Wave front sets and related topics

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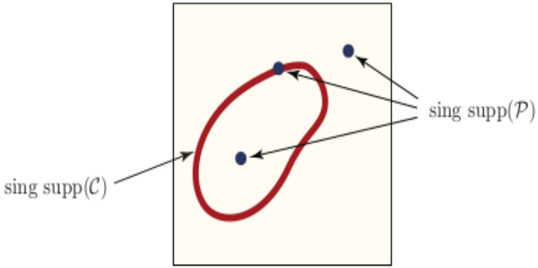
- introduction: a single notion is used for different purposes
- definition(s) of the wave front set
- the resolution of wave front set in time–frequency analysis
- concluding remarks

- One of the tasks in the analysis of cosmological data/images is the extraction of its geometrically distinct constituents.
- This is called *the geometric separation problem*. The ideal aim is to extract two "pure" images from a single image, each one containing features from only one of the two geometric constituents.
- In other words, it is desirable to separate the "pointlike" and the "curvelike" structures.
- Different image processing techniques are proposed to carry out such separation, not only in astronomy, but also in medical imaging, material science, etc.
- A mathematical framework pointing to this, and to a wider range of seemingly different "imaging" problems, is proposed in

D. Donoho and G. Kutyniok, Microlocal Analysis of the Geometric Separation Problem, Comm. Pure Appl. Math. **66 (1)**, 1-47 (2013).

- It is assumed that a distribution f is a superposition of $\mathcal{P}$ and $\mathcal{C}$ ($f = \mathcal{P} + \mathcal{C}$) where $\mathcal{P}$ has only point singularities and $\mathcal{C}$ has only curvilinear singularities.
- The aim is to find a sparse representation of f, i.e. to represent f using relatively few large coefficients.
- In order to achieve the sparsity comparable to the ideal representation, Donoho and Kutyniok proposed two (overcomplete) systems, adapted to the particular type of singularities:
  - radial wavelets - a tight frame with isotropic generating elements,
  - curvelets - a directional tight frame with increasingly anisotropic elements at fine scales.
- The heuristic and intuition behind the estimation strategy of Donoho and Kutyniok is based upon the microlocal point of view.

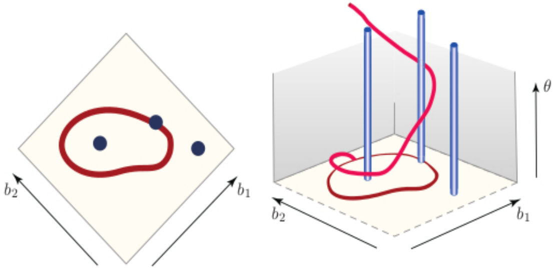


- In the geometric separation setting we have

$$\text{singsupp}(f) = \text{singsupp}(\mathcal{P} + \mathcal{C}) = \text{singsupp}(\mathcal{P}) + \text{singsupp}(\mathcal{C}),$$

where singsupp(f) is the (closure of the) set of points where f is not locally smooth (C^∞).

- To properly separate between pointlike and curvelike singularities the *phase space* indexed by position-orientation pairs (b, θ) $(\theta \in [0, \pi))$ is considered.



- Roughly speaking, $\mathrm{WF}(f)$, the wave front set of f, is the set of position-orientation pairs in the phase space at which f is nonsmoth.
- For example, $\mathrm{WF}(\mathcal{P}) = \mathrm{singsupp}(\mathcal{P}) \times [0, \pi)$, since a point singularity is singular in all directions on its singular support.

- How to resolve $\mathrm{WF}(f)$?
- For $\mathrm{WF}(\mathcal{P})$ this amounts to the resolution of $\mathrm{singsupp}(\mathcal{P})$.
- For that purpose the wavelet transform is an appropriate tool.
- In particular, in 1-D: the coefficients of the wavelet expansion of a signal f are just samples of the continuous wavelet transform at certain points. If f is smooth except at x_0, then both discrete and continuum representations are optimally sparse.

S. Jaffard, Y. Meyer, Wavelet Methods for Pointwise Regularity and Local Oscillations of Functions, AMS, Providence, 1996.

Y. Meyer, Wavelets, Vibrations and Scalings, AMS, Providence, 1998.

S. Mallat, A Wavelet Tour of Signal Processing, The Sparse Way, Academic Press, San Diego, 2009.

- Pointwise smoothness in general does not capture the geometric features of the singularity set of multidimensional functions.
- The wave front set of a curvilinear singularity $\mathrm{WF}(\mathcal{C})$, therefore can not be resolved by "isotropic" tools (such as the standard wavelet transform defined by the means of translations and dilations of appropriate wavelet).
- In other words, translations and dilations may serve well to localize the position of a singularity (of a two-dimensional signal f), but if the singularities propagate along a curve, then its orientation/direction can be resolved by including the additional rotation parameter in the corresponding integral transforms or in the family of analyzing frames.

