Convergent Star Product: Three Examples

Stefan Waldmann

Institute for Mathematics, University Würzburg, Germany

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☺This will be a non-technical overview ☺

The talk is based on the following joint works:

- Esposito, C., Stapor, P., Waldmann, S.: *Convergence of the Gutt Star Product*. J. Lie Theory **27** (2017), 579–622.
- Kraus, D., Roth, O., Schötz, M., Waldmann, S.: *A Convergent Star Product on the Poincaré Disc*. Preprint. March 2018, 29 pages.
- Schötz, M., Waldmann, S.: *Convergent star products for projective limits of Hilbert spaces*. J. Funct. Anal. **274**.5 (2018), 1381–1423.
- Waldmann, S.: *A Nuclear Weyl Algebra.* J. Geom. Phys. (2014), 10–46.

Plan of the talk

Introduction: Formal and non-formal star products

The set-up for formal star products: either finite-dimensional phase spaces with quite arbitrary geometry or particular infinite-dimensional cases with typically very simple geometry.

- Classical phase space is a symplectic manifold (M, ω), or, more general, a Poisson manifold (M, π): a manifold with a Poisson structure $\pi \in \Gamma^\infty(\Lambda^2 TM)$ with $[\![\pi, \pi]\!] = 0$ (Jacobi identity).
- This allows to define Hamiltonian vector fields $X_f = [\![f, \pi]\!]$ and dynamical systems etc.
- Observable algebra is $\mathscr{C}^\infty(M)$ with Poisson bracket

$$\{f, g\} = -[\![[\![f, \pi]\!], g]\!] = \pi(\mathrm{d} f, \mathrm{d} g)$$

- Formulate classical system using Poisson algebra $\mathscr{C}^\infty(M)$.

Examples of Poisson manifolds:

- Every symplectic manifold (M, ω) is Poisson with $\pi_\omega = \omega^{-1}$. Jacobi identity $[\![\pi, \pi]\!] = 0$ corresponds directly to $\mathrm{d}\,\omega = 0$.
- In particular, cotangent bundles T^*Q with canonical (exact) symplectic form $\omega_0 = \mathrm{d}\,\theta_0$ and $\theta_0 = p_i \,\mathrm{d}\, q^i$.
- Kähler manifolds are particular examples of symplectic manifolds in the intersection of Riemannian, complex and symplectic geometry (use techniques for all three areas. . .)
- Dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with KKS Poisson structure:

$$\{f, g\} = x_i c^i_{k\ell} \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial x_\ell}$$

  This is never symplectic.
- Every manifold admits a Poisson structure $\pi \in \Gamma^\infty_0(\Lambda^2 TM)$ with maximal rank at some point (exercise!).

Quantization: a motivating example...

- Consider $M = \mathbb{R}^2$ with coordinates (q, p).
- Canonical quantization $q \mapsto Q$ and $p \mapsto P = -\mathrm{i}\hbar \frac{\partial}{\partial q}$ on, say, $\mathscr{C}_0^\infty(\mathbb{R})$.
- Ordering necessary to quantize also higher polynomials!
- Standard ordering (most simple choice):

$$q^n p^m \mapsto \varrho_{\mathrm{Std}}(q^n p^m) = Q^n P^m = (-\mathrm{i}\hbar)^m q^n \frac{\partial}{\partial q^m}$$

  and linear extension.
- Extend to general $f \in \mathscr{C}^\infty(\mathbb{R})[p]$

$$\varrho_{\mathrm{Std}}(f) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p^r}\Big|_{p=0} \frac{\partial^r}{\partial q^r}.$$

Main idea: pull back the operator product!

- Standard-ordered star product

$$f \star_{\mathrm{Std}} g = {\varrho_{\mathrm{Std}}}^{-1}(\varrho_{\mathrm{Std}}(f)\,\varrho_{\mathrm{Std}}(g)) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p^r} \frac{\partial^r g}{\partial q^r}$$

  for $f, g \in \mathscr{C}^\infty(\mathbb{R})[p]$

- However: bad behaviour with the *-involutions:

$$\overline{f \star_{\mathrm{Std}} g} \neq \overline{g} \star_{\mathrm{Std}} \overline{f}$$

  Reason is

$$\varrho_{\mathrm{Std}}(\overline{f}) \neq \varrho_{\mathrm{Std}}(f)^\dagger = \varrho_{\mathrm{Std}}(N^2\overline{f}) \quad \text{with} \quad N = \exp\left(\frac{\hbar}{2\mathrm{i}} \frac{\partial^2}{\partial p \partial q}\right)$$

