Aharonov-Bohm superselection sectors

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AQFT: Where Operator Algebra meets Microlocal Analysis

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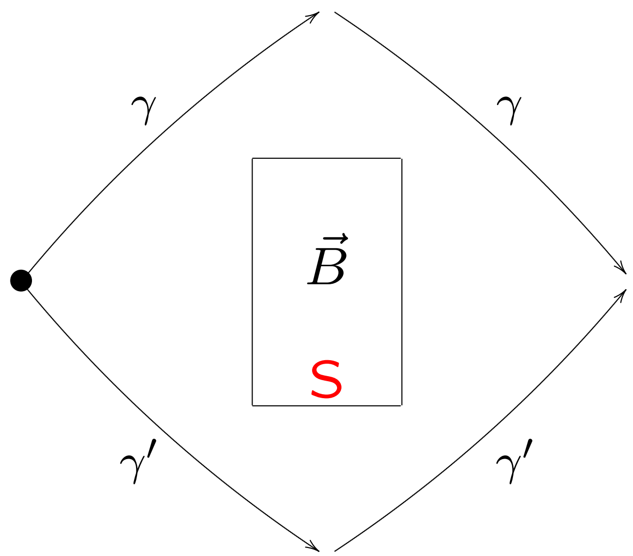
Geometry of the Aharonov-Bohm effect

Electron source



Interference pattern

- $\vec{B}$ is directed towards you $\rightsquigarrow$
- No em field outside the shielded region S (S $\sim \mathbb{R}$, ideally infinite) $\rightsquigarrow$
- the "spacetime" is $M := (\mathbb{R}^3 - \mathsf{S}) \times \mathbb{R}$, $\pi_1(M) = \mathbb{Z}$
- the em potential is $A \in Z^1_{dR}(M)$, $F = dA = 0$ $\rightsquigarrow$
- $A|_o = d\phi_o$, $\phi_o \in C^1(o, \mathbb{R})$, $\forall o \subset M$ a.s.c.

AB assumption: $\phi_o = \phi_o(t)$ for all $o \subset M$, o a.s.c.. $\rightsquigarrow$

If ψ solves the free Schroedinger eq. with supp $(\psi) \subseteq o$, then

$$\psi_o := \psi e^{-i\phi_o}$$

solves the Schroedinger eq. with interaction A.
A (homotopic invariant!) phase shift

$$\exp i \oint_{\overline{\gamma} * \gamma'} A$$

appears for coherent superpositions of states of the type ψ_o, ψ_e, with the loop $\overline{\gamma} * \gamma' \subset$ and homotopic to $o \cup e$.

The shift disappears whenever the experimenter:

- switches off $\vec{B}$ (clearly)
- or makes S "finite" (S $\supset$ S' $\rightsquigarrow M \subset M'$, $\pi_1(M') = 0$)

How geometers describe the wavefunctions ψ_o:
sections $\varsigma : M \to \mathcal{L}$, where $\mathcal{L} \to M$ is the flat line bundle with (l.c.) transition maps

$$\lambda_{hk} \ := \ e^{-i(\phi_{o_h} - \phi_{o_k})} \in \mathbb{U}(1) \quad , \quad o_h \cap o_k \neq \emptyset \ ,$$

[BM]. Actually the following objects are equivalent:

1 - $\mathcal{L} \to M$
2 - $e^{i \oint_\bullet A} : \pi_1(M) \to \mathbb{U}(1)$ ($\leftarrow$ the phase shift)
3 - $A \in Z^1_{dR}(M)$
4 - $\hat{A} \in Z^1(M_{asc}, \mathbb{R})$, $\hat{A}_{o'o} := \phi_{o'}|_o - \phi_o \in \mathbb{R}$, $\forall o \subseteq o'$ a.s.c.

- $1 \Leftrightarrow 2 \Leftarrow 3$ are well-known, [KN]
- $2 \Rightarrow 3$ [Freed], folklore
- $3 \Leftrightarrow 4$ [RRV']. $M_{asc} :=$ base (poset) of a.s.c. subsets

The phase shift can be written in terms of $\widehat{A}$:

- $\ell : [0, 1] \to M$ loop

- *poset approximation* of ℓ: a finite cover

$$p_\ell = \{o_k \in M_{asc}\} \supset \ell \,,$$

  such that there are $o_{k,0}, o_{k,1} \subset o_k$, $o_{k+1,1} = o_{k,0}$, for all $k = 1, \ldots, n$.