- Continuous curvelet and shearlet transforms are based on systems of functions which are well localized in space at various scales and with orientation controlled by an additional parameter.
- Candés and Donoho (2005) introduced the *curvelet transform* and proved that its asymptotic behavior when the scaling parameter tends to zero precisely resolves the wavefront set.

E. J. Candés and D. L. Donoho, Continuous curvelet transform: I. Resolution of the wavefront set. Appl. Comput. Harmon. Anal. **19** 162–197 (2005).

- Thereafter curvelets are used to treat the geometric separation problem.

D. Donoho and G. Kutyniok, Microlocal Analysis of the Geometric Separation Problem, Comm. Pure Appl. Math. **66 (1)**, 1-47 (2013).

- Kutyniok and Labate (2009) introduced *shearlet transform*. Its asymptotic behavior (with respect to appropriately chosen analyzing shearlet) when the scaling parameter tends to zero precisely resolves the wavefront set in 2-D.

G. Kutyniok and D. Labate, Resolution of the wavefront set using continuous shearlets, Trans. Amer. Math. Soc. **361 (5)**, 2719-2754 (2009)
and, as editors: Shearlets: Multiscale Analysis for Multivariate Data, Birkhäuser/Springer, 2012.

P. Grohs, Continuous shearlet frames and resolution of the wavefront set, Monatsh. Math. **164 (4)**, 393-426 (2011)

P. Grohs, Shearlets and microlocal analysis, 39-67 *in* Shearlets: Multiscale Analysis for Multivariate Data, Birkhäuser/Springer, 2012.

G. Alberti, S. Dahlke, F. De Mari, E. De Vito, H. Führ, Recent progress in shearlet theory: systematic construction of shearlet dilation groups, characterization of wavefront sets, and new embeddings, 127-160,
in Frames and other bases in abstract and function spaces, Birkhäuser/Springer, 2017.

- An extension to any dimension, based on abstract harmonic analysis is given in

J. Fell, H. Führ, F. Voigtlaender, Resolution of the wavefront set using general continuous wavelet transforms, J. Fourier Anal. Appl. **22 (5)**, 997-1058 (2016)

- Different aspects of the shearlet theory and its applications can be also found in

Dahlke, S., De Mari, F., Grohs, P., Labate, D.,editors: Harmonic and Applied Analysis, Birkhäuser/Springer, 2015.

- The shearlet transform can be related to the wavelet transform by the use of the Radon transform:

Bartolucci, F., De Mari, F., De Vito, E., Odone, F., The Radon transform intertwines wavelets and shearlets, Applied and Computational Harmonic Analysis (article in press), https://doi.org/10.1016/j.acha.2017.12.005

- “Quantum field theory (QFT) on curved spacetime describes quantum fields propagating under the influence of an external gravitational field. The main problem ... is an appropriate formulation of stability.”
  That is, to find a class of states with suitable stability properties...

Brunetti, R.; Fredenhagen, K.; Köhler, M., The Microlocal Spectrum Condition and Wick Polynomials of Free Fields on Curved Spacetimes, Comm. Math. Phys. 180 (3) (1996), 633-652.

Hollands, S., Wald, R.M., Local wick polynomials and time ordered products of quantum fields in curved spacetime. Commun. Math. Phys. 223 (2001), 289 –326.

- For example, the Hadamard states are used in some models since 1960. The global Hadamard condition can be formulated in terms of wavefront sets. This gives rise to a rigorous description of renormalization and complete reformulation of QFT:

Radzikowski, Marek J., Micro-local approach to the Hadamard condition in quantum field theory on curved space-time. Comm. Math. Phys. 179 (3) (1996), 529-553.

Verch, Rainer, Wavefront sets in algebraic quantum field theory. Comm. Math. Phys. 205 (1999), no. 2, 337-367.

Brunetti, R.; Fredenhagen, K., Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds. Comm. Math. Phys. 208 (3) (2000), 623-661.

see also

Pinamonti, N., Conformal generally covariant quantum field theory: the scalar field and its Wick products. Comm. Math. Phys. 288 (3) (2009), 1117-1135.

- It turns out that $\mathcal{D}'(\Gamma)$ the space of distributions having a prescribed closed cone as the set of directions of singularities, is the natural space where QFT takes place.
- In fact, the main use of the wave front set in QFT is to provide a condition for product of distributions, since it provides the precise description of the region of phase space where the product is well defined.
- For the applications in the QFT it is essential that the product should be compatible with the Leibnitz rule. Such product of distributions u and v can be defined if there is no point $(x, \xi) \in \mathrm{WF}(u)$ such that $(x, -\xi) \in \mathrm{WF}(v)$.
- This is applied e.g. to find the value of Feynmann diagram (which contains the product of propagators(distributions)) on the whole phase space by an extension procedure:

Brouder, Christian; Dang, Nguyen Viet; Hélein, Frédéric; A smooth introduction to the wavefront set. J. Phys. A 47 (2014), no. 44, 443001, 30 pp.