We can repair this: define Weyl ordering and Weyl product

$$\varrho_{\mathrm{Weyl}}(f) = \varrho_{\mathrm{Std}}(Nf) \quad \text{and} \quad f \star_{\mathrm{Weyl}} g = N^{-1}(Nf \star_{\mathrm{Std}} Ng)$$

Then we get

$$\overline{f \star_{\mathrm{Weyl}} g} = \overline{g} \star_{\mathrm{Weyl}} \overline{f} \quad \text{and} \quad \varrho_{\mathrm{Weyl}}(f)^\dagger = \varrho_{\mathrm{Weyl}}(\overline{f}).$$

Properties of $\star_{\mathrm{Std}}$ and $\star_{\mathrm{Weyl}}$:

- $\star$ is associative.
- We have $f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g)$.
- We have $f \star g = fg + \cdots$.
- We have $f \star g - g \star f = \mathrm{i}\hbar\{f, g\} + \cdots$.
- We have $f \star 1 = f = 1 \star f$.
- For $\star_{\mathrm{Weyl}}$ only: $\overline{f \star_{\mathrm{Weyl}} g} = \overline{g} \star_{\mathrm{Weyl}} \overline{f}$.

These formulas work for $f, g \in \mathscr{C}^\infty(\mathbb{R})[p]$ but what for all smooth functions on $\mathbb{R}^2$? Not so easy since too many derivatives!

Reason to use formal power series in $\hbar$: the convergence problem of (formal) star products!

Definition (BFFLS 78)

A formal star product $\star$ on a Poisson manifold (M, π) is an associative $\mathbb{C}[[\hbar]]$-bilinear product for $\mathscr{C}^\infty(M)[[\hbar]]$ such that

$$f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g)$$

with

- $C_0(f, g) = fg$,
- $C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}$,
- $C_r(1, f) = 0 = C_r(f, 1)$ for $r \geq 1$,
- C_r is a bidifferential operator.

Or the same completely algebraically:

Definition

A formal star product $\star$ for a Poisson algebra $\mathscr{A}$ is an associative $\mathbb{C}[[\hbar]]$-bilinear product for $\mathscr{A}[[\hbar]]$ such that

$$a \star b = \sum_{r=0}^{\infty} \hbar^r C_r(a, b)$$

with

- $C_0(a, b) = ab$,
- $C_1(a, b) - C_1(b, a) = \mathrm{i}\{a, b\}$,
- $C_r(\mathbb{1}, a) = 0 = C_r(a, \mathbb{1})$ for $r \geq 1$.

No convergence is required at this point!

The quest for convergence:

- Identify a (small) subalgebra $\mathscr{A} \subseteq \mathscr{C}^\infty(M)$ where the formal series converges for trivial reasons, e.g. it terminates. This is the case in several examples (polynomials!).
- Find a (locally convex) topology on this subalgebra such that the product $\star_\hbar$ is continuous. No canonical way to do this, many choices involved: the art of the story.
- Determine the completion and extend the product by continuity. If the topology is coarse the completion will hopefully contain interesting functions.
- Study the resulting locally convex algebra: dependence on $\hbar$, states, representations, self-adjointness in representations etc.

Example I: Constant Poisson Structures

Constant Poisson structure include the canonical Poisson structure on $\mathbb{R}^{2n}$ but also many examples from field theoretical applications. On polynomial functions one has convergence of the star product of exponential type.

The set-up:

- A constant Poisson tensor on a vector space V^* is equivalent to a Poisson bracket for the polynomial functions of degree -2.
- Polynomial functions correspond to (complexified) symmetric algebra $\mathrm{S}^\bullet_{\mathbb{C}}(V)$. In infinite dimensions $\mathrm{S}^\bullet_{\mathbb{C}}(V)$ is a small part of all polynomials on V^* but we stick to $\mathrm{S}^\bullet_{\mathbb{C}}(V)$.
- Poisson structure of degree -2 on $\mathrm{S}^\bullet_{\mathbb{C}}(V)$ is determined completely by its restriction

$$\{\cdot\,,\cdot\}\colon \mathrm{S}^1_{\mathbb{C}}(V) \times \mathrm{S}^1_{\mathbb{C}}(V) \longrightarrow \mathrm{S}^0_{\mathbb{C}}(V) = \mathbb{C}$$

  thanks to the Leibniz rule. We write $\mathrm{S}(V) = \mathrm{S}_{\mathbb{C}}(V)$ in the following.

