Equivalence of norms on finite linear combinations of atoms *

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Abstract

Let M be a space of homogeneous type and denote by $F^{\infty}_{\mathrm{cont}}(M)$ the space of finite linear combinations of *continuous* $(1,\infty)$-atoms. In this note we give a simple function theoretic proof of the equivalence on $F^{\infty}_{\mathrm{cont}}(M)$ of the H^1-norm and the norm defined in terms of *finite* linear combinations of atoms. The result holds also for the class of nondoubling metric measure spaces considered in previous works of A. Carbonaro and the authors.

0 Introduction

Suppose that q is in $(1,\infty]$. A function a in $L^1(\mathbb{R}^d)$ is said to be a $(1,q)$-atom if it is supported in a ball B in $\mathbb{R}^d$, and satisfies the following conditions

$$\int_{\mathbb{R}^d} a(x)\,\mathrm{d}\lambda(x) = 0 \qquad\qquad \|a\|_q \leq \lambda(B)^{-1/q'},$$

where λ denotes the Lebesgue measure on $\mathbb{R}^d$ and q' is the conjugate index to q. Denote by $F^q(\mathbb{R}^d)$ the vector space of all *finite* linear combinations of $(1,q)$-atoms, endowed with the norm $\|\cdot\|_{F^q}$, defined as follows

$$\|f\|_{F^q} = \inf\Big\{\sum_{j=1}^{N} |\lambda_j| : f = \sum_{j=1}^{N} \lambda_j\, a_j,\ a_j \text{ is a } (1,q)\text{-atom},\ N \in \mathbb{N}^+\Big\}.$$

In [MSV], the authors proved that if q is finite, then the F^q norm and the restriction to $F^q(\mathbb{R}^d)$ of the atomic $H^1(\mathbb{R}^d)$ norm (defined just below (1.1)) are equivalent. The proof hinges on the atomic decomposition and the maximal characterisation of $H^1(\mathbb{R}^d)$, and is quite technical. A similar result holds for $F^{\infty}_{\mathrm{cont}}(\mathbb{R}^d)$, a space defined much as $F^q(\mathbb{R}^d)$, but with continuous $(1,\infty)$-atoms in place $(1,q)$-atoms. In [GLY] the authors, by adapting the techniques of [MSV], succeeded to extend these results to homogeneous spaces that satisfy an additional property, called property RD. Apparently, there are serious obstructions in extending this approach to all spaces of homogeneous type.

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F. Ricci and J. Verdera [RV] complemented the analysis in [MSV] by proving that the dual of the completion of $F^\infty(\mathbb{R}^d)$ is the direct sum of $BMO(\mathbb{R}^d)$ and a nontrivial Banach space. They observed also that variations of the main argument in the proof of [RV, Thm 1] provide an alternative proof of the equivalence of the $F^q(\mathbb{R}^d)$ and the $H^1(\mathbb{R}^d)$ norms on $F^q(\mathbb{R}^d)$ for $q < \infty$. Their argument revolves around the Gelfand–Naimark theory for the commutative Banach algebra of all bounded functions on $\mathbb{R}^d$ that vanish at infinity.

The purpose of this paper is to show that there is an easy function theoretic approach to the equivalence of the $F^\infty_{\mathrm{cont}}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ norms on $F^\infty_{\mathrm{cont}}(\mathbb{R}^d)$, which holds in a very general setting, including all locally compact spaces of homogeneous type and many interesting locally doubling measured metric spaces, like those introduced in [CMM1, CMM2]. A similar argument works in the case where $q < \infty$ and gives the equivalence of the $F^q(M)$ and $H^1(M)$ norms on $F^q(M)$ when M is as above. Our proof does not make use of the Gelfand–Naimark theory for commutative Banach algebras, although our main idea, which is to prove that the dual of $\overline{F}^\infty_{\mathrm{cont}}(M)$ is just $BMO(M)$, was inspired by [RV].

For the sake of simplicity, we shall restrict to spaces of infinite measure. The case of spaces of finite measure may be treated similarly, with very few changes due to the exceptional constant atom and the slighty different definition of BMO.