- Therefore it is important to understand the corresponding continuity properties of fundamental operations between the propagators. Appropriate topology in that context is only recently introduced.

Dabrowski, Yoann; Brouder, Christian, Functional properties of Hörmander's space of distributions having a specified wavefront set. Comm. Math. Phys. 332 (2014), no. 3, 1345-1380.

Brouder, Christian; Dang, Nguyen Viet; Hélein, Frédéric; A smooth introduction to the wavefront set. J. Phys. A 47 (2014), no. 44, 443001, 30 pp.

Brouder, Christian; Dang, Nguyen Viet; Hélein, Frédéric, Continuity of the fundamental operations on distributions having a specified wave front set (with a counterexample by Semyon Alesker), Studia Mathematica 232 (2016), 201–226

- Next we introduce some notation related to the qualitative analysis of PDEs.
- Let L be a linear differential operator. For example $L = \sum_{|\alpha|\leq k} a_k \partial^k$.

Definition

A linear differential operator L with C^∞ coefficients is locally solvable at x_0 if there is a neighborhood Ω of x_0 such that for every $f \in C_c^\infty(\Omega)$ there exists $u \in \mathcal{D}'(\Omega)$ with $Lu = f$.

Treves F., On local solvability of linear partial differential equations, Bull. Amer. Math. Soc. 76 (3) (1970), 552–571.

- For example, every linear differential operator with constant coefficients is locally solvable.
- Moreover, if $\Omega = \mathbb{R}^n$ then the solution is in $C^\infty(\mathbb{R}^n)$.
- Until 1957. it was believed that the same/similar claim holds true for operators with analytic coefficients.

- H. Lewy found an example of a linear operator which is not locally solvable.[1]

Example

Consider the differential operator L defined on $\mathbb{R}^3$ with coordinates (x, y, t):

$$L = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - 2i(x+iy)\frac{\partial}{\partial t}.$$

There exist function $f \in C^\infty$ for which the equation $Lu = f$ has no solution, even in $\mathcal{D}'$.

- Along with local solvability, it is of interest to find whether the distributional solution (locally) coincides to an ”ordinary” function.

[1] Hans Lewy (1904–1988) ”An example of a smooth linear partial differential equation without solution”, Annals of Mathematics, 66 (1) (1957) 155-158.

- A linear differential operator L with C^∞ coefficients is called *hypoelliptic* if any $u \in \mathcal{D}'(\Omega)$ with $Lu \in C^\infty(\Omega)$ must be itself in $C^\infty(\Omega)$. Hypoellipticity thus comes as a refinement of local solvability.
- The Cauchy- Kowalevskaya theorem says that if the coefficients and the right hand side of the equation are analytic, then, under certain conditions on the boundary, the solution is real analytic. This is the *analytic hypoellipticity*.
- From another perspective, it is of interest to relate the singularities of the right hand side of the equation $Lu = f$ to the singularities of the solution u.
- Let f be a distribution. We say that $f \in C^\infty(U)$ (U is an open set) if there exists a function $g \in C^\infty(U)$ such that $\langle f, \varphi \rangle = \int g(x)\varphi(x)dx$ for all test functions φ supported in U.[2]
- The *singular support* of f is the complement of the union of all open sets U such that $f \in C^\infty(U)$.

[2] The support of φ is the closure of the set of points where $\varphi \neq 0$.

- The inclusion $\operatorname{singsupp} Lu \subseteq \operatorname{singsupp} u$ is obvious, since if $u \in C^\infty(U)$ then $Lu \in C^\infty(U)$ so the complement of $\operatorname{singsupp} Lu$ contains the complement of $\operatorname{singsupp} u$.
- The equality $\operatorname{singsupp} Lu = \operatorname{singsupp} u$ in fact reveals the notion of hypoellipticity.
- Recall, wavelets can be efficiently used for the resolution of $\operatorname{singsupp} u$.[3] But!
- *"... A shock can come from different directions..."*[4]
  More precisely, *"... If f is not smooth we can use the set of directions where $\hat{f}$ is not rapidly decreasing to describe which are the high frequency components of f causing the singularities..."*[5]
- If f is a compactly supported distribution, then $f \in C_0^\infty(\mathbb{R}^d)$ iff

$$|\hat{f}(\xi)| \leq C_N(1+|\xi|)^{-N}, \quad N = 1, 2, \ldots, \quad \xi \in \mathbb{R}^d.$$

[3] M. Holschneider, Wavelets: An Analysis Tool, Oxford University Press (1999).
[4] Ch. Kiselman in the speech for 1997 Schock Prize in Mathematics to Mikio Sato (1928 –).
[5] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Springer (1990).