- Conversely, every bilinear form $\Lambda\colon V \times V \longrightarrow \mathbb{C}$ determines a unique constant Poisson structure $\{\cdot\,,\,\cdot\}_\Lambda$ by

$$\{v, w\}_\Lambda = \Lambda(v, w) - \Lambda(w, v)$$

- Starting point is a choice of such a Λ, not necessarily antisymmetric.
- Indeed, define $P_\Lambda \in \operatorname{End}(\mathrm{S}^\bullet(V) \otimes \mathrm{S}^\bullet(V))$ by

$$\begin{aligned} &P_\Lambda(v_1 \cdots v_n \otimes w_1 \cdots w_m) \\ &= \sum_{k,l} \Lambda(v_k, w_l) v_1 \cdots \overset{k}{\wedge} \cdots v_n \otimes w_1 \cdots \overset{l}{\wedge} \cdots w_m \end{aligned}$$

to implement the Leibniz rule.

- The Poisson bracket on $\mathrm{S}^\bullet(V)$

$$\{a, b\} = \mu \circ (P_\Lambda - \tau \circ P_\Lambda \circ \tau)(a \otimes b)$$

with μ the symmetric tensor product and canonical flip τ.

- The (formal) Weyl product

$$a \star_{z\Lambda} b = \mu \circ \mathrm{e}^{zP_\Lambda}(a \otimes b)$$

for z formal parameter or $z \in \mathbb{C}$. Here everything converges since we are on the symmetric algebra!

- Symmetric terms in Λ give more flexibility (Wick ordering etc).

The construction of the topology:

- Assume that V is locally convex and Λ is continuous, i.e. have continuous seminorm p with

$$|\Lambda(v, w)| \leq \mathrm{p}(v)\mathrm{p}(w)$$

- Work on the tensor algebra (easier) and induce the topology for $\mathrm{S}^\bullet(V) \subseteq T^\bullet(V)$ from there.
- On each tensor power $V^{\otimes n}$ use the π-topology: for each continuous seminorm p on V define $\mathrm{p}^n = \mathrm{p} \otimes \cdots \otimes \mathrm{p}$ on $V^{\otimes n}$. Then all of these seminorms define the π-topology.

- For $a = \sum_n a_n \in T^\bullet(V)$ define new seminorm

$$\mathrm{p}_R(a) = \sum_{n=0}^{\infty} n!^R \mathrm{p}^n(a_n)$$

  and equip $T^\bullet(V)$ with all these seminorms, with parameter

$$R \geq 0.$$

- Gives a locally convex space denoted by $T^\bullet_R(V)$, inducing a locally convex topology on the symmetric algebra, denoted by $\mathrm{S}^\bullet_R(V)$.

Example: Regular polynomial functionals in Kasia's talk for $V = \mathscr{C}^\infty_0(M)$.

- As alternative in case V is pro-Hilbert: extend the continuous positive semi-definite forms to $V^{\otimes n}$ and form corresponding seminorms instead.
- The resulting topology on each $V^{\otimes n}$ is coarser than the π-topology (more Hilbert-Schmidt like).
- For $T^{\bullet}(V)$ take the orthogonal direct sum with weights as above.

In both situations:

Definition

Let $R \geq \frac{1}{2}$. The completion of the symmetric algebra $\mathrm{S}^{\bullet}_{R}(V)$ equipped with the Weyl product $\star_{z\Lambda}$ is called the locally convex Weyl algebra $\mathcal{W}_{R}(V, \star_{z\Lambda})$.

Theorem

Let $R \geq \frac{1}{2}$.

- *The Weyl product extends continuously:* $\mathcal{W}_R(V, \star_{z\Lambda})$ *is a locally convex algebra (never normable).*
- *It is first countable, iff* V *is first countable.*
- *It is nuclear iff* V *is nuclear.*
- $a \star_{z\Lambda} b$ *converges absolutely for* $a, b \in \mathcal{W}_R(V, \star_{z\Lambda})$ *and gives a entire deformation.*

- For a real vector space V and a complex-valued the complexified Weyl algebra $\mathcal{W}_R(V, \star_{\frac{\mathrm{i}\hbar}{2}\Lambda})$ with real $\hbar$ becomes a locally convex *-algebra with respect to complex conjugation if the symmetric part of Λ is imaginary and the antisymmetric part is real.
- For $R < 1$ the exponential series $\exp(v)$ for $v \in V$ belongs to the completion.
- The construction of $\mathcal{W}_R(V, \star_{z\Lambda})$ depends functorially on the pair (V, Λ).

