From Algebraic Geometry to Homological Algebra

Sepehr Jafari

Università Degli Studi di Genova
Dipartimento Di Matematica

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Outline

1. Historical Events
   - Algebraic Geometry

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2. Modern Concepts

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   - Homological Algebra
2. Modern Concepts
3. Some open Problems

The One Who Started All..

"... algebraic geometry and number theory have more open problems than solved ones..."

David Hilbert (1862-1943)

Historical Events in Geometry (400 B.C-1630 A.D)

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- Geometric construction of the roots for $x^2 = ab$.
- Greeks, in particular , used coordinates without, however, reaching the general point of view of Descartes and Fermat.

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- Classification of all cubics with respect to change of coordinates and projections. (Newton)

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- The concept of parametric representation of a curve is Newton's approach to calculus.
- The problem of intersection of two plane curves is tackled by Newton and Leibniz using elimination.

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- With Möbius, Plücker and Cayley, projective geometry received an algebraic basis by suing homogeneous coordinates.

Historical Events in Geometry

"... a mathematician, then, will be defined in what follows as someone who has published the proof of at least one nun-trivial theorem..."

Jean Dieudonné (1906-1992)

Historical Events in Homological Algebra (1857-presenet)

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- Enrico Betti was interested in the study of simply connected manifolds.

Definition

A simply connected manifold is a path-connected topological space which is closed under continues transformations of paths between to given point.

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A family of $n-$dimensional sub manifolds of V, denoted by V_i, are linearly independent if there is no homology connecting them.

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- He defined β_n to be the size of a maximal independent family of sub manifolds of V.

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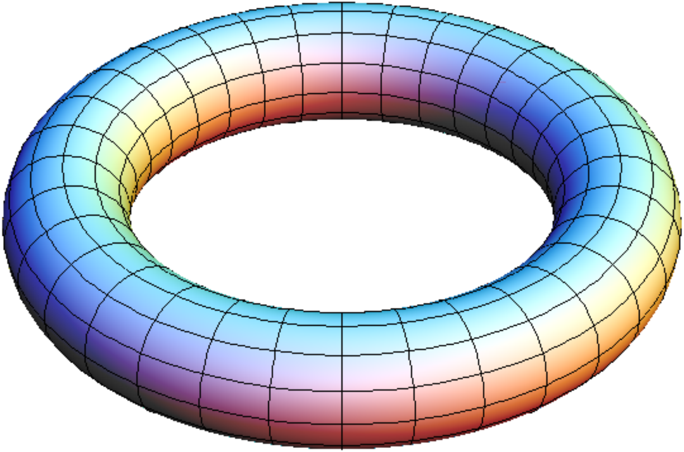

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- Mathematicians started a movement in generalizing Poincaré's ideas, which led to more variations of homology.

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- Eilenberg an Mac Lane defined Hom and Ext for the first time. (1942)

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Sheaves and Spectral Sequences

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- Koszul discovered the algebraic sides of spectral sequences.

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- Projective modules.

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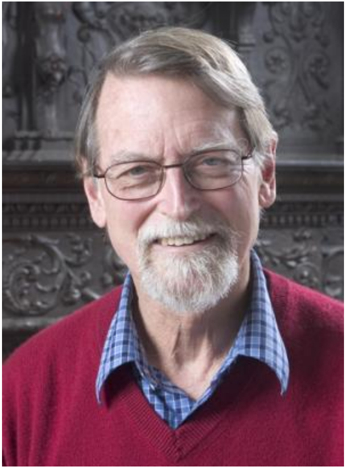
- Concepts of Tor_n and Ext^n for category of modules.
- Projective modules.
- Drived functors.

Historical Events in Geometry

”... the reason for the division is that on the one hand it is necessary to have general culture, on the other hand it is necessary to have deep knowledge of a particular field...”

Guido Castelnuovo (1865-1952)

Historical Events in Geometry



"... algebraic geometry seems to have acquired the reputation of being esoteric, exclusive and very abstract, with adherents who are secretly plotting to take over all the rest of the mathematics. In one respect, this last point is accurate..."

David Mumford (1937-)

Modern Concepts

Definition (Chain Complex)

Let R be a commutative ring. A chain complex $(\mathbf{C}_\bullet, \mathbf{d}_\bullet)$ is a family of $R-$modules $\{C_i\}_{i\in\mathbb{Z}}$ and $R-$homomorphisms $\{d_i\}_{i\in\mathbb{Z}}$ where $d_i \circ d_{i+1} = 0$.

$$\cdots \to C_{i+1} \xrightarrow{d_i+1} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$$

In other words,

$$\operatorname{Img} d_{i+1} \subseteq \operatorname{Ker} d_i$$

For the case of equality, $(\mathbf{C}_\bullet, \mathbf{d}_\bullet)$ is defined to be exact.
The $i-$th homology module of $\mathbf{C}_\bullet$ is defined as

$$\mathrm{H}_i(\mathbf{C}_\bullet) = \operatorname{Ker} d_i / \operatorname{Img} d_{i+1}$$

Graded Setting

- Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring.
  There exists a family $\{S_n\}_{n \in \mathbb{Z}}$ of subgroups of S such that

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Graded Minimal Free Resolution

Let M be a finitely generated graded $S-$ module.

$$\mathbf{F}_\bullet : \cdots \to \bigoplus_j S^{\beta(2,j)}(-j) \xrightarrow{d_2} \bigoplus_j S^{\beta(1,j)}(-j) \xrightarrow{d_1} \bigoplus_j S^{\beta(0,j)}(-j) \xrightarrow{\pi} M \to 0$$

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Theorem (Hilbert's Syzygy Theorem)

For $i > n$, $\mathrm{Syz}_i = 0$.

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Definition

Let M and N be modules and $\mathbf{F}_\bullet$ and $\mathbf{Q}_\bullet$ be a minimal free resolutions of M and N resp. The $i-$th torsion module of M and N is defined as

$$\mathrm{Tor}_i(M, N) = \mathrm{H}_i(\mathbf{F}_\bullet \otimes N) = \mathrm{H}_i(M \otimes \mathbf{Q}_\bullet)$$

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$$\dim_K \text{Tor}_i(K, M)_j = \beta_{i,j}(M)$$

Castelnuovo-Mumford Regularity

Definition

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Linear free resolution:

$$\mathbf{F}_\bullet : \cdots \to S^{\beta_{(2,j+2)}}(-j-2) \xrightarrow{d_2} S^{\beta_{(1,j+1)}}(-j-1) \xrightarrow{d_1} S^{\beta_{(0,j)}}(-j) \xrightarrow{\pi} M \to 0$$

Some Interesting Problems

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- Finding an upper bound for the regularity of product of ideals.

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