- A distribution $f \in \mathcal{D}'(U)$ is *microlocally smooth* at $(x_0, \xi_0) \in U \times \mathbb{R}^d$ if there exists $\varphi \in \mathcal{D}(U)$, $\varphi(x_0) \neq 0$ and an open cone Γ containing ξ_0 such that for every $N \in \mathbb{N}$ there is a $C_N > 0$ such that

$$|\widehat{\varphi f}(\xi)| \leq C_N(1 + |\xi|)^{-N}, \quad \forall \xi \in \Gamma.$$

- The wave front set of f, $\mathrm{WF}(f)$, is the complement of the set of points (x, ξ) in which f is microlocally smooth.
- In the above definition it is implicitly assumed that the notion is independent on the choice of φ. This fact is nontrivial, and its proof contains some basic techniques in microlocal analysis.
- $\mathrm{WF}(f)$ is a closed conic set (if $(x, \xi) \in \mathrm{WF}(f)$ then $(x, \lambda\xi) \in \mathrm{WF}(f)$ for every $\lambda > 0$).
- If $(x, \xi) \in \mathrm{WF}(f)$ then $x \in \mathrm{singsupp} f$.

- Dirac's delta is defined by $\langle \delta, \varphi \rangle = \varphi(0)$ when $\varphi \in C_0^\infty$. Obviously $\operatorname{supp} \delta = \operatorname{singsupp} \delta = \{0\}$. Next, $\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \langle 1, \varphi \rangle$, so that $\mathrm{WF}(\delta) = \{(0, \xi) \mid \xi \neq 0\}$.
- $\operatorname{singsupp} \chi_{y>0} = \{(x, 0),\ x \in \mathbb{R}\}$.
  Let φ be "localized" around $(x_0, 0)$.

$$\widehat{\varphi\chi_{y>0}}(\xi, \eta) = \int \int_0^\infty e^{-2\pi i(x\xi + y\eta)} \varphi(x, y) dy dx = \int_0^\infty \mathcal{F}_1 \varphi(\xi, y) e^{-2\pi i y \eta} dy.$$

When $t \to \infty$ and $\xi \neq 0$ we have $\widehat{\varphi\chi_{y>0}}(t\xi, t\eta) \leq C_N \langle t \rangle^{-N}$, $N \in \mathbb{N}$.
It remains to check the points of the form $(x_0, 0, 0, \eta)$, $\eta \neq 0$.
Let $\varphi(x_0, 0) = \varphi_1(x_0)\varphi_2(0)$, so that $\varphi_1(x_0) = 1$, $\varphi_2(0) = 1$. Then, $\varphi\chi_{y>0}(x, y) = \varphi_1(x)\chi(y)$ ($\chi(y) = \varphi_2(y)$ when $y > 0$ and 0 otherwise) wherefrom

$$\widehat{\varphi\chi_{y>0}}(0, \eta) = \widehat{\varphi_1}(0)\widehat{\chi}(\eta),$$

which is not rapidly decreasing, since χ has jump discontinuity at $y = 0$.
If X is an open set in $\mathbb{R}^d$ and S a closed conic subset of $X \times (\mathbb{R}^d \setminus 0)$ then one and find $f \in \mathcal{D}'(X)$ with $\mathrm{WF}(f) = S$.[6]

[6] Hörmander, Thm 8.1.4

- As already mentioned a product of distributions whose singular supports do not overlap can be defined. Even if the singular supports of f and g overlap their product $f \cdot g$ can be defined if the folowing separation condition holds:
  We never had both $(x, \xi) \in \mathrm{WF}(f)$ and $(x, -\xi) \in \mathrm{WF}(g)$.
- An operator L is a microlocal operator if $\mathrm{WF}(Lu) \subseteq \mathrm{WF}(u)$. Microlocal hypoellipticity is described by $\mathrm{WF}(Lu) = \mathrm{WF}(u)$.
- For different classes of operators, distributions, and appropriate wave front sets, *propagation of singularities* takes the following form:

$$\mathrm{WF}(Lu) \subseteq \mathrm{WF}(u) \subseteq \mathrm{WF}(Lu) \cup \mathrm{Char}(L),$$

where $L = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$,

$$\mathrm{Char}_{x_0}(L) = \Big\{ (x_0, \xi) \in U \times \mathbf{R}^d \backslash \{0\} \mid L_m(x_0, \xi) = \sum_{|\alpha| = m} a_\alpha(x_0) \xi^\alpha = 0 \Big\},$$

