Construction of Haag-Kastler nets for factorizing S-matrices with poles

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Towards more 2d QFTs

Construct Haag-Kastler nets for integrable models for scalar factorizing S-matrices with **poles** (bound states).
Massive, **non-perturbative, interacting** quantum field theories in $d = 2$.

Methods and results

Take the conjectured S-matrix with **poles** as an input, construct first **observables localized in wedges**, then prove the existence of local observables indirectly.

- **Observables in wedge**: $\widetilde{\phi}(\xi) = z^\dagger(\xi) + \chi(\xi) + z(\xi)$ (c.f. Lechner '08, $\phi(f) = z^\dagger(f^+) + z(f^+)$ for S-matrix without poles).
- Observables in double cones by intersection.

Duality, solitons, bound states, quantum groups...

Overview of the strategy

- Haag-Kastler net $(\{\mathcal{A}(O)\}, U, \Omega)$: local observables $\mathcal{A}(O)$, spacetime symmetry U and the vacuum Ω.
- Wedge-algebras first: construct $\mathcal{A}(W_{\mathrm{R}}), U, \Omega$ from **wedge-local fields**, then take the intersection

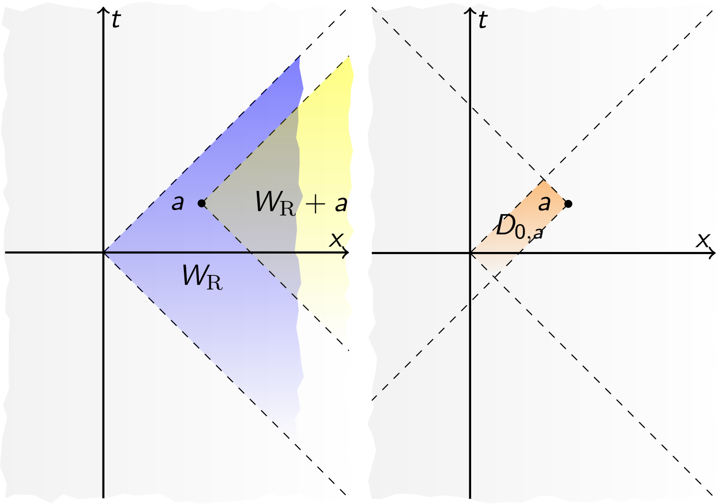
$$\mathcal{A}(D_{a,b}) = U(a)\mathcal{A}(W_{\mathrm{R}})U(a)^* \cap U(b)\mathcal{A}(W_{\mathrm{R}})'U(b)^*$$

  The intersection is large enough if modular nuclearity or wedge-splitting holds.
- Wedge-local observables: $\widetilde{\phi}, \widetilde{\phi}'$ such that $[e^{i\widetilde{\phi}(\xi)}, e^{i\widetilde{\phi}'(\eta)}] = 0$.

Examples: scalar analytic factorizing S-matrix (Lechner '08), twisting by inner symmetry (T. '14), diagonal S-matrix (Alazzawi-Lechner '17)...

More example? **S-matrices with poles**.

Standard wedge and double cone



Analytic factorizing S-matrix

- Pointlike fields are hard. Larger regions contain better observables.
- **Wedge**: $W_{\mathrm{R}/\mathrm{L}} := \{(t,x) : x > \pm|t|\}$.

Wedge-local fields in integrable models (Schroer, Lechner)

- S: factorizing S-matrix (**without poles**).
- $z^\dagger, z$: Zamolodchikov-Faddeev algebra (creation and annihilation operators defined on S-**symmetric Fock space**).
- $\phi(f) = z^\dagger(f^+) + z(f^+)$, $\operatorname{supp} f \subset W_{\mathrm{L}}$, is localized in W_{L}.

The full QFT

- The observables $\mathcal{A}(W_{\mathrm{L}})$ in W_{L} are generated by $\phi(f)$.
- For diamonds $D_{a,b}$, define $\mathcal{A}(D_{a,b}) = \mathcal{A}(W_{\mathrm{L}} + a) \cap \mathcal{A}(W_{\mathrm{R}} + b)$.
- Examine the **boost operator** to show the existence of local operators (**modular nuclearity** (Buchholz, D'antoni, Longo, Lechner)).