We mention that our result is related to the extendability of linear operators defined on finite linear combinations of atoms. Denote by Y a Banach space and suppose that q is in $(1, \infty]$. We say that Y has the *q-extension property* if every Y-valued linear operator T, defined on *finite* linear combinations of $(1, q)$-atoms and satisfying the condition

$$\sup\{\|Ta\|_Y : a \text{ is a } (1, q)\text{-atom}\} < \infty, \tag{0.1}$$

extends to a bounded operator from $H^1(M)$ to Y. Similarly, we say that Y has the *continuous ∞-extension property* if T satisfies the condition above with continuous $(1, \infty)$-atoms in place of $(1, q)$-atoms.

On the one hand, no Banach space Y has the ∞-extension property. Indeed, a direct consequence of a recent result of M. Bownik [Bow] is that for every Banach space Y there exists a Y-valued operator B, defined on finite linear combinations of $(1, \infty)$-atoms that satisfies

$$\sup\{\|Ba\|_Y : a \text{ is a } (1, \infty)\text{-atom}\} < \infty, \tag{0.2}$$

but does not admit an extension to a bounded operator from $H^1(\mathbb{R}^d)$ to Y. On the other hand, every Banach space Y has the q-extension property for all q in $(1, \infty)$, and the continuous ∞-extension property. This is proved in [YZ] in the case where $q = 2$, and, independently, in [MSV] for all q in $(1, \infty)$ and in the continuous ∞ case.

Our analysis is limited to the Hardy space $H^1(M)$, because the applications we have in mind are to the space H^1 rather than to H^p with p in $(0, 1)$. We leave the investigation of the interesting case p in $(0, 1)$ to further studies. An elegant analysis of the case $p \leq 1$ in the Euclidean setting may be found in [RV]. See also the papers [MSV, YZ].

1 Notation and background information

Suppose that (M, ρ, μ) is a space of homogeneous type. In particular, ρ is a quasi-distance on M and μ is a regular Borel measure on M. We shall assume that $\mu(M) = \infty$ and that $\mu(B) < \infty$ for all balls in M. We refer the reader to [CW] for any unexplained notation concerning spaces of homogeneous type.

Definition 1.1 Suppose that q is in $(1, \infty]$. A $(1, q)$-*atom* a associated to a ball B is a function in $L^1(M)$ supported in B with the following properties:

(i) $\|a\|_q \leq \mu(B)^{-1/q'}$, where q' denotes the index conjugate to q;

(ii) $\displaystyle\int_B a \, \mathrm{d}\mu = 0$.

The *Hardy space* $H^{1,q}(M)$ is the space of all functions g in $L^1(M)$ that admit a decomposition of the form

$$g = \sum_{k=1}^{\infty} \lambda_k \, a_k, \tag{1.1}$$

where a_k is a $(1, q)$-atom and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. The norm $\|g\|_{H^{1,q}}$ of g is the infimum of $\sum_{k=1}^{\infty} |\lambda_k|$ over all decompositions (1.1) of g.

For each q in $(1, \infty)$ we denote by $F^q(M)$ the vector space of all finite linear combination of $(1, q)$-atoms. A natural norm on $F^q(M)$ is defined as follows

$$\|f\|_{F^q} = \inf\Big\{\sum_{j=1}^{N} |\lambda_j| : f = \sum_{j=1}^{N} \lambda_j \, a_j, \ a_j \text{ is a } (1, q)\text{-atom}, \ N \in \mathbb{N}^+\Big\}. \tag{1.2}$$

Similarly, we denote by $F^\infty_{\mathrm{cont}}(M)$ the vector space of all finite linear combination of *continuous* $(1, \infty)$-atoms. The $F^\infty_{\mathrm{cont}}(M)$ norm is defined as the $F^q(M)$ norm above, but with $(1, \infty)$ continuous atoms in place of $(1, q)$-atoms. Obviously

$$\|f\|_{H^1} \leq \|f\|_{F^q} \qquad \forall f \in F^q(M). \tag{1.3}$$

Clearly, $F^q(M)$ is a dense subspace of $H^1(M)$ with respect to the norm of $H^1(M)$. Similar observations apply to $F^\infty_{\mathrm{cont}}(M)$.

