Connectivity for dual graphs of projective schemes

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PhD seminars

Università degli Studi di Genova

Outline

Minimal primes

Along these slides, we will denote by S the polynomial ring $k[x_1, x_2, \ldots, x_n]$, for some field k.

Let I be a homogeneous ideal of S and consider the radical of I:

$$\sqrt{I} = \{x \in S \mid x^n \in I, \text{ for some } n \in \mathbb{N}\}.$$

- The radical of I can be written as

$$\sqrt{I} = p_1 \cap p_2 \cap \ldots \cap p_s,$$

  where $p_1, \ldots, p_s$ are distinct prime ideals such that $p_i \not\subseteq p_j$ for all i and j. $p_1, \ldots, p_s$ are called minimal primes of I.
- The height of p_i is the maximum length of a chain of primes contained in p_i:

$$\text{height}(p_i) = \max\{n \mid \exists\, p'_0 \subsetneq \ldots \subsetneq p'_n = p_i, p'_j \text{ prime for all } j\}.$$

  The height of I is the minimum of the heights of the p_i, $i = 1, \ldots, s$.

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Construction of the dual graph

Algebraically

$I \subseteq S = k[x_1, \ldots, x_n]$ homogeneous equidimensional ideal,

$$\sqrt{I} = p_1 \cap \ldots \cap p_s$$

$$\downarrow$$

$$G(I) = (V, E),$$

where $V = \{1, \ldots, s\}$ and $\{i, j\} \in E$ if height$(p_i + p_j) =$height$(I) + 1$.

Geometrically

X equidimensional projective scheme in $\mathbb{P}^{n-1}$,

$$X = X_1 \cup \ldots \cup X_s$$

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Examples: arrangements of lines in $\mathbb{P}^3$

$\mathbb{P}^3$ $\qquad$ $k[x, y, z, t]$

$L_1 : \begin{cases} x = z \\ y = 4t \end{cases}$ $\qquad$ $p_1 = (x - z, y - 4t)$

$L_2 : \begin{cases} x = z \\ y = t \end{cases}$ $\qquad$ $p_2 = (x - z, y - t)$

$L_3 : \begin{cases} x = -y \\ z = -t \end{cases}$ $\qquad$ $p_3 = (x + y, z + t)$

$L_4 : \begin{cases} x = 2z \\ y = 2t \end{cases}$ $\qquad$ $p_4 = (x - 2z, y - 2t)$

$X = L_1 \cup L_2 \cup L_3 \cup L_4$ $\qquad$ $I = p_1 \cap p_2 \cap p_3 \cap p_4$

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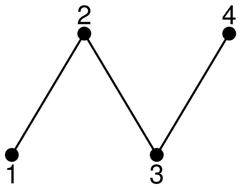
$G(I)$



Connectivity in codimension one

- A noetherian topological space X (in our case a projective scheme) is connected in codimension one if whenever we take a Y such that

$$Y \subset X, \qquad Y \text{ closed }, \qquad \operatorname{codim}(Y, X) > 1,$$

$X \setminus Y$ is connected.

- A graph $G = (V, E)$ is connected if for every couple of vertices $v', v'' \in V$ there exists a finite sequence of vertices

$$v' = v_1, v_2, \ldots, v_r = v''$$

such that for each i, $(v_i, v_{i+1}) \in E$.

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Dimension and depth

- Let S be as before, and let $I \subseteq S$ be a homogeneous ideal. Consider the chains of prime ideals contained in S/I:

$$C : p_0 \subsetneq p_1 \subsetneq \ldots \subsetneq p_{n_C} = S/I$$

The dimension of S/I is the maximum of the lengths n_C of these chains.

- A regular sequence $(a_1, \ldots, a_l) \subseteq S/I$ is an ordered sequence such that:
  1. $a_1 \neq 0$ is a non-zero divisor ($a_1 x \neq 0$ for all $x \neq 0, x \in S/I$);
  2. $(S/I)/(a_1, \ldots, a_i) \neq 0$ and a_i is a non zero-divisor in $(S/I)/(a_1, \ldots, a_{i-1})$, for all $i = 2, \ldots, l$.

The depth of S/I is the maximum length l of a regular sequence

$$(a_1, \ldots, a_l) \subseteq (\overline{x}_1, \ldots, \overline{x}_n).$$

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Cohen-Macaulay rings

The ring S/I is Cohen-Macaulay if

$$\dim(S/I) = \operatorname{depth}(S/I).$$

Some advantages of Cohen-Macaulay rings:

1. All the associated primes of I have the same height.
2. You can quotient a CM ring by a regular element and still get a CM ring of dimension one less.

