Extremal Behaviour in Sectional Matrices

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joint work with Anna Maria Bigatti[1] and Michele Torielli[2]

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From Wikipedia I

A Computer Algebra System

is a software program that allows computation over mathematical expressions in a way which is similar to the traditional manual computations of mathematicians and scientists.

The development of the computer algebra systems started in the second half of the 20th century and this discipline is called "computer algebra" or "symbolic computation".

Computer algebra systems may be divided in two classes:

- The specialized ones are devoted to a specific part of mathematics, such as number theory, group theory, etc. [CoCoA, Macaulay2, Singular, ...]
- General purpose computer algebra systems aim to be useful to a user working in any scientific field that requires manipulation of mathematical expressions. [Matlab, Maple, Magma, ...]

From Wikipedia II

A Gröbner basis

is a particular kind of generating set of an ideal in a polynomial ring over a field $K[x_1, \dots, x_n]$.

Gröbner basis computation is one of the main practical tools for solving systems of polynomial equations and computing the images of algebraic varieties under projections or rational maps.

Gröbner basis computation can be seen as a multivariate, non-linear generalization of both Euclidian algorithm for computing polynomial greatest common divisors, and Gaussian elimination for linear systems.

What you need to compute a Gröbner Basis:

- polynomial ring
- monomial ordering
- reduction algorithm

Hilbert Function

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Definition

Let K be a field of characteristic 0. Given a homogeneous ideal I in $P = K[x_1, \ldots, x_n]$, we define the *Hilbert function of I* to be the function

$$H_I(d) := \dim_K(I_d).$$

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Similarly, we define the *Hilbert function of P/I* to be the function

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The Hilbert function is important in computational algebraic geometry, as it is the easiest known way for computing the dimension and the degree of an algebraic variety defined by explicit polynomial equations.

Question

Can the Hilbert function characterize also some of the geometrical behaviour of algebraic variety in the projective space?

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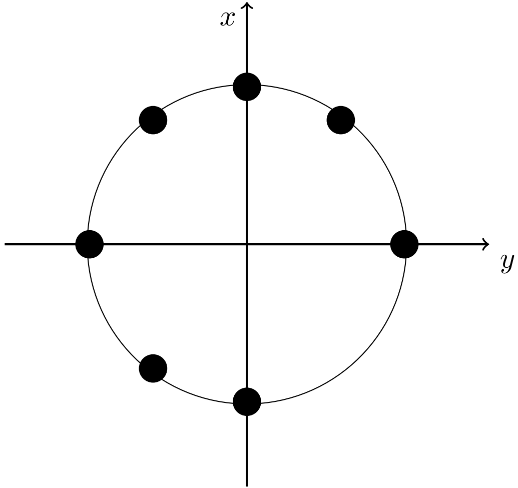
Answer

In general no.

Example in $\mathbb{P}^2$



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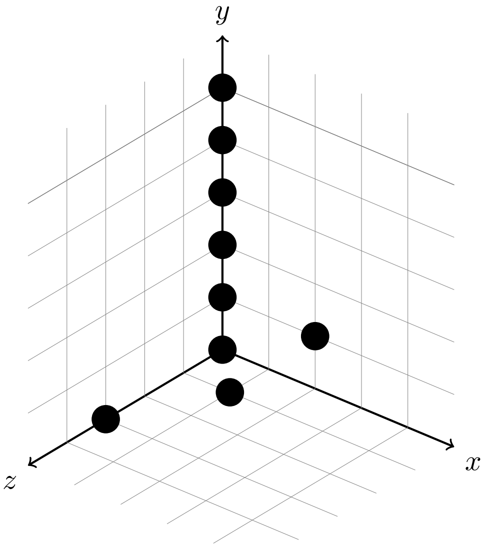

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Answer

Yes, the sectional matrix.

Sectional Matrix

Definition

Let K be a field of characteristic 0. Given a homogeneous ideal I in $P = K[x_1, \ldots, x_n]$, we define the *sectional matrix of* I to be the function

$$\mathcal{M}_I(i,d) := \dim_K(I + (L_1, \ldots, L_{n-i})/(L_1, \ldots, L_{n-i}))_d$$

where $L_1, \ldots, L_{n-i}$ are general linear forms, $i = 1, \ldots, n$ and $d \geq 0$.