Some more results:

- In pro-Hilbert case, the construction has a fairly general uniqueness statement: $R = \frac{1}{2}$ is optimal.
- In pro-Hilbert case, we have many states for $\hbar \geq 0$.
- In pro-Hilbert case, we have many automatic self-adjointness results in continuous *-representations (as e.g. GNS for continuous states): up to quadratic Hermitian elements are represented essentially self-adjoint.
- In the nuclear case one can use the results from the pro-Hilbert case (every nuclear space is pro-Hilbert).

Example II: Linear Poisson Structures

Linear Poisson structures on a vector space correspond to Lie algebra structures on the dual space. In infinite dimensions some additional care is necessary.

Gutt star product 1983:

For linear Poisson structure on $\mathfrak{g}^*$ with Lie algebra $\mathfrak{g}$ one gets a star product as follows: First consider $\mathrm{S}^\bullet(\mathfrak{g}) \subseteq \mathrm{Pol}^\bullet(\mathfrak{g}^*)$. Then take PBW isomorphism

$$\mathrm{S}^\bullet(\mathfrak{g}) \ni \xi_1 \vee \cdots \vee \xi_k \ \mapsto \ \frac{(\mathrm{i}\hbar)^k}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)} \in \mathfrak{U}(\mathfrak{g})$$

to universal enveloping algebra of $\mathfrak{g}$. Pull-back of the product turns out to be a star product on $\mathrm{S}^\bullet(\mathfrak{g})$, extends to formal star product on $\mathscr{C}^\infty(\mathfrak{g}^*)$. One has ($z = \mathrm{i}\hbar$)

$$\exp(z\xi) \star_z \exp(z\eta) = \exp(\mathrm{BCH}(z\xi, z\eta)).$$

In finite dimensions one has $\mathrm{S}^\bullet(V) = \mathrm{Pol}^\bullet(V^*)$.

Need formula for the Gutt star product:

$$\xi^k \star_z \eta^\ell = \frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \exp\left(\frac{1}{z}\mathrm{BCH}(zt\xi, sz\eta)\right)$$

$$= \sum_{r=0}^{k+\ell} \frac{1}{r!}\frac{z^{k+\ell}}{z^r} k!\ell! \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\cdots+a_r=k\\ b_1+\cdots+b_r=\ell}} \mathrm{BCH}_{a_i,b_i}(\xi,\eta)\cdots\mathrm{BCH}_{a_r,b_r}(\xi,\eta),$$

where $\mathrm{BCH}_{a,b}(\xi,\eta)$ denotes the homogeneous part of the BCH series containing a times ξ and b times η.
Then need to polarize the formula to get $(\xi_1\cdots\xi_k)\star_z(\eta_1\cdots\eta_\ell)$.
No need to see the details...

Need to estimate the contributions of the polarized version $\mathrm{BCH}_{a,b}(\xi_1, \ldots, \xi_k; \eta_1, \ldots, \eta_\ell)$:

- One aspect comes from the combinatorial coefficients of the BCH series. Explicit formulas are very ugly, but reasonable universal estimates available.
- One aspect comes from iterated Lie brackets. Here we need some continuity of the Lie bracket.

Definition (Asymptotic estimate algebra)

Let $\mathcal{A}$ be a Hausdorff locally convex algebra (not necessarily associative) with $\cdot$ denoting the multiplication, and let p be a continuous seminorm.

- A continuous seminorm q is said to be an asymptotic estimate for p, if

  $$\mathrm{p}\left(w_n(x_1, \ldots, x_n)\right) \leq \mathrm{q}(x_1) \cdots \mathrm{q}(x_n)$$

  for all words $w_n(x_1, \ldots, x_n)$ made out of $n-1$ products of the elements $x_1, \ldots, x_n \in \mathcal{A}$ with arbitrary position of placing brackets.
- A locally convex algebra is said to be an asymptotic estimate algebra (AE-algebra), if every continuous seminorm has an asymptotic estimate.

Example

- Every finite dimensional algebra is AE, we even have submultiplicative norms.
- More general: every Banach (Lie) algebra is AE.
- A submultiplicative seminorm p is its own asymptotic estimate: lmc Lie algebras are AE. Here one has several infinite-dimensional examples of interest.
- Commutators in lmc associative algebras give lmc and hence AE Lie algebras.

Theorem

Let $\mathfrak{g}$ be an AE-Lie algebra and let $R \geq 1$.

- *The Gutt star product $\star_z$ is continuous with respect to the S_R-topology for every $z \in \mathbb{K}$.*
- *The completion $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ becomes a locally convex Hopf algebra with respect to the Gutt star product and the undeformed coproduct, antipode, and counit.*
- *The Gutt star product is convergent as series in $z \in \mathbb{K}$.*
- *The construction is functorial for continuous Lie algebra homomorphisms.*

Theorem

Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra. Then all the above statements hold for all $R \geq 1$ and for the projective limit $R \to 1^-$.