Wedge observables for analytic S-matrix

- Input: **analytic** function $S : \mathbb{R} + i(0, \pi) \to \mathbb{C}$,

$$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i), \quad \theta \in \mathbb{R}.$$

- S-symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-**symmetric** functions:

$$\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

- S-symmetrized creation and annihilation operators (ZF-algebra): $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- **Wedge-local field** (Lechner '03): $\phi(f) = z^\dagger(f^+) + z(J_1 f^-)$,

$$f^\pm(\theta) = \int dx\, e^{\pm i x \cdot p(\theta)} f(x), \quad p(\theta) = (m\cosh\theta, m\cosh\theta),$$

J_1 is the one-particle CPT operator, $\phi'(g) = J\phi(g_j)J$, $g_j(x) = \overline{g(-x)}$. If $\mathrm{supp}\, f \subset W_{\mathrm{L}}, \mathrm{supp}\, g \subset W_{\mathrm{R}}$, then $[e^{i\phi(f)}, e^{i\phi'(g)}] = 0$.

If S has a pole:

$$[\phi(f), \phi'(g)]\Psi_1(\theta_1) =$$
$$-\int d\theta\, (f^+(\theta)g^-(\theta)S(\theta_1 - \theta) - f^+(\theta + \pi i)g^-(\theta + \pi i)S(\theta_1 - \theta + \pi i))$$
$$\times \Psi_1(\theta_1)$$

obtains the **residue** of S and does not vanish.

- Example (the Bullough-Dodd model): poles at $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, residues $-R, R$

$$S_\varepsilon(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)} \cdot \frac{\tanh \frac{1}{2}\left(\theta - \frac{(1-\varepsilon)\pi}{3}\right)}{\tanh \frac{1}{2}\left(\theta + \frac{(1-\varepsilon)\pi i}{3}\right)} \frac{\tanh \frac{1}{2}\left(\theta - \frac{(1+\varepsilon)\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta + \frac{(1+\varepsilon)\pi i}{3}\right)},$$

where $0 < \varepsilon < \frac{1}{2}$. $S_\varepsilon(\theta) = S_\varepsilon\left(\theta + \frac{\pi i}{3}\right) S_\varepsilon\left(\theta - \frac{\pi i}{3}\right)$.

New wedge-local field?

The bound state operator

S: two-particle S-matrix, poles $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, $S(\theta) = S\left(\theta + \frac{\pi i}{3}\right) S\left(\theta - \frac{\pi i}{3}\right)$

P_n: S-symmetrization, $\mathcal{H} = \bigoplus P_n \mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$\mathrm{Dom}(\chi_1(\xi)) : \text{ to be defined}$$

$$(\chi_1(\xi))\Psi_1(\theta) := \sqrt{2\pi|R|}\xi\left(\theta + \frac{\pi i}{3}\right)\Psi_1\left(\theta - \frac{\pi i}{3}\right), R = \mathrm{Res}_{\zeta=\frac{2\pi i}{3}} S(\zeta)$$

New observables :

$$\chi(\xi) := \bigoplus \chi_n(\xi), \qquad \chi_n(\xi) = nP_n\left(\chi_1(\xi)\otimes \mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)P_n,$$

$$\widetilde{\phi}(\xi) := \phi(\xi) + \chi(\xi) \qquad (= z^\dagger(\xi) + \chi(\xi) + z(\xi)),$$

$$\widetilde{\phi}'(\eta) := J\widetilde{\phi}(J_1\eta)J, \qquad \chi'(\eta) = J\chi(J_1\eta)J.$$

Theorem (Cadamuro-T. arXiv:1502.01313)

ξ: L^2 bounded analytic in $\mathbb{R} + i(0,\pi)$ "real", η: L^2 bounded analytic in $\mathbb{R} + i(-\pi, 0)$ "real", then $\langle\widetilde{\phi}(\xi)\Phi, \widetilde{\phi}'(\eta)\Psi\rangle = \langle\widetilde{\phi}'(\eta)\Phi, \widetilde{\phi}(\xi)\Psi\rangle$ on a dense domain.

The one-particle bound state operator

- $\xi(\zeta)$: analytic in $\mathbb{R} + i(0,\pi)$, $\overline{\xi(\theta+\pi i)} = \xi(\theta)$ ("real").
- $\mathcal{H}_1 = L^2(\mathbb{R})$
- $\mathscr{D}_0 = H^2(-\frac{\pi}{3}, \frac{\pi}{3})$: L^2-analytic functions in $\mathbb{R} + i(-\frac{\pi}{3}, \frac{\pi}{3})$
- $(\chi_1(\xi))\Psi_1(\theta) := \sqrt{2\pi|R|}\xi(\theta + \frac{\pi i}{3})\Psi_1(\theta - \frac{\pi i}{3})$

What are self-adjoint extensions of $\chi_1(\xi)$?