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$$\mathcal{M}_{P/I}(i,d) := \dim_K(P/(I + (L_1, \dots, L_{n-i})/(L_1, \dots, L_{n-i})))_d$$

$$= \binom{d+i-1}{i-1} - \mathcal{M}_I(i,d)$$

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Strongly Stable Ideal

Definition

Let I be a homogenous ideal in $P = K[x_1, \dots, x_n]$. We say that I is a *strongly stable ideal* if $T = x_1^{i_1} \cdots x_n^{i_n} \in I$, then $x_i \cdot T/x_j \in I$ for all $i < j \leq \max\{k \mid i_k \neq 0\}$.

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Example

The ideal $I = (x^3, x^2y, xy^2, xyz)$ is not a strongly stable in $\mathbb{Q}[x, y, z]$ because $x \cdot xyz/y = x^2z \notin I$. The ideal $I + (x^2z)$ is strongly stable.

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Remark

If I is a strongly stable ideal, then in the definition of the sectional matrix we can take $L_i = x_{n-i+1}$.

Generic Initial Ideal

Theorem (Galligo '74)

Let I be a homogeneous ideal in $K[x_1, \dots, x_n]$, σ a term-ordering such that $x_1 >_\sigma x_2 >_\sigma \cdots >_\sigma x_n$. Then there exists a Zariski open set $U \subseteq \mathrm{GL}(n)$ and a strongly stable ideal J such that for each $g \in U$, $\mathrm{LT}_\sigma(g(I)) = J$.

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Definition

The strongly stable ideal J given in the previous Theorem will be called the *generic initial ideal with respect to* σ of I and it will be denoted by $\mathrm{gin}_\sigma(I)$. In particular, $\mathrm{gin}_{\mathrm{DegRevLex}}(I)$ is denoted by many authors with $\mathrm{rgin}(I)$.

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Example

Consider the ideal $I = (z^5, xyz^3)$ in $\mathbb{Q}[x, y, z]$, then $\mathrm{rgin}(I) = (x^5, x^4y, x^3y^3)$.

Properties of the Sectional Matrix

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3. $\mathcal{M}_{P/I}(k, d+1) \leq \sum_{i=1}^{k} \mathcal{M}_{P/I}(i, d)$*, for all* k *and* d*. If we have* $\mathcal{M}_{P/I}(k, \delta+1) = \sum_{i=1}^{k} \mathcal{M}_{P/I}(i, \delta)$*, then* $\mathcal{M}_{P/I}(s, d+1) = \sum_{i=1}^{s} \mathcal{M}_{P/I}(i, d)$*, for all* $s \leq k$ *and* $d \geq \delta$.

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4. *If* I *is a strongly stable ideal* ($I = \mathrm{rgin}(I)$), *then* $\mathcal{M}_{P/I}(k, d+1) = \sum_{i=1}^{k} \mathcal{M}_{P/I}(i, d)$, *for all* $d > \delta$ *and for all* k.

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5. *If* $\delta = \mathrm{reg}(I)$, *then* $\mathcal{M}_{P/I}(k, d+1) = \sum_{i=1}^{k} \mathcal{M}_{P/I}(i, d)$, *for all* $d > \delta$ *and for all* k.

Example

Let I be the zero-dimensional homogeneous ideal

$$(x^2+y^2-25z^2, y^4-3xy^2z-4y^3z+12xyz^2-25y^2z^2+100yz^3, xy^3-16xyz^2)$$

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The sectional matrix of I is

$$\begin{array}{rcccccccc} & & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ H_{I+\langle L_1,L_2\rangle}(d) = \mathcal{M}_I(1,d): & & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ H_{I+\langle L_1\rangle}(d) = \mathcal{M}_I(2,d): & & 0 & 0 & 1 & 2 & 5 & 6 & \dots \\ H_I(d) = \mathcal{M}_I(3,d): & & 0 & 0 & 1 & 3 & 8 & 14 & \dots \end{array}$$

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The sectional matrix of P/I is

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Additional Algebraic Properties I

Proposition (Bigatti-P.-Torielli)

Let I be a homogeneous ideal in $P = K[x_1, \dots, x_n]$ and $\delta = \operatorname{reg}(I)$. Suppose that

- $\mathcal{M}_{P/I}(i, \delta) \neq 0$ *but* $\mathcal{M}_{P/I}(i-1, \delta) = 0$ *for some* $i = 2, \dots, n$.