and

$$\mathrm{Char}(L) = \bigcup_{x_0 \in U} \mathrm{Char}_{x_0}(L).$$

- If the coefficients $a_\alpha(x)$ are non-smooth (distributions) then it is natural to develop a Colombeau-type microlocal analysis. We refer to the series of papers by G. Hörmann, M. Oberguggenberger, S. Pilipović, and others....
- Different definitions of $\mathrm{WF}(f)$ are introduced, studied and applied to various problems. For example,
    - Analytic wave front sets and wave front sets with respect to smoothness classes (L. Hörmander in The Analysis of Linear Partial Differential Operators I, Springer (1990) and *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer, Berlin, 1997).
    - Gevrey wave-front sets (L. Rodino, *Linear Partial Differential Operators in Gevrey Spaces*, World Scientific, Singapore, 1993).
    - Global wave-front sets (Hörmander, *Quadratic hyperbolic operators* 1991 and L. Rodino, P. Wahlberg, *The Gabor wave front set*, 2013).
    - S-wave-front sets (with "exit-components") (S. Coriasco and L. Maniccia, in Ann. Glob. Anal. Geom. 2003).

- It turned out that Hörmander's global WFS, Nakamura's homogeneous WFS, and Gabor WFS are equal. Such WFS are given by cones taken with respect to the whole of the phase space variables and are recently successfully applied to the study of Schrödinger equations:

L. Hörmander, *Quadratic hyperbolic operators*, 118-160, in *Microlocal analysis and applications (Montecatini Terme, 1989)*, Lecture Notes in Math. 1495, Springer, 1991.

L. Rodino, P. Wahlberg, *The Gabor wave front set*, Monatsh. Math. **173 (4)** (2014), 625-655.

E. Cordero, F. Nicola, L. Rodino, *Schrödinger equations with rough Hamiltonians* Discrete Contin. Dyn. Syst. **35 (10)** (2015), 4805-4821. and *Propagation of the Gabor wave front set for Schrodinger equations with non-smooth potentials*, Rev. Math. Phys. **27 (1)** (2015), 33 pages

R. Schulz, P. Wahlberg, *Equality of the homogeneous and the Gabor wave front set*, Comm. Partial Differential Equations **42 (5)** (2017), 703-730.

K. Pravda-Starov, L. Rodino, P. Wahlberg, *Propagation of Gabor singularities for Schrödinger equations with quadratic Hamiltonians*, Mathematische Nachrichten, **291 (1)** (2018), 128-159

- Gevrey classes describe regularity between C^∞ and analyticity. Let $s \geq 1$.

$$G^s(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \ (\exists C > 0) \left|\partial^\beta f(x)\right| \leq C^{1+|\beta|} \beta!^s, \ \ \beta \in \mathbb{N}_0^d\}.$$

- Such classes are important e.g. in regularity theory (well posedness)in PDEs

M.D. Bronštein, The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity, Tr. Mosk. Mat. Obs. 41 (1980) 83–99.

- For example, a non-analytic solution of PDE which belongs to a class contained in every Gevrey Class:

Hörmander, Lars: A counterexample of Gevrey class to the uniqueness of the Cauchy problem, Math. Research Letters 7 (2000) 615–624

- Micro-local analysis in the context of Gevrey classes:

Rodino, Luigi: Linear partial differential operators in Gevrey spaces. World Scientific, 1993.

Definition

Let $s > 1$ and $u \in \mathcal{D}'_s(U)$, and $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0, \xi_0) \notin \mathrm{WF}_s(u)$ if there exists $\varphi \in G_0^s(U)$, $\varphi(x) = 1$ in an open neighborhood $\Omega \subset U$ of x_0, and a conic neighborhood Γ of ξ_0 such that

$$|\widehat{(\varphi u)}(\xi)| \leq A \, \frac{h^N N!}{|\xi|^{N/s}}, \quad N \in \mathbf{N}, \, \xi \in \Gamma$$

holds for some constants $A, h > 0$.

- If $u \in \mathcal{D}'(U)$ we have:

$$\mathrm{WF}(u) \subset \mathrm{WF}_s(u) \subset \mathrm{WF}_A(u).$$

- An extension to the Gevrey regularity is proposed in

S. Pilipović, N. Teofanov, F. Tomić, Beyond Gevrey regularity, Journal of Pseudo-Differential Operators and Applications, 7 (2016), 113–140.