- **Many extensions**: $n_\pm(\chi_1(\xi)) =$ "half of the zeros" of ξ
- **Choose** $\xi = \xi_0^2$, no zeros, no singular part (Beurling decomposition). Set $\xi_+(\theta + \frac{\pi i}{3}) = \exp\left(\int d\theta\, P(\theta + \frac{2\pi i}{3}) \log|\xi(\theta + \frac{\pi i}{3})|\right)$, where $P(\theta)$ is the Poisson kernel for $\{\zeta : \frac{\pi}{3} < \mathrm{Re}\,\zeta < \frac{2\pi}{3}\}$.
  $\chi_1(\xi) := M_{\xi_+}^* \Delta_1^{\frac{1}{6}} M_{\xi_+}$ is self-adjoint and a natural extension of the above, M_{ξ_+} is unitary, $(\Delta_1^{\frac{1}{6}}\Psi_1)(\theta) = \Psi_1(\theta - \frac{\pi i}{3})$.

Towards proof of strong commutativity

Note: $\chi_1(\xi) = M^*_{\xi_+}\Delta_1^{\frac{1}{6}} M_{\xi_+}$ **have different domains for different** ξ.

$$\chi(\xi) := \bigoplus \chi_n(\xi), \qquad \chi_n(\xi) = nP_n\left(\chi_1(\xi)\otimes \mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)P_n$$
$$= nM^{*\otimes n}_{\xi_+}P_n\left(\Delta_1^{\frac{1}{6}}\otimes \mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)P_n M^{\otimes n}_{\xi_+}.$$

If $\chi(\xi)+\chi'(\eta)$ **is self-adjoint**, then...

- $\chi(\xi)+\chi'(\eta)+cN$ is self-adjoint.
- $T(\xi,\eta) := \widetilde{\phi}(\xi)+\widetilde{\phi}'(\eta)+cN$ is self-adjoint by Kato-Rellich.
  $(= \chi(\xi)+\chi'(\eta)+cN+\phi(\xi)+\phi'(\eta))$
- $[T(\xi,\eta),\widetilde{\phi}(\xi)] = [cN,\widetilde{\phi}(\xi)] = [cN,\phi(\xi)]$ is small,
  $\|\widetilde{\phi}(\xi)\Psi\| \le \|T(\xi,\eta)\Psi\|$.
- use Driessler-Fröhlich theorem (weak $\Rightarrow$ **strong commutativity**:
  $[e^{i\widetilde{\phi}(\xi)}, e^{i\widetilde{\phi}'(\eta)}] = 0$) with $T(\xi,\eta)$ as the reference operator.

Self-adjointness of $\chi_n(\xi) + \chi'_n(\eta)$

We exhibit the proof for

$$\chi_2(\xi) \cong P_2(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1})P_2 \quad \subset \quad \Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + M_S(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_S^* \quad \text{on } \mathcal{H}_1 \otimes \mathcal{H}_1.$$

$\mathrm{Dom} =$ L^2-functions $\Psi(\theta_1, \theta_2)$ analytic in θ_1 in $\mathbb{R} + i(-\frac{\pi i}{3}, 0)$ and s.t. $S(\theta_1 - \theta_2)\Psi(\theta_1, \theta_2)$ analytic in θ_2 in $\mathbb{R} + i(-\frac{\pi i}{3}, 0)$.

Lemma (Kato-Rellich+)

If $A, B, A + B$ are self-adjoint, and assume that there is $\delta > 0$ such that $\mathrm{Re}\,\langle A\Psi, B\Psi\rangle > (\delta - 1)\|A\Psi\|\|B\Psi\|$ for $\Psi \in \mathrm{Dom}(A + B)$. If T is a symmetric operator such that $\mathrm{Dom}(A) \subset \mathrm{Dom}(T)$ and $\|T\Psi\|^2 < \delta\|A\Psi\|^2$, then $A + B + T$ is self-adjoint.

$\Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + \mathbb{1} \otimes \Delta_1^{\frac{1}{6}}$ is self-adjoint. $\mathrm{Dom} =$ L^2-functions $\Psi(\theta_1, \theta_2)$ both analytic in θ_1 and in θ_2.

Self-adjointness of $\chi_n(\xi) + \chi'_n(\eta)$

$C(\theta_2 - \theta_1)$: function with the same poles and zeros as S in $0 < \mathrm{Im}\,(\theta_2 - \theta_1) < \frac{\pi}{3}$, bounded above/below if $-\frac{\pi i}{3} < \mathrm{Im}\,(\theta_2 - \theta_1) < 0$.