$\rightsquigarrow$

$$\oint_\ell A \;=\; \sum_{k=1}^{n} \left\{ \widehat{A}_{o_k o_{k,0}} - \widehat{A}_{o_k o_{k,1}} \right\} \;.$$

Dirac fields interacting with background AB-potentials

- M glob.hyp. 4d spacetime
- $A \in Z^1_{dR}(M)$ $(dA = 0)$
- $\exists$ a Clifford bundle and a Dirac bundle $DM \to M$
- There is a spin connection ∇
- Clifford bundle $\rightsquigarrow$ one can define $\not\nabla$ and $\not A$

Task: construct a Dirac field ψ_{int} such that

$$\{i\not\nabla + \not A - m\}\psi_{int} \; = \; 0 \; .$$

Remark: on any $o \in M_{asc}$ we have $A = d\phi_o \rightsquigarrow$

$$\psi_{int}(e^{i\phi_o}s) \quad , \quad s \in \mathcal{S}_o(DM) \ ,$$

must be a solution of the free Dirac equation $\rightsquigarrow$

Idea [Vas]: take a free Dirac field $\psi : \mathcal{S}(DM) \to B(H)$ ([Dimock]), and for any $o \in M_{asc}$ define

$$\psi_o : \mathcal{S}_o(DM) \to B(H) \quad , \quad \psi_o(s) := \psi(e^{-i\phi_o}s)$$

✓ One has $\psi_o((i\nabla\!\!\!/ + A\!\!\!/ - m)s) = 0$ for all $s \in \mathcal{S}_o(DM)$

- But, $\psi_{o'}(s) = e^{-i\widehat{A}_{o'o}}\psi_o(s)$ for $s \in \mathcal{S}_o(DM)$ and $o \subseteq o'$

- Let $\varsigma \in \mathcal{S}_o(DM \otimes \mathcal{L})$ and $\pi_{o'} : \mathcal{L}|_{o'} \to o' \times \mathbb{C}$ be local charts for all $o' \supseteq o$.
- Set $\varsigma_{o'} := \{id_{DM} \otimes \pi_{o'}\}\varsigma \in \mathcal{S}_{o'}(DM)$.

- $\pi_{o'}\pi_o^{-1} = e^{i\widehat{A}_{o'o}} \Rightarrow \varsigma_{o'} = e^{i\widehat{A}_{o'o}}\varsigma_o \rightsquigarrow$
- $\psi_{o'}(\varsigma_{o'}) = \psi_o(\varsigma_o) \rightsquigarrow$

✓ $\psi_{int} : \mathcal{S}(DM \otimes \mathcal{L}) \to B(H)$, $\psi_{int}(\varsigma) := \psi_o(\varsigma_o) \rightsquigarrow$

Theorem. Given $A \in Z^1_{dR}(M)$ and a free Dirac field ψ, there exists the interacting field

$$\psi_{int} \overset{1-1}{\leftrightarrow} \{\psi_o : \psi_{o'}(s) = e^{-i\widehat{A}_{o'o}}\psi_o(s)\}\,.$$

Interacting Dirac fields vs. sectors

An "interacting net": for all o, set

$$\mathcal{F}(o) := \{\psi_o(s), s \in \mathcal{S}_o(DM)\}'' = \mathcal{F}_{free}(o) \subset B(H)\,,$$

$$\mathcal{R}(o) := \mathcal{F}^{\alpha}(o) \quad , \quad \alpha : \mathbb{U}(1) \to \mathrm{Aut}\mathcal{F}\,.$$

Inclusion maps:

- dictated by $\psi_{o'} = e^{-i\widehat{A}_{o'o}}\psi_o$, $o \subseteq o' \rightsquigarrow$
- $\alpha(e^{-i\widehat{A}_{o'o}}) : \mathcal{F}(o) \to \mathcal{F}(o') \rightsquigarrow$
- $(\mathcal{F}, \alpha(e^{-i\widehat{A}}))$ precosheaf (more general than a net) $\rightsquigarrow$

- $\mathcal{R} = \mathcal{R}_{free}$ is a net
- $\mathcal{R}$ is represented as $\pi = \oplus_{\kappa\in\mathbb{Z}}\pi^{\kappa} : \mathcal{R} \to B(H)$

! π^0 fulfils Borchers [dAH] and Haag duality [V]

- Borchers property $\rightsquigarrow \pi^\kappa \simeq \pi^\kappa_o := \text{ad}\, u^\kappa_o$
- $u^\kappa_o \in \mathcal{U}(\mathcal{F}(o))$ a "phase" of $\psi_o(s)^\kappa$