Theorem (Hartshorne's Connectedness Theorem, 1962)

If S/I_X is Cohen-Macaulay, then $G(I_X)$ is connected.

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If S/I_X is Cohen-Macaulay, then $G(I_X)$ is connected.

For example, if $I_X = (f_1, \ldots, f_c)$ is a complete intersection, i.e.

$$\text{height}(I_X)(= \text{codim}(X)) = c,$$

then S/I_X is Cohen-Macaulay and hence $G(I_X)$ is connected.

Example (Subspace arrangements)

Let

$$X = X_1 \cup X_2 \cup \ldots \cup X_s,$$

where X_i is a projective vector subspace of $\mathbb{P}^{n-1}$ and

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If we can write the associated ideal I_X as

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Consider S/I with the induced usual grading given on polynomials.

- The Hilbert series of S/I is

$$HS_{S/I}(t) = \sum_{i \in \mathbb{N}} \dim_k (S/I)_i t^i \quad \in \quad \mathbb{Z}[[t]]$$

- (Hilbert-Serre) If $d = \dim(S/I)$, then

$$HS_{S/I}(t) = \frac{h_{S/I}(t)}{(1-t)^d},$$

where $h_{S/I}(t)$ is a polynomial with integer coefficients.

- If S/I is Cohen-Macaulay, the Castelnuovo-Mumford regularity can be defined as

$$\operatorname{reg}(S/I) := \deg(h_{S/I}(t)).$$

- The degree (or multiplicity) of S/I is $\deg(S/I) = e(S/I) := h_{S/I}(1)$.

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r-connectivity for graphs

A graph $G = (V, E)$ is r-connected if $|V| > r$ and the removal of a set of $r - 1$ vertices does not disconnect G (1-connected=connected).

Example

the graph is not connected

the graph is connected but not 2-connected

the graph is 2-connected but not 3-connected

the graph is 3-connected but not 4-connected

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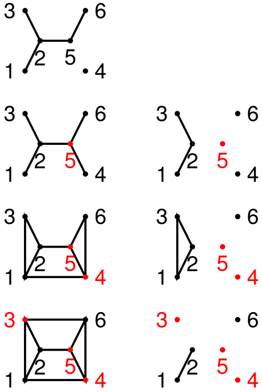

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How can the properties of S/I (complete intersection, height, degree, regularity) be translated into properties of $G(I)$ (number of vertices, connectivity, diameter)?

- (Benedetti-Bolognese-Varbaro 2015) Let S/I be Gorenstein (e.g. complete intersection) of regularity r and

$$I = p_1 \cap \ldots \cap p_s,$$

for some prime ideals $p_1, \ldots, p_s$ ($I = \sqrt{I}$). If

$$\sum_{i \in A} \deg(p_i) \leq r - 1 \text{ for some } A \subseteq [s],$$

then $G \setminus A$ is connected.
In particular, if $deg(p_i) \leq D$ for all i, then $G(I)$ is $\lfloor \frac{r+D-1}{r} \rfloor$-connected.

- Tools: local cohomology and sheaf cohomology, liaison theory.

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$$I = p_1 \cap \ldots \cap p_s,$$

for some prime ideals $p_1, \ldots, p_s$ ($I = \sqrt{I}$). If

$$\sum_{i \in A} \deg(p_i) \leq r - 1 \text{ for some } A \subseteq [s],$$

then $G \setminus A$ is connected.
In particular, if $deg(p_i) \leq D$ for all i, then $G(I)$ is $\left\lfloor \frac{r+D-1}{r} \right\rfloor$-connected.

- Tools: local cohomology and sheaf cohomology, liaison theory.

Question (2)

Does the following relation hold for S/I Cohen-Macaulay and I generated by quadrics:

$$diam(G(I)) \leq height(I)?$$

- (Adiprasito-Benedetti 2014) Yes when I is a monomial ideal.
- Conjectured to be true in general (Benedetti-Varbaro 2015).

Possible approach

Try to prove that if there exists some monomial order $\prec$ such that $in_{\prec}(I)$ is Cohen-Macaulay and quadratic, then

$$\text{diam}(G(I)) \leq \text{diam}(G(in_{\prec}(I))).$$

(Not true in general!)