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Then $\dim(P/I) = n - i + 1$ *and* $\deg(P/I) = \mathcal{M}_{P/I}(i, \delta)$.

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Proposition (Bigatti-P.-Torielli)

Let I be a homogeneous ideal in $P = K[x_1, \ldots, x_n]$ such that the minimal generators of I have degree $\leq \delta$. Suppose that

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Then the ideals $(I)_{\leq \delta}$ and $(I)_{\leq \delta+1}$ are saturated and their elements have a GCD of degree $\mathcal{M}_{P/I}(2, \delta)$.

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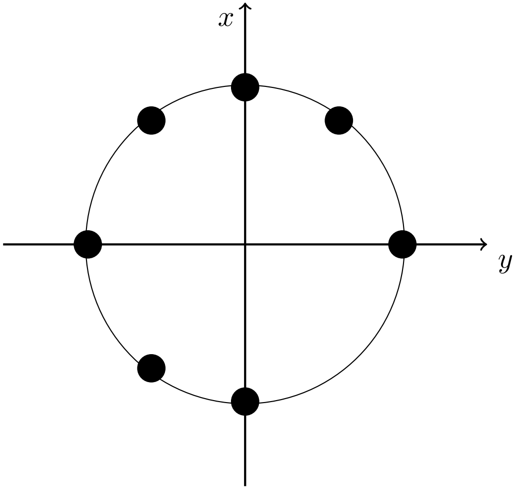
Then the ideal $(I)_{\leq \delta}$ *is a saturated,* $\dim(P/(I)_{\leq \delta}) = (n - i + 1)$, *and of degree* $\mathcal{M}_{P/I}(i, \delta)$ *and it is* δ*-regular. Moreover,* $\dim(P/I) \leq n - i + 1$.

Example in $\mathbb{P}^2$



Consider the points

$$(0,5),(0,-5),(5,0),(-5,0)$$
$$(-3,4),(3,4),(-3,-4)$$

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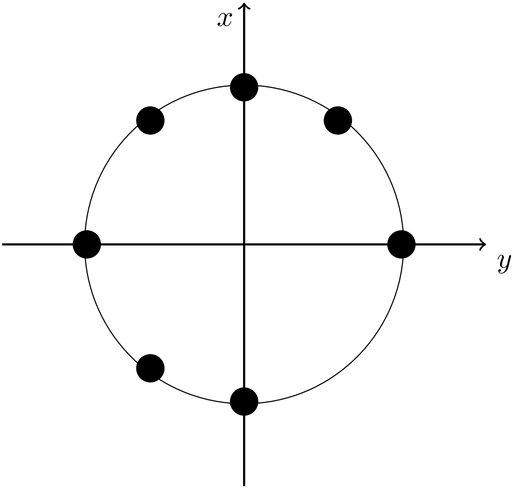

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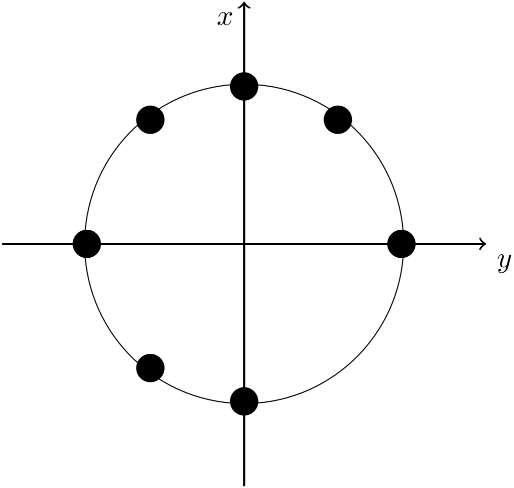

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The sectional matrix is

0	1	2	3	4	5	$\cdots$
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1	2	2	2	0	0	...
1	3	5	7	7	7	...