For each q in $(1, \infty)$, denote by $\overline{F}^q(M)$ the completion of $F^q(M)$ with respect to the norm $\|\cdot\|_{F^q}$. The completion of $F^\infty_{\mathrm{cont}}(M)$ with respect to the norm $\|\cdot\|_{F^\infty_{\mathrm{cont}}(M)}$ will be denoted by $\overline{F}^\infty_{\mathrm{cont}}(M)$.

The dual of the Banach space A will be denoted by A^*.

2 The main result

The main result of this paper is the following.

Theorem 2.1 *For each q in $(1, \infty)$ there exists a constant C such that*

$$\|f\|_{H^1} \leq \|f\|_{F^q} \leq C \, \|f\|_{H^1} \qquad \forall f \in F^q(M). \tag{2.1}$$

Similarly, if M is locally compact, then there exists a constant C such that

$$\|f\|_{H^1} \leq \|f\|_{F^\infty_{\rm cont}} \leq C\,\|f\|_{H^1} \qquad \forall f \in F^\infty_{\rm cont}(M). \tag{2.2}$$

Steps of the proof. Note that (2.1) does not require the assumption that M be locally compact, which is used, instead, in the proof of (2.2) to identify the dual of the space of continuous functions with support contained in a closed ball B with the space of all complex Radon measures on the interior of B. The proof of each of the two statements of the theorem is divided into three steps. We illustrate those needed to prove (2.2). The proof of (2.1) is similar, perhaps easier, and is omitted.

(I) Prove (see Lemma 2.2) that $BMO(M)$ is isomorphic as a Banach space to the dual of $\overline{F}^\infty_{\rm cont}(M)$.

(II) The identity is a continuous linear map from $F^\infty_{\rm cont}(M)$ to $H^1(M)$ by (1.3). Hence it extends (uniquely) to a continuous linear map, $\iota_{\rm cont}$ say, from the completion $\overline{F}^\infty_{\rm cont}(M)$ of $F^\infty_{\rm cont}(M)$ to $H^1(M)$. Prove (see Theorem 2.4) that $\iota_{\rm cont} : \overline{F}^\infty_{\rm cont}(M) \to H^1(M)$ is injective, and its transpose map $\iota^t_{\rm cont}$ is a Banach space isomorphism between $H^1(M)^*$ and $\overline{F}^\infty_{\rm cont}(M)^*$.

(III) Use Lemma 2.3 to conclude that $\iota_{\rm cont}$ is Banach space isomorphisms between $\overline{F}^\infty_{\rm cont}(M)$ and $H^1(M)$.

Since the restriction of $\iota_{\rm cont}$ to $F^\infty_{\rm cont}(M)$ is the identity, the required conclusion follows. □

Lemma 2.2 *The following hold:*

(i) if M is locally compact, then $\overline{F}^\infty_{\rm cont}(M)^*$ is isomorphic to $BMO(M)$.

(ii) $\overline{F}^q(M)^*$ is isomorphic to $BMO(M)$ for every q in $(1, \infty)$;

Proof. First we prove (i). If g is in $BMO(M)$, then the functional F, defined on $F^\infty_{\rm cont}(M)$ by

$$F(f) = \int_M f\, g \,\mathrm{d}\mu \qquad \forall f \in F^\infty_{\rm cont}(M)$$

satisfies the estimate

$$\begin{aligned} |F(f)| &\leq \|f\|_{H^1}\, \|g\|_{BMO} \\ &\leq \|f\|_{F^\infty_{\rm cont}}\, \|g\|_{BMO} \qquad \forall f \in F^\infty_{\rm cont}(M), \end{aligned}$$

hence it extends to a continuous linear functional on $\overline{F}^\infty_{\rm cont}(M)$ with norm at most $\|g\|_{BMO}$.