- For given $\tau > 0$ and $\sigma > 1$ the spaces $\mathcal{E}_{\{\tau,\sigma\}}(U)$ are defined via the estimates of the form:

$$(\exists h > 0)|\partial^\alpha \phi(x)| \le h^{|\alpha|^\sigma+1}|\alpha|^{\tau|\alpha|^\sigma}, \ \alpha \in \mathbf{N}^d.$$

- $\mathcal{E}_{\{s,1\}}(U) = G^s(U)$ and $\mathcal{E}_{\{1,1\}}(U) = \mathcal{A}(U)$.
- Let $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, and let $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ if and only if there exists a conic neighborhood Γ_0 of ξ_0, a compact set $K \subset\subset U$ and $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$, $\phi = 1$ on a neighborhood of x_0, and such that

$$|\widehat{\phi u}(\xi)| \le A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \Gamma_0,$$

  for some $A, h > 0$.
- Let $u \in \mathcal{D}'(U)$, $t > 1$, $\tau > 0$ and $\sigma > 1$. Then

$$\mathrm{WF}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(u) \subseteq \bigcap_{t>1} \mathrm{WF}_t(u) \subseteq \mathrm{WF}_A(u).$$

- The wavefront set of a distribution is exactly identified by the decay properties of Continuous Curvelet transform and Continuous Shearlet transform.
- Moreover, the shearlet system allows a natural digitalization of the continuum world, leading, for example, to optimally sparse approximations of cartoon-like images.
- One of the main tools in time–frequency analysis is the short–time Fourier transform, and frames related to its ”discretization”. Appropriate functional analysis background is formulated in terms of *modulation spaces*.
- Gabor frames already posses translation and rotation parameters. For micro-local analysis we therefore need to introduce an additional parameter which corresponds to different scales. This is in the background of our definition of a Gabor pair.
- It turned out that the modulation spaces are (micro)locally equivalent to Fourier Lebesgue spaces.

- Warning: very technical part, not easy to follow. Serves as an illustration of results on my microlocal analysis in the context of TFA.
- A positive and measurable function ω is v-moderate weight if $(\exists C > 0)$ such that
$$\omega(x+y) \leq C\omega(x)v(y), \quad x, y \in \mathbb{R}^d$$
If v can be chosen as a polynomial, then $\omega \in \mathcal{P}(\mathbb{R}^d)$.
- Let $q \in [1, \infty]$ and $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then the (weighted) Fourier Lebesgue space $\mathcal{F}L^q_{(\omega)}(\mathbb{R}^d)$ is the Banach space of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that
$$\|f\|_{\mathcal{F}L^q_{(\omega)}} \equiv \|\widehat{f} \cdot \omega\|_{L^q} < \infty.$$

- Let $p,q \in [1,\infty]$, $\phi \in \mathcal{S}(\mathbb{R}^d)$ and let $\omega \in \mathcal{P}(\mathbb{R}^{2d})$. Then the modulation space $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |V_\phi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty$$

  where

$$V_\phi f(x,\xi) \equiv \mathcal{F}(f\,\overline{\phi(\,\cdot\, - x)})(\xi)$$

  is the short-time Fourier transform of f w.r.t. ϕ.
- The definitions of wave-front sets with respect to Fourier Lebesgue types and modulation space types depend on semi-norms which are defined in a similar way as the norms of these spaces.

- Let $p, q \in [1, \infty], f \in \mathcal{S}'(\mathbb{R}^d)$ and let $\Gamma \subseteq \mathbb{R}^d \setminus 0$ be an open cone. Set

$$|f|_{\mathcal{B}(\Gamma)} \equiv \Big(\int_\Gamma |\widehat{f}(\xi)\omega(x,\xi)|^q \, d\xi \Big)^{1/q},$$

$$|f|_{\mathcal{C}(\Gamma)} \equiv \Big(\int_\Gamma \Big(\int_{\mathbb{R}^d} |V_\phi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q}, \qquad (1)$$

when $\quad \mathcal{B} = \mathcal{F}L^q_{(\omega)}(\mathbb{R}^d) \quad$ and $\quad \mathcal{C} = M^{p,q}_{(\omega)}(\mathbb{R}^d)$.

respectively.

- The wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$, respectively $\mathrm{WF}_{\mathcal{C}}(f)$, of $f \in \mathcal{D}'(U)$, U is open in $\mathbb{R}^d$, consists of all pairs $(x_0, \xi_0) \in U \times (\mathbb{R}^d \setminus 0)$ such that for each $\varphi \in C_0^\infty(U)$ with $\varphi(x_0) \neq 0$ and each conical neighborhood Γ of ξ_0 it holds

$$|\varphi f|_{\mathcal{B}(\Gamma)} = +\infty, \quad \text{respectively} \quad |\varphi f|_{\mathcal{C}(\Gamma)} = +\infty.$$

- Information on regularity in the background of wave-front sets of Fourier Lebesgue types might be more detailed comparing to classical wave-front sets, because we may play with the exponent $q \in [1, \infty]$ and the weight function ω in $\mathcal{F}L^q_{(\omega)}(\mathbb{R}^d)$.
- By choosing $q = 1$ and $\omega(\xi) = \langle \xi \rangle^N$, where $N \geq 0$ is an integer, $\mathcal{F}L^1_{(\omega)}(\mathbb{R}^d)$ locally contains $C^{N+d+1}(\mathbb{R}^d)$, and is contained in $C^N(\mathbb{R}^d)$. Consequently, our wave-front sets can be used to investigate micro-local properties which, in some sense, are close to C^N-regularity.
- For each distribution f we have

$$\mathrm{WF}_{\mathcal{B}}(f) = \mathrm{WF}_{\mathcal{C}}(f), \tag{2}$$

where

$$\mathcal{B} = \mathcal{F}L^q_{(\omega)}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{C} = M^{p,q}_{(\omega)}(\mathbb{R}^d), \quad p, q \in [1, \infty]. \tag{3}$$