- Charge transport $\rightsquigarrow z^\kappa_{o'o} \in \mathcal{R}(o')$: $z^\kappa_{o'o}\pi^\kappa_o(\cdot) = \pi^\kappa_{o'}(\cdot) z^\kappa_{o'o}$
- $z^\kappa_{o'o}$ phase of $\psi_{o'}(s')^\kappa \psi_{o'}(s)^{\kappa,*} = \psi_{o'}(s')^\kappa \, e^{i\kappa \hat{A}_{o'o}} \, \psi_o(s)^{\kappa,*}$
- $z^\kappa_{o'o} = u^\kappa_{o'} \, e^{i\kappa \hat{A}_{o'o}} \, u^{\kappa,*}_o$

Theorem. Pairs (π^κ, z^κ) are sectors with s.d.$= 1$ ($=:\mathbf{sect}^1(\mathcal{R})$) in the sense of [BR], with holonomy

$$z^\kappa(p_\ell) := {z^\kappa_{o_n o_{0n}}}^* \cdots z^\kappa_{o_1 o_{11}} \;=\; \exp i\kappa \oint_\ell A \; .$$

- Better: $\mathbf{sect}^1(\mathcal{R}) \ni (\pi, z) \overset{1-1}{\leftrightarrow} (\kappa; A, \psi_{int})$
- Sectors as in [GLRV]: $A = d\varphi \Rightarrow z(p_\ell) \equiv 1$

Non-abelian phases

- ∘ From topology ($\pi_1(M)$ non-Ab):

- $(\pi^\rho, z^\rho) \in \mathbf{sect}^{>1}(\mathcal{R}) \rightsquigarrow \rho : \pi_1(M) \to \mathbb{U}(d)$
- [Barrett] $\rightsquigarrow A^\rho \in \Omega_{flat}(M, \mathfrak{u}(d))$
- $u^\rho_{o,1} \dots u^\rho_{o,d}$ Borchers' isometries, $\pi^\rho_o = \sum_i u^\rho_{o,i} \cdot u^{\rho,*}_{o,i} \rightsquigarrow$

$$z^\rho(p_\ell) \ = \ \sum_{ij} \left(\mathcal{P} \, \mathrm{exp} \, i \oint_\ell A^\rho \right)_{ij} u^\rho_{o,i} u^{\rho,*}_{o,j}$$

! No suitable local primitives ϕ_o of $A^\rho \rightsquigarrow$

! There is no immediate way to construct ψ_{int}

- The test space should be $\mathcal{S}(DM \otimes \mathcal{E}_\rho)$, $\mathcal{E}_\rho := \widehat{M} \times_\rho \mathbb{C}^d$

∘ From gauge symmetry (G cp Lie non-Ab, $G \subseteq \mathbb{U}(n)$):

- $\psi_G : \mathcal{S}(DM \otimes \mathbb{C}^n) \to B(H)$ free field $\leadsto \mathcal{F}_G$, $\mathcal{R}_G$
- $(\pi^\sigma, z^\sigma) \in \mathbf{sect}^{>1}(\mathcal{R}_G)$, $\sigma \in \mathbf{irr}(G)$
- $u^\sigma_{o,i}$ Borchers' isometries, $\pi^\sigma_o = \sum_i u^\sigma_{o,i} \cdot u^{\sigma,*}_{o,i} \leadsto$

$$z^\sigma(p_\ell) \;=\; \sum_{ij} \sigma \left(\mathcal{P} \, \mathrm{exp} \, i \oint_\ell A^\sigma_{\mathfrak{g}} \right)_{ij} u^\sigma_{o,i} u^{\sigma,*}_{o,j}$$

! There is no immediate way to construct $\psi_{G,int}$

Conclusions and outlooks

✓ Given $A \in Z^1_{dR}(M)$, $\exists\ \psi_{int}$ s.t. $\{i\nabla\!\!\!/ + A\!\!\!/ - m\}\psi_{int} = 0$
✓ $\{\kappa; A, \psi_{int}\} \leftrightarrow \mathbf{sect}^1(\mathcal{R})$
✓ $A^\rho \in \Omega_{flat}(M, \mathfrak{u}(d)) \leftrightarrow \mathbf{sect}^{>1}(\mathcal{R})$ $(\leftarrow \pi_1(M)$ n.a.)

✓ $\{\sigma; A^\sigma_{\mathfrak{g}}\} \leftrightarrow \mathbf{sect}(\mathcal{R}_G)$

- Interpretation of A^ρ for $\pi_1(M)$ n.a.
  (e.g. two shielded solenoids $\Rightarrow \pi_1(M) = \mathbb{F}_2$)
- A more complete formulation should involve lgt's
- Non-flat background potential $A \in \Omega(M, \mathbb{R})$,
  we should get connections as in [RRV,CRV]
- Non-relativistic case, relation with [MS]

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