Next we assume that F is in $\overline{F}^\infty_{\rm cont}(M)^*$. Then

$$\sup\{|Fa| : a \text{ is a continuous } (1,\infty)\text{-atom}\} \leq \|F\|. \tag{2.3}$$

For each closed ball B in M, we denote by $C_B(M)$ the space of all continuous functions on M which are supported in B and by $C_{B,0}(M)$ the subspace of those that have integral 0. Since M is locally compact, the dual of $C_B(M)$ is the space of finite Radon measures on the interior of B. Furthermore, the annihilator of $C_{B,0}(M)$ in the dual of $C_B(M)$ is $\mathbb{C}\mu$. Indeed, μ annihilates $C_{B,0}(M)$ by definition and, if ν is a Radon measure on the interior of B that annihilates $C_{B,0}(M)$, then $\nu = \alpha\mu$ for some $\alpha \in \mathbb{C}$, because $\ker(\mu) \subset \ker(\nu)$ (here we slightly abuse the notation and denote by μ the restriction of μ to the interior of B). Thus the dual of $C_{B.0}(M)$ is the quotient $C_B^*(M)/\{\mathbb{C}\mu\}$.

Fix a reference point o in M and for each positive integer k denote by B_k the ball centred at o with radius k. For each ball B_k, and each f in $C_{B_k,0}(M)$, the function $f/\big(\mu(B_k)\,\|f\|_\infty\big)$ is a continuous $(1,\infty)$-atom, so that

$$|Ff| \leq \|\!|F|\!\|\,\mu(B_k)\,\|f\|_\infty \qquad \forall f \in C_{B_k,0}(M) \tag{2.4}$$

by (2.3). Hence the restriction of F to $C_{B_k,0}(M)$ is a bounded linear functional on $C_{B_k,0}(M)$ for each k. Therefore there exists a Radon measure μ_k such that

$$Ff = \int_{B_k} f \,\mathrm{d}\mu_k \qquad \forall f \in C_{B_k,0}(M) \tag{2.5}$$

and two such measures differ only by a complex multiple of μ on the interior of B_k. Thus, we may choose the measures μ_k so that the restriction of μ_k to the interior of B_{k-1} coincides with μ_{k-1}.

We claim that μ_k is absolutely continuous with respect to the restriction of μ to the interior of B_k. To prove this, we first show that

$$\big|\mu_k(B)\big| \leq \|\!|F|\!\|\,\mu(B) \tag{2.6}$$

for every open ball B contained in the interior of B_k. By inner regularity of Radon measures

$$\big|\mu_k(B)\big| = \sup\Big\{\Big|\int_{B_k} f\,\mathrm{d}\mu_k\Big| : f \in C_{B_k}(M),\ f \prec B\Big\}.$$

Now, (2.4) and (2.5) imply that

$$\Big|\int_{B_k} f\,\mathrm{d}\mu_k\Big| \leq \|\!|F|\!\|\,\mu(B),$$

and (2.6) follows.

Now, (2.6) implies that there exists a constant C such that

$$\big|\mu_k(U)\big| \leq C\,\|\!|F|\!\|\,\mu(U)$$

for every open set U in contained in the interior of B_k. Indeed, there exists a positive integer N, depending on the doubling constant of the space, such that for every $\varepsilon > 0$ there exists a sequence $\{B^j\}$ of mutually disjoint open balls such that

$$\bigcup_j B^j \subset U \subset \bigcup_j (NB^j).$$

Then

$$\begin{aligned}\big|\mu_k(U)\big| &\leq \sum_j \big|\mu_k(NB^j)\big| \\ &\leq \|\!|F|\!\|\, \sum_j \big|\mu(NB^j)\big| \\ &\leq C\, \|\!|F|\!\|\, \sum_j \big|\mu(B^j)\big| \\ &\leq C\, \|\!|F|\!\|\, \mu(U),\end{aligned}$$

as required.

Since μ_k is a finite Radon measure, it is outer regular on all Borel sets, whence

$$\big|\mu_k(E)\big| \leq C\, \|\!|F|\!\|\, \mu(E)$$

for every Borel set E contained in the interior of B_k. This inequality implies that μ_k is absolutely continuous with respect to μ. By the Radon–Nykodim Theorem, there exists a function g_k in $L^1(B_k, \mu)$ such that

$$\mu_k(E) = \int_E g_k \,\mathrm{d}\mu$$

for every Borel set E contained in B_k. We may choose g_k so that its restriction to B_{k-1} agrees with g_{k-1}. Denote by g_F the function on M such that its restriction to B_k agrees with g_k. Clearly

$$Ff = \int_M f\, g_F \,\mathrm{d}\mu \qquad \forall f \in C_{c,0}(M). \tag{2.7}$$

In particular, this holds whenever f is a continuous $(1,\infty)$-atom.