- Therefore, a compactly supported distribution f belongs to $\mathcal{B} = \mathcal{F}L^q_{(\omega)}(\mathbb{R}^d)$ or $\mathcal{B} = M^{p,q}_{(\omega)}(\mathbb{R}^d)$ iff $\mathrm{WF}_{\mathcal{B}}(f) = \emptyset$.

S. Pilipović, N. Teofanov and J. Toft, Micro-local analysis in Fourier Lebesgue and modulation spaces. Part I, J. Fourier Anal. Appl. **17 (3)**, 374–407 (2011)

- The classical wave front set can be characterized within our approach.
- The same is true for the Gevrey wave front set, if we use weights of almost exponential growth instead.

K. Johansson, S. Pilipović, N. Teofanov and J. Toft, A note on wave-front sets of Roumieu type Ultradistributions, in Operator Theory: Advances and Applications, **231**, 229–242, Sringer, Basel, 2013.

- A discrete analogue of $\mathrm{WF}(f)$, called *toroidal* wave-front sets ($\mathrm{WF}_T(f)$) is introduced by Ruzhansky and Turunen

M. Ruzhansky and V. Turunen, *Pseudo-Differential Operators and Symmetries: Background Analysis and Advanced Topics*, Birkhäuser, Boston, 2010.

Note that $\mathrm{WF}_T(f)$ can capture rational directions of singularities.

- A discrete analogue of $\mathrm{WF}_{\mathcal{B}}(f)$ is introduced in

K. Johansson, S. Pilipović, N. Teofanov and J. Toft, Gabor pairs, and a discrete approach to wave-front sets, Monatsh. Math. **166 (2)**, 181–199 (2012).

- Warning: Even more technical part. (The good news: we are close to the end of the lecture.)
- Let $b_1, \dots, b_d$ be a basis in $\mathbb{R}^d$. Then

$$\Lambda = \{\, a_1 b_1 + \cdots + a_d b_d \,;\, a_1, \dots, a_d \in \mathbf{Z} \,\}$$

  is a lattice in $\mathbb{R}^d$.
- Let $e_1, \dots, e_d$ in $\mathbb{R}^d$ be a basis for Λ, i. e. for some $x_0 \in \Lambda$ we have

$$\Lambda = \{\, x_0 + a_1 e_1 + \cdots + a_d e_d \,;\, a_1, \dots, a_d \in \mathbb{Z} \,\}.$$

- Assume that Λ_1 and Λ_2 are lattices in $\mathbb{R}^d$ with bases $e_1, \dots, e_d$ and $\varepsilon_1, \dots, \varepsilon_d$ respectively. Then the pair (Λ_1, Λ_2) is called a *strongly admissible lattice pair* if $0 < \langle e_j, \varepsilon_j \rangle < 2\pi$ and $\langle e_j, \varepsilon_k \rangle = 0$ when $j \neq k$.

- Assume that $q \in [1,\infty]$, $\omega \in \mathcal{P}(\mathbb{R}^d)$ and that $H \subseteq \mathbb{R}^d$ is a discrete set. Then we set

$$|f|^{(D)}_{\mathcal{B}(H)} \equiv \Big(\sum_{\{\xi_k\} \in H} |\widehat{f}(\xi_k)\omega(\xi_k)|^q \Big)^{1/q},$$

(with obvious modifications when $q = \infty$).

Definition

Let $q \in [1,\infty], f \in \mathcal{D}'(U)$, U is open in $\mathbb{R}^d$, $x_0 \in U$, and let (Λ_1, Λ_2) be a strongly admissible lattice pair in $\mathbb{R}^d$ such that $x_0 \notin \Lambda_1$. Moreover, let $\omega \in \mathcal{P}(\mathbb{R}^d)$ and $\mathcal{B} = \mathcal{F}L^q_{(\omega)}(\mathbb{R}^d)$. Then the discrete wave-front set $\mathrm{DF}_{\mathcal{B}}(f)$ consists of all $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ such that for each $\varphi \in C_0^\infty(U)$ with $\varphi(x_0) \neq 0$ and each open conical neighborhood Γ of ξ_0, it holds