Let x be an invertible element in $\mathcal{B}(\mathcal{H})$, A be a self-adjoint operator on $\mathcal{H}$ and assume that Ax^* is densely defined. Then xAx^* is self-adjoint.

$M_C(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + \mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^* = M_C(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1})M_C^* + M_C(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^*$ is self-adjoint. If ε is small enough, and K large enough,

$\Rightarrow M_C^{\frac{k}{K}}(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1})M_C^{\frac{k}{K}*} + M_C(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^*$ is self-adjoint by KR+.

$\Rightarrow \Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + M_C M_{\mathfrak{I}}^{\frac{k}{K}}(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_{\mathfrak{I}}^{\frac{k}{K}*}M_C^*$ is self-adjoint by KR+, where $C(\theta)\mathfrak{I}(\theta) = S(\theta)$.

$\Rightarrow \Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + M_S(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_S^*$ is self-adjoint by KR+.

For a fixed ε, $\chi_{\varepsilon_2,2}(\xi)$ is a perturbation of $\chi_{\varepsilon_1,2}(\xi)$ if $\varepsilon_2 - \varepsilon_1$ is sufficiently small (by intertwining P_{ε_1} and P_{ε_2}).

Similar arguments work for n and $\chi_n(\xi) + \chi'_n(\eta)$ (as long as $\varepsilon_2 < \frac{\pi}{6}$)) (after computations of 30 pages long...).

(sample computations of crossing terms)

$$
\begin{aligned}
&\left\langle M_{C_\varepsilon}^{\frac{k}{K}}(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1}) M_{C_\varepsilon}^{\frac{k}{K}*}\Psi,\ M_{C_\varepsilon}\left(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}}\right) M_{C_\varepsilon}^{*}\Psi \right\rangle \\
&= \int d\boldsymbol{\theta}\ \overline{C_\varepsilon(\theta_2-\theta_1)^{\frac{k}{K}} C_\varepsilon\left(\theta_2-\theta_1-\frac{\pi i}{3}\right)^{\frac{k}{K}}}\ \overline{\Psi\left(\theta_1-\frac{\pi i}{3},\theta_2\right)} \\
&\qquad \times C_\varepsilon(\theta_2-\theta_1)\ \overline{C_\varepsilon\left(\theta_2-\theta_1+\frac{\pi i}{3}\right)}\Psi\left(\theta_1,\theta_2-\frac{\pi i}{3}\right) \\
&= \int d\boldsymbol{\theta}\ C_\varepsilon(\theta_2-\theta_1)\ \overline{\Psi\left(\theta_1-\frac{\pi i}{6},\theta_2-\frac{\pi i}{6}\right)} \\
&\quad \times C_\varepsilon(\theta_2-\theta_1)^{\frac{k}{K}}\ \overline{C_\varepsilon\left(\theta_2-\theta_1-\frac{\pi i}{3}\right)^{\frac{k}{K}}} C_\varepsilon\left(\theta_2-\theta_1+\frac{\pi i}{3}\right) C_\varepsilon(\theta_2-\theta_1)^{-1} \\
&\quad \times \overline{C_\varepsilon(\theta_2-\theta_1)}\Psi\left(\theta_1-\frac{\pi i}{6},\theta_2-\frac{\pi i}{6}\right) + \text{residue}
\end{aligned}
$$

and the factor in the middle has positive real part, the residue is small if ε is small...

Existence of local operators: modular nuclearity

- $\mathcal{N} \subset \mathcal{M}$: inclusion of von Neumann algebras, Ω: cyclic and separating for both, Δ: the modular operator for $\mathcal{M}$.
- **Modular nuclearity** (Buchholz-D'Antoni-Longo): if the map

$$\mathcal{N} \ni A \longmapsto \Delta^{\frac{1}{4}} A\Omega \in \mathcal{H}$$

  is nuclear, then the inclusion $\mathcal{N} \subset \mathcal{M}$ is split.
- (sketch of proof) By assumption, the map

$$\mathcal{N} \ni \mathcal{A} \longmapsto \langle JA\Omega, \cdot\, \Omega\rangle = \langle \Delta^{\frac{1}{2}} A^* \Omega, \cdot\, \Omega\rangle \in \mathcal{M}_*$$

  is nuclear. $\langle JBJ\Omega, A\Omega\rangle = \sum \varphi_{1,n}(A)\varphi_{2,n}(B)$ and one may assume that $\varphi_{k,n}$ are normal. This defines a normal state on $\mathcal{N} \otimes \mathcal{M}'$ which is equivalent to $\mathcal{N} \vee \mathcal{M}'$.
- **Bisognano-Wichmann property**: for $\mathcal{M} = \mathcal{A}(W_{\mathrm{R}})$, Δ^{it} is Lorentz boost (follows if one assumes strong commutativity)

Towards modular nuclearity

$\xi = \xi_0^2$. Strong commutativity + Bisognano-Wichmann (Δ^{it} = boosts).