To conclude the proof of the claim it suffices to prove that g_F belongs to $BMO(M)$ and that

$$\|g_F\|_{BMO} \leq \|\!|F|\!\|. \tag{2.8}$$

This is a classical argument, which works also in our setting. We give the details for the reader's convenience. Suppose that B is a ball and observe that

$$\int_B |g_F - (g_F)_B| \,\mathrm{d}\mu = \sup_{\|\varphi\|_\infty = 1} \Big| \int_B \varphi\,\big(g_F - (g_F)_B\big) \,\mathrm{d}\mu \Big|,$$

where the supremum is taken with respect to all functions φ in $C_{B,0}(M)$. Since

$$\int_B \varphi\,\big(g_F - (g_F)_B\big) \,\mathrm{d}\mu = \int_B \varphi\, g_F \,\mathrm{d}\mu$$

and $\varphi/\mu(B)$ is a continuous $(1,\infty)$-atom,

$$\Big| \int_B \varphi\; g_F \,\mathrm{d}\mu \Big| \leq \|\!|F|\!\|\, \mu(B).$$

Combining the above three formulae, we conclude that for every ball B

$$\frac{1}{\mu(B)} \int_B |g_F - (g_F)_B| \,\mathrm{d}\mu \leq \|\!|F|\!\|,$$

and (2.8) follows.

The proof of (ii) follows the same lines. We simply replace $C_{B,0}(M)$ by the space $L^q_0(M)$ of all functions in $L^q(M)$ that are supported in B and have integral 0. $\square$

Lemma 2.3 *Suppose that A and B are Banach spaces and that $\iota : A \to B$ is an injective continuous linear map such that $\iota^t : B^* \to A^*$ is an isomorphism. Then $\iota : A \to B$ is an isomorphism. In particular, if $\iota : A \to B$ is the inclusion map, then $A = B$ with equivalent norms.*

Proof. Suppose that a is in A. By the Hahn-Banach theorem there exists λ in A^* such that $\|\lambda\|_{A^*} = 1$ and $\lambda(a) = \|a\|_A$. Since ι^t is an isomorphism between B^* and A^*, there exists a unique element μ in B^* such that $\iota^t(\mu) = \lambda$. Then, on the one hand

$$\begin{aligned}\|a\|_A &= \lambda(a) = \iota^t(\mu)(a) = \mu\big(\iota(a)\big)\\ &\leq \|\mu\|_{B^*}\, \|\iota(a)\|_B = \|(\iota^t)^{-1}\lambda\|_{B^*}\, \|\iota(a)\|_B\\ &\leq |||(\iota^t)^{-1}|||\, \|\iota(a)\|_B.\end{aligned}$$

On the other hand $\|\iota(a)\|_B \leq |||\iota|||\, \|a\|_A$, because ι is assumed to be continuous. Therefore $\|a\|_A$ and $\|\iota(a)\|_B$ are equivalent norms on A and $\iota(A)$ respectively. Hence $\iota : A \to \iota(A)$ is an isomorfism and $\iota(A)$ is closed in B.

To conclude the proof it remains to prove that $\iota(A) = B$. We argue by contradiction. Suppose that $\iota(A)$ is properly included in B, and choose b in $B \setminus \iota(A)$. By the Hahn-Banach theorem there exists a linear functional μ in B^* such that $\mu(b) = 1$ and $\mu(\iota(a)) = 0$ for every a in A. Then $\iota^t(\mu) = 0$, thereby contradicting the injectivity of ι^t. $\square$

For each q in $(1,\infty)$, the identity is a continuous linear map from $F^q(M)$ to $H^1(M)$. Denote by ι_q its extension from $\overline{F}^q(M)$ into $H^1(M)$. The map ι_{cont} from $\overline{F}^\infty_{\mathrm{cont}}(M)$ to $H^1(M)$ is defined similarly.