Example III: Poincaré Disc

The Poincaré disc $\mathbb{D}$ is topologically still trivial but carries a curved Kähler structure allowing for a Wick type star product. Phase space reduction allows to move the Wick star product on $\mathbb{C}^2$ to $\mathbb{D}$ and use convergence results for the functions on $\mathbb{D}$ arising from polynomials on $\mathbb{C}^2$.

Consider $\mathbb{C}^2$ with symplectic structure $\frac{\mathrm{i}}{2}(\mathrm{d}\,z^0 \wedge \mathrm{d}\,\overline{z}^0 + z^1 \wedge \mathrm{d}\,\overline{z}^1)$ and $C \subseteq \mathbb{C}^2$ defined by

$$Z = \left\{ z \in \mathbb{C}^2 \;\middle|\; h(z) = |z^0| - |z^1| > 0 \right\}$$

Then $U(1)$ acts on $\mathbb{C}^2$ in a Hamiltonian way with Hamiltonian h and leaves Z invariant. The Poincaré disc is then

$$\mathbb{D} = Z / U(1)$$

as Marsden-Weinstein reduced symplectic manifold. This turns out to be Kähler with the usual complex structure and the usual Kähler metric of constant negative curvature.

Construction of the star product on $\mathbb{D}$:

- Take the canonical Wick star product $\star$ on $\mathbb{C}^2$ using the above pseudo Kähler structure.
- The $U(1)$-invariant polynomials form a subalgebra.
- Divide the $U(1)$-invariant polynomials by the (two-sided) $\star$-ideal generated by the polynomial $h - 1$.
- Gives polynomial-like functions on the disc with very explicit (though complicated) formulas for $\star_\hbar$.
- Everything works for $\hbar \in \mathbb{C}$. However, only for

$$\hbar \in \mathbb{C} \setminus \{-\frac{1}{2n} | n \in \mathbb{N}\}$$

  the quotient algebra stays infinite-dimensional.
- We also have to exclude $\hbar = 0$ as a boundary point to get good behaviour: allowed values of $\hbar$ are then called $H \subseteq \mathbb{C}$.

Theorem

For $\hbar \in H$ one has:

- *The quotient algebra inherits the $\mathrm{S}_{R=\frac{1}{2}}$-topology of the Wick star product on polynomials on $\mathbb{C}^2$.*
- *The completion of the polynomial-like functions on $\mathbb{D}$ can be explicitly described as all holomorphic functions on a certain Stein manifold extension of $\mathbb{D} \times \mathbb{D}$. The $\mathrm{S}_{R=\frac{1}{2}}$-topology then becomes the locally uniform topology.*
- *The star product $\star_\hbar$ on $\mathbb{D}$ depends holomorphically on $\hbar \in H$ with simple poles at $\hbar = -\frac{1}{2n}$ with $n \in \mathbb{N}$.*

Theorem

For $\hbar > 0$ one has:

- *The complex conjugation is a continuous *-involution.*
- *Every classical state of $\mathscr{C}^\infty(\mathbb{D})$ is positive for $\star_\hbar$ without further corrections.*
- *We have automatic essential self-adjointness in every continuous *-representation of linear and of semi-bounded quadratic elements in the algebra (images of quadratic/quartic $U(1)$-invariant polynomials on $\mathbb{C}^2$).*
- *Every continuous *-representation carries a unitary $SU(1,1)$-representation for which it is covariant. The classical momentum map for the $SU(1,1)$-action on $\mathbb{D}$ yields the (self-adjoint) generators of the $SU(1,1)$-representation.*

Some remarks:

- There are higher-dimensional analogs starting with $\mathbb{C}^{n+1}$ as well.
- Also for the sphere $\mathbb{S}^2 = \mathbb{CP}^1$ and the higher $\mathbb{CP}^n$ one has explicit star products for which the convergence can be studied along these lines. However, in this case the poles are at the positive $\hbar = \frac{1}{2n}$. The deformation is still holomorphic in $\hbar$ but for $\hbar > 0$ one has not enough positive functionals to separate algebra elements. Hence physically not very reasonable.

Outlook

- More examples? Here the hope is to have similar approaches for other Hermitian symmetric spaces of non-compact type.
- More detailed study in infinite-dimensional cases: both constant and linear Poisson structures...
- More infinite-dimensional examples?
- Beyond locally convex algebras: separately continuous products, bornological context instead of locally convex, etc.?