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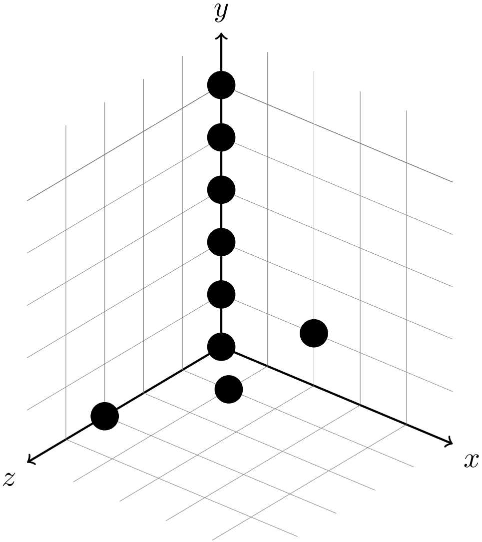

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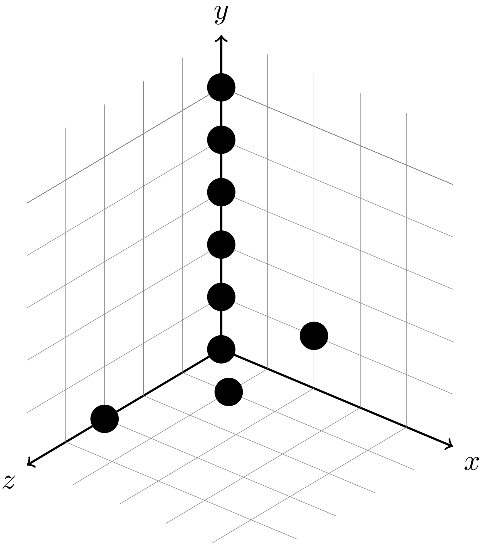

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$$(2,1,0),(1,1,0),(0,0,3)$$

The defining ideal I is such that $\text{rgin}(I) =$

$$(x^2, xy, xz, y^2, yz^2, z^6).$$

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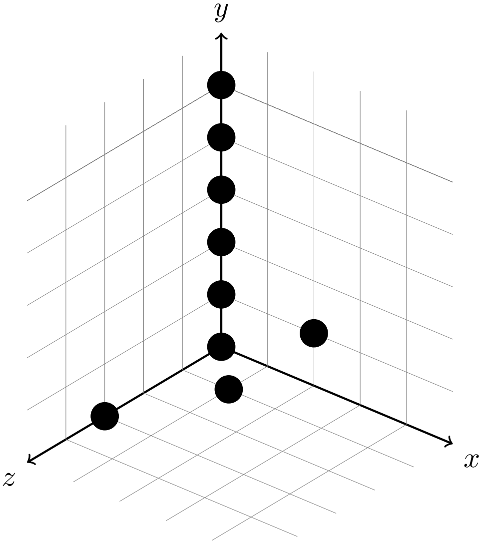

Consider the points

$$(0,0,0),(0,1,0),(0,2,0)$$
$$(0,3,0),(0,4,0),(0,5,0)$$
$$(2,1,0),(1,1,0),(0,0,3)$$

The defining ideal I is such that $\text{rgin}(I) =$

$$(x^2, xy, xz, y^2, yz^2, z^6).$$

The sectional matrix is

0	1	2	3	4	5	6	.
1	1	0	0	0	0	0	.
1	2	0	0	0	0	0	.
1	3	2	1	1	1	0	.
1	4	6	7	8	9	9	.

- Hilbert function
- sectional matrix
- generic initial ideal (gin)
- resolution

(*Example*)

- Hilbert function
- sectional matrix (*Example*)
- generic initial ideal (gin)
- resolution

The End