Theorem 2.4 *The following hold:*

(i) if M is locally compact, then ι_{cont} is a Banach space isomorphism between $\overline{F}^\infty_{\mathrm{cont}}(M)$ and $H^1(M)$.

(ii) for each q in $(1,\infty)$ the map ι_q is a Banach space isomorphism between $\overline{F}^q(M)$ and $H^1(M)$.

Proof. We first prove (i). Clearly the transpose map ι^t_{cont} is a continuous linear mapping from the dual of $H^1(M)$ to the dual of $\overline{F}^\infty_{\mathrm{cont}}(M)$.

Step I: ι^t_{cont} is injective. Indeed, suppose that F is in the dual of $H^1(M)$ and that $\iota^t_{\mathrm{cont}}(F) = 0$. Then for every f in $\overline{F}^\infty_{\mathrm{cont}}(M)$

$$0 = \iota^t_{\mathrm{cont}}(F)(f) = F\big(\iota_{\mathrm{cont}}(f)\big) = (F \circ \iota_{\mathrm{cont}})(f).$$

Hence $F \circ \iota_{\mathrm{cont}}$ is the null functional on $\overline{F}^\infty_{\mathrm{cont}}(M)$. Since the restriction of ι_{cont} to $F^\infty_{\mathrm{cont}}(M)$ is the identity map, F is a continuous linear functional on $H^1(M)$

that vanishes on $F^\infty_{\mathrm{cont}}(M)$. Then there exists a function g_F in $BMO(M)$ such that

$$0 = \int_M f\, g_F \,\mathrm{d}\mu \qquad \forall f \in C_{c,0}(M)$$

for $F^\infty_{\mathrm{cont}}(M)$ coincides with $C_{c,0}(M)$. This implies that g_F is constant almost everywhere, so that F is the null functional, as required.

Step II: ι^t_{cont} is surjective. Suppose that F is a continuous linear functional on $\overline{F}^\infty_{\mathrm{cont}}(M)$. We need to prove that there exists a continuous linear functional G on $H^1(M)$ such that $\iota^t_{\mathrm{cont}}(G) = F$, i.e.,

$$F(f) = G\big(\iota_{\mathrm{cont}}(f)\big) \qquad \forall f \in \overline{F}^\infty_{\mathrm{cont}}(M). \tag{2.9}$$

By Lemma 2.2 there exists a function g_F in $BMO(M)$ such that

$$Ff = \int_M f\, g_F \,\mathrm{d}\mu \qquad \forall f \in F^\infty_{\mathrm{cont}}(M). \tag{2.10}$$

Denote by G the continuous linear functional on $H^1(M)$ associated to the $BMO(M)$ function g_F. Since the restriction of ι_{cont} to $F^\infty_{\mathrm{cont}}(M)$ is the identity, (2.9) holds for all f in $F^\infty_{\mathrm{cont}}(M)$. Then F and $G \circ \iota_{\mathrm{cont}}$ are continuous linear functionals on $\overline{F}^\infty_{\mathrm{cont}}(M)$ that agree on $F^\infty_{\mathrm{cont}}(M)$. Since $F^\infty_{\mathrm{cont}}(M)$ is dense in $\overline{F}^\infty_{\mathrm{cont}}(M)$, they agree on $\overline{F}^\infty_{\mathrm{cont}}(M)$, as required.

Step III: conclusion. We have proved that ι^t_{cont} is a continuous bijective operator from the dual of $H^1(M)$ onto the dual of $\overline{F}^\infty_{\mathrm{cont}}(M)$. Therefore ι^t_{cont} is an isomorphism between the dual of $H^1(M)$ and the dual of $\overline{F}^\infty_{\mathrm{cont}}(M)$. In view of Lemma 2.3, to conlude that ι_{cont} is an isomorphism between $\overline{F}^\infty_{\mathrm{cont}}(M)$ and $H^1(M)$ it suffices to prove that ι_{cont} is injective.