Consider $\mathcal{A}(W_\mathrm{R} + a) \subset \mathcal{A}(W_\mathrm{R})$, where $a = (0, a_1)$ and the vacuum Ω.
Modular nuclearity: $\mathcal{A}(W_\mathrm{R}) \ni A \mapsto \Delta^{\frac{1}{4}} U(a)A\Omega \in \mathcal{H}$,

$$(\Delta^{\frac{1}{4}} U(a)A\Omega)_n(\boldsymbol{\theta}) = e^{-ia_1 \sum_k \sinh(\theta_k - \frac{\pi i}{2})}(A\Omega)_n \left(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2}\right),$$

which contains a strongly damping factor $e^{-c\sum_k \cosh\theta_k}$.

- (1) Bounded analytic extension. (2) Cauchy integral.

$A \in \mathcal{A}(W_\mathrm{R}) \Longrightarrow A\Omega \in \mathrm{Dom}(\widetilde{\phi}(\xi)) \Longrightarrow (A\Omega)_n \in \mathrm{Dom}(\chi_n(\xi))$, where
$\chi_1(\xi) = M_{\xi_+}\Delta_1^{\frac{1}{6}} M_{\xi_+}^*$.

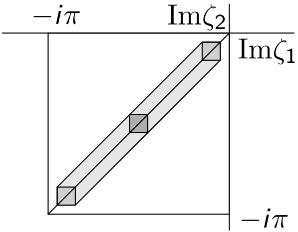
$$\begin{aligned}\langle \chi_n(\xi)(A\Omega)_n, (A\Omega)_n\rangle &= n\|(\Delta_1^{\frac{1}{12}} M_{\xi_+}^* \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}) \cdot (A\Omega)_n\|^2 \\ &= \langle (\widetilde{\phi}(\xi) - \phi(\xi))(A\Omega)_n, (A\Omega)_n\rangle \\ &= \langle (A\xi - \phi(\xi)A\Omega)_n, (A\Omega)_n\rangle \le 3\sqrt{n+1}\|\xi\| \cdot \|A\Omega\|^2\end{aligned}$$

Towards modular nuclearity

Choose a **nice** ξ so that $|\xi_+(\theta + i\lambda)| > |e^{-ia_1 \sinh \frac{\theta}{2}}|$ for $\lambda > \delta > 0$.
$\implies$ Estimate of $(U(\frac{a}{2})A\Omega)_n$ around $\left(\theta_1 - \frac{\pi i}{6}, \theta_2, \cdots, \theta_n\right)$ by $\|A\|$
$\implies$ By S-symmetry and the flat tube theorem, $(U(\frac{a}{2})A\Omega)_n$ has an analytic continuation in all variables in the cube.

- $(A\Omega)_n \in \mathrm{Dom}(\Delta_n^{\frac{1}{2}}) = \mathrm{Dom}(\Delta_1^{\frac{1}{2}\otimes n})$ so it is analytic on the diagonal.
- By $\Delta^{\frac{1}{2}} A\Omega = JA^*\Omega$, $(U(\frac{a}{2})A\Omega)_n$, it is analytic on the lower cube.

$\implies$ Estimate of $(U(\frac{a}{2})A\Omega)_n$ around $\left(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2}\right)$ by $\|A\|$
$\implies$ nuclearity for minimal distance (Alazzawi-Lechner '17).



Summary

- input: two-particle factorizing S-matrix with **poles**
- **new observables** $\widetilde{\phi}(\xi) = \phi(\xi) + \chi(\xi)$
- strong commutativity + modular nuclearity $\Rightarrow$ interacting net

Open problems

- ☑ Bullough-Dodd (scalar)
- □ $Z(N)$-Ising, Sine-Gordon, Gross-Neveu, Toda field theories...
- Equivalence with other constructions (exponential interaction by Hoegh-Krohn): what about other examples?
  - sinh-Gordon (Hoegh-Krohn vs Lechner)
  - Federbush (Ruijsenaars vs T.)
  - sine-Gordon ((Fröhlich-)Park(-Seiler) / Bahns-Rejzner vs ??)
- Relations with CFT (scaling limit, integrable perturbation...)
- quantum group symmetry?