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r-weighted connectivity

Let G_δ be a graph on the set of vertices $[s] = \{1, 2, \ldots, s\}$, weighted on the vertices and having total weight δ:

$$i \rightsquigarrow \text{ weight } w_i \in \mathbb{N} \text{ and } \sum_{i=1}^{s} w_i = \delta.$$

G_δ is *r*-weighted-connected if given $A \subseteq [s]$ such that $\sum_{i \in A} w_i \leq r - 1$, $G_\delta \setminus A$ is connected.

Example

this graph is 4*-weighted connected (but not* 5*-w.c.).*

Note: A 4-weighted connected graph could be not 2-connected!

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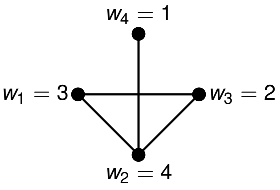

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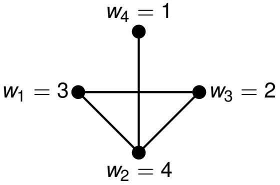

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Outline

Menger's theorem

A graph $G = (V, E)$ is r-connected iff for each couple of vertices v', v'' there exist at least r pairwise disjoint paths from v' to v''.

If $|V| = s$ and G is r-connected, then

$$\operatorname{diam}(G) \leq \left\lfloor \frac{s-2}{r} \right\rfloor + 1$$

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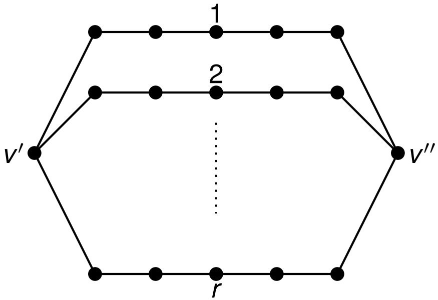

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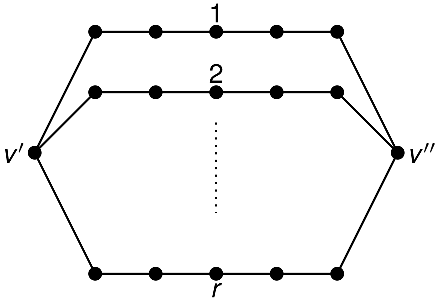

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Proposition

For every couple of vertices $v', v'' \in V$ there exist at least r paths such that for any two of them their intersection is contained in B.

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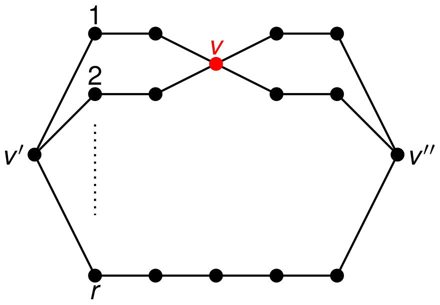

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Bound for the diameter of a r-weighted connected graph

Suppose that the vertices of G_δ are indexed in a non-decreasing order with respect to their weights:

$$w_1 \leq w_2 \leq \ldots \leq w_s.$$

If G_δ is r-weighted connected, for each i, the set of maximum cardinality from which we can remove $i-1$ vertices without disconnecting G is:

$$A_i := \left\{1, 2, \ldots, k : \sum_{j=k-i+2}^{k} w_j \leq r-1\right\}.$$

Let $h = \max\{i \ s.t. \ A_i \neq \emptyset\}$.

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Actual bound and future goals

The diameter of G_δ then satisfies:

$$\operatorname{diam}(G) \leq \left\lfloor \frac{s-2+\sum_{i=2}^{h} b_i}{h} \right\rfloor + 1,$$

where $b_i = |V \setminus A_i|$.

This bound depends on:

- The total weight $\delta =$ the degree of S/I;
- The weight distribution $w_1, \ldots, w_s =$ the degree distribution on the p_i.

Note: The number of weight distributions of the total weight δ is equal to the number of partitions of δ.

Question

Is there a good upper bound which depends only on $r = reg(S/I)$ and δ?

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Example (Complete intersection of 3 quadrics)

Let $I = (f_1, f_2, f_3)$, where $f_1, f_2, f_3 \in (k[x, y, z, t])_2$.
Then $\delta = deg(S/I) = 8$ and $r = reg(S/I) = 3$.
Recall the general bound:

$$diam(G) \leq \lfloor \frac{s - 2 + \sum_{i=2}^{h} b_i}{h} \rfloor + 1$$

s	Partition	h	b_2	b_3	Bound
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