Suppose that $\overline{f}$ is in $\overline{F}^\infty_{\mathrm{cont}}(M)$ and that $\iota_{\mathrm{cont}}(\overline{f}) = 0$. Pick any Cauchy sequence $\{f_n\}$ of functions in $F^\infty_{\mathrm{cont}}(M)$ such that

$$\lim_{n\to\infty} \|f_n - \overline{f}\|_{\overline{F}^\infty_{\mathrm{cont}}} = 0.$$

Denote by $\overline{F}$ a continuous linear functional on $\overline{F}^\infty_{\mathrm{cont}}(M)$ such that $\overline{F}(\overline{f}) = \|\overline{f}\|_{\overline{F}^\infty_{\mathrm{cont}}}$. Then

$$\begin{aligned}
\|\overline{f}\|_{\overline{F}^\infty_{\mathrm{cont}}} &= \overline{F}(\overline{f}) \\
&= \lim_{n\to\infty} \overline{F}(f_n) \\
&= \lim_{n\to\infty} \langle f_n, g_{\overline{F}} \rangle ,
\end{aligned}$$

where $g_{\overline{F}}$ is the function in $BMO(M)$ which, by Lemma 2.2 represents the restriction of the functional $\overline{F}$ to $F^\infty_{\mathrm{cont}}(M)$. But $\{f_n\}$ converges to $\iota_{\mathrm{cont}}(\overline{f})$ in $H^1(M)$, by the definition of ι_{cont}. Since we assumed that $\iota_{\mathrm{cont}}(\overline{f}) = 0$, and

$$\lim_{n\to\infty} \langle f_n, g_{\overline{F}} \rangle = \big\langle \iota_{\mathrm{cont}}(\overline{f}), g_{\overline{F}} \big\rangle = 0,$$

we may conclude that $\|\overline{f}\|_{\overline{F}^\infty_{\mathrm{cont}}} = 0$, i.e, that $\overline{f} = 0$, as required.

The proof of (ii) is almost *verbatim* the same as the proof of (i), and is omitted. $\square$

3 Measured metric spaces

As mentioned in the introduction, the main result we presented in the last section in the case of spaces of homogenous type, may be generalised to a variety of settings. In this section we describe the generalisation to the case of the atomic Hardy spaces on certain measured metric spaces introduced in [CMM1, CMM2]. We restrict to the case where the space has infinite measure. Again, slight modifications will also cover the finite measure case.

Suppose that (M, ρ) is a locally compact metric space, and that μ is a regular Borel measure on (M, ρ). We shall also assume that (M, ρ) possesses that approximate midpoint property [CMM1, Section 2.1].

Denote by $H^1(M)$ the atomic Hardy space defined in [CMM1]. We recall that the definition of a $H^1(M)$-atom is exactly as in the case of spaces of homogeneous type, but, unlike in the classical case, $H^1(M)$ is the space of all functions g in $L^1(M)$ that admit a decomposition of the form

$$g = \sum_{k=1}^{\infty} \lambda_k \, a_k, \tag{3.1}$$

where a_k is a $H^1(M)$-atom *supported in a ball B of radius at most* 1, and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. The norm $\|g\|_{H^1}$ of g is the infimum of $\sum_{k=1}^{\infty} |\lambda_k|$ over all decompositions (3.1) of g.

As in the case of spaces of homogeneous type we may define the spaces $F^q(M)$, $F^{\infty}_{\mathrm{cont}}(M)$ and their completions $\overline{F}^q(M)$ and $\overline{F}^{\infty}_{\mathrm{cont}}(M)$. Clearly $F^q(M)$, $F^{\infty}_{\mathrm{cont}}(M)$ will involve only atoms supported in balls of radius at most 1.

Straightforward adaptations of the arguments of the previous section yield the following.

Theorem 3.1 *For each q in $(1, \infty)$ there exists a constant C such that*

$$\|f\|_{H^1} \leq \|f\|_{F^q} \leq C \, \|f\|_{H^1} \qquad \forall f \in F^q(M).$$

Similarly, if M is locally compact, then there exists a constant C such that

$$\|f\|_{H^1} \leq \|f\|_{F^{\infty}_{\mathrm{cont}}} \leq C \, \|f\|_{H^1} \qquad \forall f \in F^{\infty}_{\mathrm{cont}}(M).$$

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