal{I}$ (= **past light observation set** of p). Manifold and causal structure of (\mathcal{M}, g) can be recovered from future or past light observation sets if p is not conjugate to any point in them (Engelhardt-Horowitz (2017)). Recently, Hintz and Uhlmann (2018) showed that one needs **only a small piece** of the future or past light observation sets in order to perform the same reconstruction.
- **Fefferman-Graham boundary data** – gravitational time delay comes at leading order from the rescaled electric Weyl tensor E at \mathcal{I} (Woolgar (1994), Page-Sorkin-Woolgar *ibid.*), which together with g_0 allow one to rebuild the bulk metric near \mathcal{I} as a formal power series in z and (possibly) $z \log(z)$ (Fefferman-Graham (1985,2012), Graham (2000)). Kichenassamy (2004) showed that if g_0 and E are **real analytic**, then this series has a **positive** radius of convergence.



CODA

Taking stock:

- Starting from a sufficiently general and intrinsic definition of causal wedges, we were able to define algebraic holography for any b -globally hyperbolic space-time.
- We provided a very simple example in the form of the Weyl system of (off-shell) C^* -algebras for the Klein-Gordon field. Going on shell allows one to identify field solutions with boundary data, in the spirit of AdS/CFT duality;
- States satisfying Wrochna's holographic Hadamard condition, when restricted to the boundary, encode information about the bulk geometry, which in principle can be recovered in a number of ways.

Future challenges:

- Complete analysis of on-shell quotient of off-shell Weyl algebras, compare with Fredenhagen-Sommer (2006) = **universal construction** of observable nets over non-globally hyperbolic space-times;
- “Holographic” perturbation theory (Dütsch-Rehren (2011)) \Rightarrow extend to AAdS space-times, rewrite in functional language;
- How do we know which boundary subsets are “admissible” light observation (sub)sets for bulk points?
- How to recover rescaled electric Weyl tensor at \mathcal{I} from gravitational time delay?
 \Rightarrow AdS/CFT as an **inverse problem**





Thanks a lot for your attention!