$$|\varphi f|^{(D)}_{\mathcal{B}(\Gamma)} = \infty.$$

Definition

Assume that $\varepsilon \in (0,1]$, $\{x_j\}_{j\in J} = \Lambda_1 \subseteq \mathbb{R}^d$ and $\{\xi_k\}_{k\in J} = \Lambda_2 \subseteq \mathbb{R}^d$ are lattices and let $\Lambda_1(\varepsilon) = \varepsilon\Lambda_1$. Also let $\phi, \psi \in C_0^\infty(\mathbb{R}^d)$ be non-negative, and set

$$
\begin{aligned}
\phi^\varepsilon &= \phi(\,\cdot\,/\varepsilon), & \psi^\varepsilon &= \psi(\,\cdot\,/\varepsilon), \\
\phi^\varepsilon_{j,k} &= \phi^\varepsilon(\,\cdot\, - \varepsilon x_j)e^{i\langle\,\cdot\,,\xi_k\rangle}, & \psi^\varepsilon_{j,k} &= \psi^\varepsilon(\,\cdot\, - \varepsilon x_j)e^{i\langle\,\cdot\,,\xi_k\rangle},
\end{aligned} \tag{4}
$$

when $\varepsilon x_j \in \Lambda_1(\varepsilon)$ (i. e. $x_j \in \Lambda_1$) and $\xi_k \in \Lambda_2$. Then the pair

$$
(\{\phi^\varepsilon_{j,k}\}_{j,k\in J}, \{\psi^\varepsilon_{j,k}\}_{j,k\in J}) \tag{5}
$$

is called a *Gabor pair* with respect to the lattices Λ_1 and Λ_2 if for each $\varepsilon \in (0,1]$, the sets $\{\phi^\varepsilon_{j,k}\}_{j,k\in J}$ and $\{\psi^\varepsilon_{j,k}\}_{j,k\in J}$ are dual Gabor frames.

- If the pair in (5) is a Gabor pair, then the lattice pair (Λ_1, Λ_2) in Definition 5 is strongly admissible.

- For the definition of discrete wave-front sets of modulation spaces, we consider Gabor pairs $(\{\phi^{\varepsilon}_{j,k}\}_{j,k\in J}, \{\psi^{\varepsilon}_{j,k}\}_{j,k\in J})$, $\varepsilon \in (0,1]$, and let
$$J_{x_0}(\varepsilon) = J_{x_0}(\varepsilon,\phi,\psi) = J_{x_0}(\varepsilon,\phi,\psi,\Lambda_1)$$
be the set of all $j \in J$ such that $x_0 \in \operatorname{supp} \phi^{\varepsilon}_{j,k} \quad \text{or} \quad x_0 \in \operatorname{supp} \psi^{\varepsilon}_{j,k}$.
- Let $p,q \in [1,\infty]$, $f \in \mathcal{D}'(U)$, $U \subseteq \mathbb{R}^d$ be open, $x_0 \in U$, and let $(\{\phi^{\varepsilon}_{j,k}\}_{j,k\in J}, \{\psi^{\varepsilon}_{j,k}\}_{j,k\in J})$ be a Gabor pair with respect to the lattices Λ_1 and Λ_2 in $\mathbb{R}^d$. Moreover, let $\omega \in \mathcal{P}(\mathbb{R}^{2d})$ and $\mathcal{C} = M^{p,q}_{(\omega)}(\mathbb{R}^d)$. Then the discrete wave-front set $\mathrm{DF}_{\mathcal{C}}(f)$ consists of all $(x_0,\xi_0) \in U \times (\mathbb{R}^d \setminus 0)$ such that for each $\varepsilon \in (0,1]$ and each open conical neighbourhood Γ of ξ_0, it holds
$$\Bigg(\sum_{\{\xi_k\}\in\Gamma\cap\Lambda_2} \Bigg(\sum_{j\in J_{x_0}(\varepsilon)} |c_{j,k}(\varepsilon)\omega(\xi_k)|^p \Bigg)^{q/p} \Bigg)^{1/q} = \infty,$$
where $f = \sum_{j,k\in J} c_{j,k(\varepsilon)}\phi^{\varepsilon}_{j,k}, \quad c_{j,k}(\varepsilon) = C_{\phi,\psi}(f,\psi^{\varepsilon}_{j,k})_{L^2(\mathbb{R}^d)}$ and the constant $C_{\phi,\psi}$ depends on ϕ and ψ only.

Theorem

Let $X \subseteq \mathbb{R}^d$ be open and let $f \in \mathcal{D}'(X)$. Then

$$\mathrm{WF}_{\mathcal{B}}(f) = \mathrm{WF}_{\mathcal{C}}(f) = \mathrm{DF}_{\mathcal{B}}(f) = \mathrm{DF}_{\mathcal{C}}(f).$$

- microlocal analysis is a refinement of local analysis. It is useful for different purposes such as the geometric separation, the product of distributions and propagation of singularities.
- different problems ask for modified versions of the notion of microlocal regularity.
- a common task in diverse applications might be the study of coherence between the discrete and continuum variants of the problem.